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Geometry of Mus-Sasaki metric

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Abstract. In this paper, we introduce the Mus-Sasaki metric on the tangent bundle TM as a new natural metric non-rigid on TM. First we investigate the geometry of the Mus-Sasakian metrics and we characterize the sectional curvature and the scalar curvature.

1 Introduction

The geometry of the tangent bundle TM equiped with Sasaki metric has been studied by many authors Sasaki [16], K. Yano and S. Ishihara [18], P. Dombrowski [7], A. Salimov, A. Gezer and N. Cengiz (see [2], [8], [12], [13], [14]) etc. The rigidity of Sasaki metric has incited some geometers to construct and study other metrics on TM. J. Cheeger and D. Gromoll has introced the notion of Cheeger-Gromoll metric [3], this metric has been studied also by many authors (see [1], [4], [5], [6], [9], [10], [11], [15], [17]).

In this paper, we introduce the Mus-Sasaki metric on the tangent bundle TM as a new natural metric non-rigid on TM. First we investigate the geometry of the Mus-Sasakian metrics and we characterize the sectional curvature (Theorem 4 and Theorem 5) and the scalar curvature (Theorem 6 and Theorem 7).

1.1 Basic Notions and Definition on TM

Let (M, g) be an m-dimensional Riemannian manifold and (TM, π, M) be its tangent bundle. A local chart $(U, x^i)_{i=1,\dots,n}$ on M induces a local chart

$$(\pi^{-1}(U), x^i, y^i)_{i=1,\dots,n}$$

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on TM. Denote by Γ_{ij}^k the Christoffel symbols of g and by ∇ the Levi-Civita connection of g.

We have two complementary distributions on TM, the vertical distribution $\mathcal{V}$ and the horizontal distribution $\mathcal{H}$, defined by:

$$\begin{aligned}\mathcal{V}_{(x,u)} &= \operatorname{Ker}\big(d\pi_{(x,u)}\big) = \Big\{a^i \frac{\partial}{\partial y^i}\Big|_{(x,u)} : a^i \in \mathbb{R}\Big\}\\ \mathcal{H}_{(x,u)} &= \Big\{a^i \frac{\partial}{\partial x^i}\Big|_{(x,u)} - a^i u^j \Gamma_{ij}^k \frac{\partial}{\partial y^k}\Big|_{(x,u)} : a^i \in \mathbb{R}\Big\},\end{aligned}$$

where $(x,u) \in TM$, such that $T_{(x,u)}TM = \mathcal{H}_{(x,u)} \oplus \mathcal{V}_{(x,u)}$.

Let $X = X^i \frac{\partial}{\partial x^i}$ be a local vector field on M. The vertical and the horizontal lifts of X are defined by

$$X^V = X^i \frac{\partial}{\partial y^i} \tag{1}$$

$$X^H = X^i \frac{\delta}{\delta x^i} = X^i \Big\{\frac{\partial}{\partial x^i} - y^j \Gamma_{ij}^k \frac{\partial}{\partial y^k}\Big\} \tag{2}$$

For consequences, we have $\big(\frac{\partial}{\partial x^i}\big)^H = \frac{\delta}{\delta x^i}$ and $\big(\frac{\partial}{\partial x^i}\big)^V = \frac{\partial}{\partial y^i}$, then $\big(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\big)_{i=1,\ldots,n}$ is a local adapted frame on TTM.

Proposition 1. *(see [18]) Let (M,g) be a Riemannian manifold and R its tensor curvature, then for all vector fields $X, Y \in \Gamma(TM)$ and $p \in T^2M$ we have:*

$$\begin{aligned}&[X^H, Y^H]_p = [X,Y]_p^H - (R_x(X,Y)u)^V,\\ &[X^H, Y^V]_p = (\nabla_X Y)_p^V,\\ &[X^V, Y^V]_p = 0,\end{aligned}$$

where $p = (x,u)$.

Definition 1. Let (M,g) be a Riemannian manifold and $K\colon TM \to TM$ be a smooth bundle endomorphism of TM. Then the vertical and horizontal vector fields VK and HK are defined on TM by

$$\begin{aligned}VK\colon TM &\to TTM\\ (x,u) &\mapsto \big(K(u)\big)^V,\\ HK\colon TM &\to TTM\\ (x,u) &\mapsto \big(K(u)\big)^H,\end{aligned}$$

locally we have

$$VK = y^i K_i^j \frac{\partial}{\partial y^j} = y^i \Big(K\Big(\frac{\partial}{\partial x^i}\Big)\Big)^V \tag{3}$$

$$HK = y^i K_i^j \frac{\partial}{\partial x^j} - y^i y^k K_i^j \Gamma_{jk}^s \frac{\partial}{\partial y^s} = y^i \Big(K\Big(\frac{\partial}{\partial x^i}\Big)\Big)^H. \tag{4}$$

2 Mus-Sasaki metric.

Definition 2. Let (M,g) be a Riemannian manifold and $f\colon M\times\mathbb{R}\to(0,+\infty)$. On the tangent bundle TM, we define a Mus-Sasaki metric noted g_f^S by

$$\begin{aligned} g_f^S(X^H,Y^H)_{(x,u)} &= g_x(X,Y)\\ g_f^S(X^H,Y^V)_{(x,u)} &= 0\\ g_f^S(X^V,Y^V)_{(x,u)} &= f(x,r)g_x(X,Y)\end{aligned}$$

where $X,Y\in\Gamma(TM)$, $(x,u)\in TM$ and $r=g(u,u)$. The function f is called twisting function.

Note that, if $f=1$ then g_f^S is the Sasaki metric [18].

Lemma 1. *Let (M,g) be a Riemannian manifold, then for all $x\in M$ and $u=u^i\frac{\partial}{\partial x^i}\in T_xM$, we have the following*

$$\begin{aligned} X^H(g(u,u))_{(x,u)} &= 0\\ X^H(g(Y,u))_{(x,u)} &= g(\nabla_X Y,u)_x\\ X^V(g(u,u)_{(x,u)} &= 2g(X,u)_x\\ X^V(g(Y,u)_{(x,u)} &= g(X,Y)_x\end{aligned}$$

Proof. Locally, Lemma 1 follows from formulas (1) and (2). □

From Lemma 2 we obtain

Lemma 2. *Let (M,g) be a Riemannian manifold, $F\colon\mathbb{R}^2\to[0,+\infty]$, $\alpha\colon M\to(0,+\infty)$ and $\beta\colon\mathbb{R}\to(0,+\infty)$ be smooth functions. If $f(x,r)=F(\alpha(x),\beta(r))$, then we have the following*

$$\begin{aligned} X^V(f)_{(x,u)} &= 2\beta'(r)g_x(X,u)\frac{\partial F}{\partial t}\big(\alpha(x),\beta(r)\big)\\ X^H(f)_{(x,u)} &= g_x(\operatorname{grad}_M\alpha,X)\frac{\partial F}{\partial s}\big(\alpha(x),\beta(r)\big)=X(\alpha)\frac{\partial F}{\partial s}\big(\alpha(x),\beta(r)\big)\end{aligned}$$

where $(x,u)\in TM$ and $r=g_x(u,u)$.

In the following, we consider $f(x,r)=F\big(\alpha(x),\beta(r)\big)$, where $F\colon\mathbb{R}^2\to[0,+\infty]$, $\alpha\colon M\to(0,+\infty)$ and $\beta\colon\mathbb{R}\to(0,+\infty)$ are smooth functions.

Lemma 3. *Let (M,g) be a Riemannian manifold. If $f(x,r)=F(\alpha(x),\beta(r))$ and ∇ (resp. $\widetilde{\nabla}$) denote the Levi-Civita connection of (M,g) (resp. (TM,g_f^S)), then we have:*

$$\begin{aligned} g_f^S(\widetilde{\nabla}_{X^H}Y^H,Z^H) &= g_f^S\big((\nabla_X Y)^H,Z^H\big)\\ g_f^S(\widetilde{\nabla}_{X^H}Y^H,Z^V) &= -\frac{1}{2}g_f^S\big((R(X,Y)u)^V,Z^V\big)\end{aligned}$$

$$g_f^S(\widetilde{\nabla}_{X^H}Y^V, Z^H) = g_f^S\Big(\frac{f}{2}\big(R(u,Y)X\big)^H, Z^H\Big)$$
$$g_f^S(\widetilde{\nabla}_{X^H}Y^V, Z^V) = g_f^S\Big(\frac{1}{2}X(\alpha)\frac{\partial \ln F}{\partial s}Y^V + (\nabla_X Y)^V, Z^V\Big)$$
$$g_f^S(\widetilde{\nabla}_{X^V}Y^H, Z^H) = g_f^S\Big((\frac{f}{2}R(u,X)Y\big)^H, Z^H\Big)$$
$$g_f^S(\widetilde{\nabla}_{X^V}Y^H, Z^V) = g_f^S\Big(\frac{1}{2}Y(\alpha)\frac{\partial \ln F}{\partial s}X^V, Z^V\Big)$$
$$g_f^S(\widetilde{\nabla}_{X^V}Y^V, Z^H) = g_f^S\Big(-\frac{1}{2}g(X,Y)\frac{\partial F}{\partial s}\big(\mathrm{grad}(\alpha)^H, Z^H\big)\Big)$$
$$g_f^S(\widetilde{\nabla}_{X^V}Y^V, Z^V) = \beta'\frac{\partial \ln F}{\partial t}g_f^S\big(g(X,u)Y^V + g(Y,u)X^V - g(X,Y)U^V, Z^V\big)$$

The proof of Lemma 3 follows directly from Kozul formula, Definition 2, Lemma 1 and Lemma 2.

As a direct consequence of Lemma 3, we get the following theorem.

Theorem 1. *Let (M,g) be a Riemannian manifold. If $f(x,r) = F(\alpha(x),\beta(r))$ and ∇ (resp. $\widetilde{\nabla}$) denote the Levi-Civita connection of (M,g) (resp. (TM, g_f^S)), then we have:*

$$(\widetilde{\nabla}_{X^H}Y^H)_p = (\nabla_X Y)_p^H - \frac{1}{2}\big(R_x(X,Y)u\big)^V$$
$$(\widetilde{\nabla}_{X^H}Y^V)_p = (\nabla_X Y)_p^V + \frac{f(x,r)}{2}\big(R_x(u,Y)X\big)^H + \frac{1}{2}X(\alpha)\frac{\partial \ln F}{\partial s}\big(\alpha(x),\beta(r)\big)Y_p^V$$
$$(\widetilde{\nabla}_{X^V}Y^H)_p = \frac{f(x,r)}{2}\big(R_x(u,X)Y)\big)^H + \frac{1}{2}Y(\alpha)\frac{\partial \ln F}{\partial s}\big(\alpha(x),\beta(r)\big)X_p^V$$
$$\begin{aligned}(\widetilde{\nabla}_{X^V}Y^V)_p = {} & \beta'(r)\frac{\partial \ln F}{\partial t}\big(\alpha(x),\beta(r)\big)\big[g_x(Y,U)X_p^V + g_x(X,U)Y_p^V - g_x(X,Y)U_p^V\big] \\ & - \frac{1}{2}g_x(X,Y)\frac{\partial F}{\partial s}\big(\alpha(x),\beta(r)\big)(\mathrm{grad}_M\,\alpha)_p^H\end{aligned}$$

for all vector fields $X, Y \in \Gamma(TM)$ and $p = (x,u) \in TM$, where R denotes the curvature tensor of (M,g).

From Definition 1 and Theorem 1 we have:

Proposition 2. *Let (M,g) be a Riemannian manifold and $\widetilde{\nabla}$ be the Levi-Civita connection of the tangent bundle (TM, g_f^s) equipped with the Mus-Sasaki metric. If K is a tensor field of type $(1,1)$ on M, then:*

$$(\widetilde{\nabla}_{X^H}HK)_{(x,u)} = H(\nabla_X K)(x,u) - \frac{1}{2}\Big(R_x\big(X_x, K_x(u)\big)u\Big)^V$$
$$\begin{aligned}(\widetilde{\nabla}_{X^H}VK)_{(x,u)} = {} & V(\nabla_X K)(x,u) + \frac{f(x,r)}{2}\Big(R_x\big(u, K_x(u)\big)X_x\Big)^H \\ & + \frac{1}{2}X(\alpha)\frac{\partial \ln F}{\partial s}(\alpha(x),\beta(r))(VK)_p\end{aligned}$$

$$
\begin{aligned}
(\widetilde{\nabla}_{X^V} HK)_{(x,u)} &= (K(X))^H_{(x,u)} + \frac{f(x,r)}{2}\Big(R_x\big(u, X_x\big)K_x(u)\Big)^H \\
&+ \frac{1}{2} g_x(\operatorname{grad}_M \alpha, K_x(u))\frac{\partial \ln F}{\partial s}(\alpha(x), \beta(r))X^V_p \\
(\widetilde{\nabla}_{X^V} VK)_{(x,u)} &= (K(X))^V_{(x,u)} - \frac{1}{2} g_x\big(X, K_x(u)\big)\frac{\partial F}{\partial s}\big(\alpha(x), \beta(r)\big)(\operatorname{grad}_M \alpha)^H_p \\
&+ \beta'(r)\frac{\partial \ln F}{\partial t}(\alpha(x), \beta(r))\Big[g_x\big(K_x(u), u\big)X^V_p) \\
&+ g_x(X, U)(VK)_p - g_x\big(X, K_x(u)\big)U^V_p\Big]
\end{aligned}
$$

where $(x,u) \in TM$ *and* $X \in \Gamma(TM)$.

3 Curvatures of Mus-Sasaki metric.

Theorem 2. *Let* (M,g) *be a Riemannian manifold and* (TM, g^S_f) *its tangent bundle equipped with the Mus-Sasaki metric. If* R *(resp.* $\widetilde{R}$*) denote the Riemann curvature tensor of* M *(resp.* TM*), then we have the following formulae.*

$$
\begin{aligned}
\widetilde{R}_{(x,u)}(X^H, Y^H)Z^H &= \Big\{\frac{1}{2}(\nabla_Z R)(X,Y)u + \frac{1}{2}Z(\alpha)\frac{\partial F}{\partial s}(\alpha(x), \beta(r))R(X,Y) \\
&- \frac{1}{4}X(\alpha)\frac{\partial \ln F}{\partial s}(R(Y,Z)u + \frac{1}{4}Y(\alpha)\frac{\partial \ln F}{\partial s}R(X,Z)u\Big\}^V_x \\
&+ \Big\{R(X,Y)Z + \frac{f(x,r)}{4}R(u, R(Z,Y)u)X \\
&+ \frac{f(x,r)}{4}R(u, R(X,Z)u)Y + \frac{f(x,r)}{2}R(u, R(X,Y)u)Z\Big\}^H_x
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
\widetilde{R}(X^H, Y^H)Z^V &= \Big\{R(X,Y)Z - \frac{f(x,r)}{4}R\big(X, R(u,Z)Y\big)u \\
&+ \frac{f(x,r)}{4}R(Y, R(u,Z)X)u \\
&+ \beta'(r)\frac{\partial \ln F}{\partial t}\big[g(Z,u)R(X,Y)u - g(Z, R(X,Y)u)u\big]\Big\}^V \\
&+ \Big\{\frac{f(x,r)}{2}\Big[(\nabla_X R)(u,Z)Y - (\nabla_Y R)(u,Z)X \\
&+ \frac{1}{2}X(\alpha)\frac{\partial \ln F}{\partial s}R(u,Z)Y - \frac{1}{2}Y(\alpha)\frac{\partial \ln F}{\partial s}R(u,Z)X\Big] \\
&- \frac{1}{2}g(R(X,Y)u, Z)\frac{\partial \ln F}{\partial s}\operatorname{grad}_M(\alpha)\Big\}^H
\end{aligned} \tag{6}
$$

$$\begin{aligned}
\widetilde{R}(X^H, Y^V)Z^V = \Big\{&\beta' X(\alpha)\frac{\partial^2 \ln F}{\partial s \partial t}\Big[g(Z,U)Y + g(Y,U)Z - g(Y,Z)U\Big] \\
&- \beta' \frac{\partial \ln F}{\partial t} g_x(Y,Z)(\nabla_X U) - \frac{f}{4} g\big(\operatorname{grad}_M \alpha, R(u,Z)X\big)\frac{\partial \ln F}{\partial s} Y \\
&- \beta' X(\alpha) g(Y,u)\frac{\partial^2 \ln F}{\partial t \partial s} Z + \frac{1}{4} g(Y,Z)\frac{\partial F}{\partial s}\big(R(X, \operatorname{grad}_M \alpha)u\big)\Big\}^V \\
&+ \Big\{\frac{f}{2}\beta' \frac{\partial \ln F}{\partial t}\Big[g(Z,u)(R(u,Y)X) + g(Y,u)(R(u,Z)X)\Big] g \\
&- \frac{1}{2} X(\alpha) g_x(Y,Z)\frac{\partial^2 F}{\partial s^2}(\operatorname{grad}_M \alpha) \\
&- \frac{1}{2} g(Y,Z)\frac{\partial F}{\partial s}(\nabla_X \operatorname{grad}_M \alpha) - \beta' g(Y,u)\frac{\partial F}{\partial t}\big(R(u,Z)X\big) \\
&- \frac{f}{2}\big(R(Y,Z)X\big)^H - \frac{f^2}{4}\big(R(u,Y)R(u,Z)X\big) \\
&+ \frac{f}{4} X(\alpha) g(Y,Z)\Big(\frac{\partial \ln F}{\partial s}\Big)^2 (\operatorname{grad}_M \alpha)\Big\}^H
\end{aligned} \tag{7}$$

$$\begin{aligned}
\widetilde{R}(X^H, Y^V)Z^H = \Big\{&\frac{1}{2}R(X,Z)Y - \frac{f(x,r)}{4} R\big(X, R(u,Y)Z\big)u + \frac{1}{2} X\big(Z(\alpha)\big)\frac{\partial \ln F}{\partial s} Y \\
&+ \frac{1}{2} X(\alpha) Z(\alpha)\Big(\frac{\partial^2 \ln F}{\partial s^2} + \frac{1}{2}\Big(\frac{\partial \ln F}{\partial s}\Big)^2\Big) Y - \frac{1}{2}(\nabla_X Z)(\alpha)\frac{\partial \ln F}{\partial s} Y \\
&+ \frac{\beta'(r)}{2}\frac{\partial \ln F}{\partial t}\Big[g(Y,u)R(X,Z)u - g\big(Y, R(X,Z)u\big)U\Big]\Big\}^V \\
&+ \Big\{\frac{1}{2} X(\alpha)\frac{\partial F}{\partial s} R(u,Y)Z + \frac{f(x,r)}{4} Z(\alpha)\frac{\partial \ln F}{\partial s} R(u,Y)X \\
&+ \frac{f(x,r)}{2}(\nabla_X R)(u,Y)Z - \frac{1}{4} g\big(Y, R(X,Z)u\big)\frac{\partial F}{\partial s}\operatorname{grad}_M(\alpha)\Big\}^H
\end{aligned} \tag{8}$$

$$\begin{aligned}
\widetilde{R}_{(x,u)}(X^V, Y^V)Z^H = \Big\{&\beta' \frac{\partial F}{\partial t}\Big[g(X,U)R(u,Y)Z - g(Y,U)R(u,X)Z\Big] + fR(X,Y)Z \\
&+ \frac{f^2}{4}\Big[R(u,X)R(u,Y)Z - R(u,Y)R(u,X)Z\Big]\Big\}^H_x \\
&+ \Big\{\beta' Z(\alpha)\frac{\partial^2 \ln F}{\partial s \partial t}\Big[g(X,u)Y - g(Y,u)X\Big] \\
&+ \frac{f}{4}\frac{\partial \ln F}{\partial s}\Big[g(\operatorname{grad}_M \alpha, R(u,Y)Z)X \\
&- g(\operatorname{grad}_M \alpha, R(u,X)Z)Y\Big]\Big\}^V_x
\end{aligned} \tag{9}$$

$$\begin{aligned}
\widetilde{R}_{(x,u)}(X^V,Y^V)Z^V &= \Big\{2\Big[\beta''\frac{\partial \ln F}{\partial t} + (\beta')^2\frac{\partial^2 \ln F}{\partial t^2}\Big]\Big[g(X,u)g(Z,U)Y \\
&- g(X,u)g(Y,Z)U - g(Y,u)g(Z,U)X + g(Y,u)g(X,Z)U\Big] \\
&+ \beta'\frac{\partial \ln F}{\partial t}\Big[g(X,Z)Y - g(Y,Z)X\Big] \\
&+ (\beta')^2\Big(\frac{\partial \ln F}{\partial t}\Big)^2\Big[g(Y,u)g(Z,U)X - g(Y,U)g(X,Z)U \\
&- g(X,u)g(Z,U)Y + g(X,u)g(Y,Z)U \\
&- g(Y,Z)\|u\|^2X + g(X,Z)\|u\|^2Y\Big] \\
&+ \frac{f}{4}\Big(\frac{\partial \ln F}{\partial s}\Big)^2\|\operatorname{grad}_M \alpha\|^2\Big[g(X,Z)Y - g(Y,Z)X\Big]\Big\}^V_x \\
&+ \Big\{\beta'\frac{\partial \ln F}{\partial t}\frac{\partial F}{\partial s}\Big[g(Y,Z)g(X,u) - g(X,Z)g(Y,u)\Big]\operatorname{grad}_M \alpha \\
&+ \frac{f}{4}\frac{\partial F}{\partial s}\Big[g(X,Z)R(u,Y)\operatorname{grad}_M \alpha - g(Y,Z)R(u,X)\operatorname{grad}_M \alpha\Big] \\
&+ \beta'\frac{\partial^2 \ln F}{\partial s\partial t}\Big[g(Y,u)g(X,Z) - g(X,u)g(Y,Z)\Big]\operatorname{grad}_M \alpha\Big\}^H_x
\end{aligned} \tag{10}$$

for all $(x,u) \in TM$ and $X,Y,Z \in \Gamma(TM)$.

Proof. Let

$$\begin{aligned} K_1 \colon TM &\to TM \\ u &\mapsto R(Y,Z)u\,. \end{aligned}$$

Then we have:

$$\begin{aligned}
\widetilde{\nabla}_{X^H}\widetilde{\nabla}_{Y^H}Z^H &= \widetilde{\nabla}_{X^H}(\nabla_Y Z)^H - \frac{1}{2}\widetilde{\nabla}_{X^H}VK_1 \\
&= (\nabla_X\nabla_Y Z)^H - \frac{1}{2}\big(R(X,\nabla_Y Z)u\big)^V - \frac{1}{2}\big(\nabla_X R(Y,Z)u\big)^V \\
&\quad - \frac{f}{4}\big(R_x(u,R_x(Y,Z)u)X\big)^H - \frac{1}{4}X(\alpha)\frac{\partial \ln F}{\partial s}VK_1
\end{aligned}$$

$$\begin{aligned}
\widetilde{\nabla}_{X^H}\widetilde{\nabla}_{Y^H}Z^H &= (\nabla_X\nabla_Y Z)^H - \frac{1}{2}(R(X,\nabla_Y Z)u)^V - \frac{1}{2}\big[(\nabla_X R)(Y,Z)u\big]^V \\
&- \frac{1}{2}\big[R(\nabla_X Y,Z)u + R(Y,\nabla_X Z)u\big]^V \\
&- \frac{f}{4}\big(R_x(u,R_x(Y,Z)u)X\big)^H - \frac{1}{4}X(\alpha)\frac{\partial \ln F}{\partial s}\big(R(Y,Z)u\big)^V
\end{aligned} \tag{11}$$

$$
\begin{aligned}
\widetilde{\nabla}_{Y^H}\widetilde{\nabla}_{X^H}Z^H &= (\nabla_Y\nabla_X Z)^H - \frac{1}{2}\big(R(Y,\nabla_X Z)u\big)^V - \frac{1}{2}\big[(\nabla_Y R)(X,Z)u\big]^V \\
&- \frac{1}{2}\big[R(\nabla_Y X,Z)u + R(X,\nabla_Y Z)u\big]^V \\
&- \frac{f}{4}\big(R_x(u,R_x(X,Z)u)Y\big)^H - \frac{1}{4}Y(\alpha)\frac{\partial \ln F}{\partial s}\big(R(X,Z)u\big)^V
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
\widetilde{\nabla}_{[X,Y]^H}Z^H &= (\nabla_{[X,Y]}Z)^H - \frac{1}{2}\big[R([X,Y],Z)u\big]^V - \frac{f(x,r)}{2}\big(R(u,R(X,Y)u)Z\big)^H \\
&- \frac{1}{2}Z(\alpha)\frac{\partial \ln F}{\partial s}\big(R(X,Y)u\big)^V.
\end{aligned} \tag{13}
$$

From (11), (12), (13) and the second Bianchi identity

$$(\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) + (\nabla_Z R)(X,Y) = 0$$

we obtain the formula (5).

The other formulae are obtained by a similar calculation. □

From Definition 2 and Theorem 2 we get

Theorem 3. *Let (M,g) be a Riemannian manifold and (TM, g_f^S) its tangent bundle equipped with the Mus-Sasaki metric. If R (resp. $\widetilde{R}$) denote the Riemann curvature tensor of M (resp. TM), then we have*

$$
\begin{aligned}
g_f^S(\widetilde{R}(X^H,Y^H)Y^H,X^H) &= g(R(X,Y)Y,X) - \frac{3f}{4}\|R(Y,X)u\|^2 \\
g_f^S(\widetilde{R}(X^H,Y^V)Y^V,X^H) &= \frac{f}{4}|X(\alpha)|^2\|Y\|^2\Big(\frac{\partial \ln F}{\partial s}\Big)^2 - \frac{1}{2}|X(\alpha)|^2\|Y\|^2\frac{\partial^2 F}{\partial s^2} \\
&- \frac{1}{2}\|Y\|^2\frac{\partial F}{\partial s}g(\nabla_X \operatorname{grad}_M \alpha, X) + \frac{f^2}{4}\|R(u,Y)X\|^2 \\
g_f^S(\widetilde{R}(X^V,Y^V)Y^V,X^V) &= 2f\Big[\beta''\frac{\partial \ln F}{\partial t} + (\beta')^2\frac{\partial^2 \ln F}{\partial t^2}\Big]\Big[2g(X,u)g(Y,u)g(X,Y) \\
&- \|Y\|^2|g(X,u)|^2 - \|X\|^2|g(Y,u)|^2\Big] \\
&+ f\beta'\frac{\partial \ln F}{\partial t}\Big[|g(X,Y)|^2 - \|X\|^2\|Y\|^2\Big] \\
&+ f(\beta')^2\Big(\frac{\partial \ln F}{\partial t}\Big)^2\Big[\|X\|^2|g(Y,u)|^2 - 2g(X,u)g(Y,u)g(X,Y) \\
&+ \|Y\|^2|g(X,u)|^2 - \|X\|^2\|Y\|^2\|u\|^2 + |g(X,Y)|^2\|u\|^2\Big] \\
&+ \frac{f^2}{4}\Big(\frac{\partial \ln F}{\partial s}\Big)^2\|\operatorname{grad}_M \alpha\|^2\Big[|g(X,Y)|^2 - \|X\|^2\|Y\|^2\Big].
\end{aligned}
$$

for any $X,Y \in \Gamma(TM)$.

By Theorem 3 and definition of sectional curvature tensor we get

Theorem 4. *Let (M, g) be a Riemannian manifold and (TM, g_f^S) its tangent bundle equipped with the Mus-Sasaki metric. If K (resp. $\widetilde{K}$) denote the sectional curvature tensor of M (resp. TM), then for any orthonormal vector fields $X, Y \in \Gamma(TM)$, we have*

$$\widetilde{K}(X^H, Y^H) = K(X,Y) - \frac{3f}{4}\|R(X,Y)u\|^2.$$

$$\begin{aligned}\widetilde{K}(X^H, Y^V) &= \frac{f}{4}\|R(u,Y)X\|^2 + \frac{1}{4}|X(\alpha)|^2\Big(\frac{\partial \ln F}{\partial s}\Big)^2 - \frac{1}{2f}|X(\alpha)|^2\frac{\partial^2 F}{\partial s^2}\\ &\quad - \frac{1}{2f}\frac{\partial F}{\partial s}g(\nabla_X \operatorname{grad}_M \alpha, X)\end{aligned}$$

$$\begin{aligned}\widetilde{K}(X^V, Y^V) &= -\frac{2}{f}\beta''\frac{\partial \ln F}{\partial t}\Big[|g(X,u)|^2 + |g(Y,u)|^2\Big] - \frac{1}{4f^2}\Big(\frac{\partial F}{\partial s}\Big)^2\|\operatorname{grad}_M \alpha\|^2\\ &\quad - \frac{(\beta')^2}{f}\Big(\frac{\partial \ln F}{\partial t}\Big)^2\Big[\|u\|^2 - |g(X,u)|^2 - |g(Y,u)|^2\Big] - \frac{\beta'}{f}\frac{\partial \ln F}{\partial t}\end{aligned}$$

Theorem 5. *Let (M, g) be a Riemannian manifold of constant sectional curvature λ and (TM, g_f^S) its tangent bundle equipped with the Mus-Sasaki metric. If $\widetilde{K}$ denote the sectional curvature tensor of TM, then for any orthonormal vector fields $X, Y \in \Gamma(TM)$, we have*

$$\widetilde{K}(X^H, Y^H) = \lambda - \frac{3f\lambda^2}{4}\Big[g(X,u)^2 + g(Y,u)^2\Big]$$

$$\begin{aligned}\widetilde{K}(X^H, Y^V) &= \frac{f\lambda^2}{4}g(u,X)^2 + \frac{1}{4}|X(\alpha)|^2\Big(\frac{\partial \ln F}{\partial s}\Big)^2 - \frac{1}{2f}|X(\alpha)|^2\frac{\partial^2 F}{\partial s^2}\\ &\quad - \frac{1}{2f}\frac{\partial F}{\partial s}g(\nabla_X \operatorname{grad}_M \alpha, X)\end{aligned}$$

$$\begin{aligned}\widetilde{K}(X^V, Y^V) &= -\frac{2}{f}\Big[\beta''\frac{\partial \ln F}{\partial t} + (\beta')^2\frac{\partial^2 \ln F}{\partial t^2}\Big]\Big[|g(X,u)|^2 + |g(Y,u)|^2\Big] - \frac{\beta'}{f}\frac{\partial \ln F}{\partial t}\\ &\quad - \frac{(\beta')^2}{f}\Big(\frac{\partial \ln F}{\partial t}\Big)^2\Big[\|u\|^2 - |g(X,u)|^2 - |g(Y,u)|^2\Big]\\ &\quad - \frac{1}{4}\Big(\frac{\partial \ln F}{\partial s}\Big)^2\|\operatorname{grad}_M \alpha\|^2\end{aligned}$$

The proof of Theorem 5 is deduced from the Theorem 4 and the following equations

$$\begin{aligned}R(X,Y)Z &= \lambda(g(X,Z)Y - g(Y,Z)X)\\ \|R(X,Y)u\|^2 &= \lambda^2\big[g(X,u)^2 + g(Y,u)^2\big]\\ \|R(u,Y)X\|^2 &= \lambda^2 g(u,X)^2\end{aligned}$$

for all $(x,u) \in TM$ and $X, Y, Z \in \Gamma(TM)$

Theorem 6. *Let (M,g) be a Riemannian manifold and (TM, g_f^S) its tangent bundle equipped with the Mus-Sasaki metric. If σ (resp. $\widetilde{\sigma}$) denote the scalar curvature of M (resp. TM), then for any local orthonormal frame $(E_1,\ldots,E_m)$ on M, we have*

$$\begin{aligned}\widetilde{\sigma} &= \sigma + \frac{2-3f}{4}\sum_{i,j=1}^{m}\|R(E_i,E_j)u\|^2 - \frac{m}{f^2}\frac{\partial F}{\partial s}\operatorname{trace}[g(\nabla_{\operatorname{grad}_M}\alpha, *)] \\ &+ \frac{m}{f^2}\|\operatorname{grad}(\alpha)\|^2\Big[\frac{1}{2f}\Big(\frac{\partial F}{\partial s}\Big)^2 - \frac{\partial^2 F}{\partial s^2} - \frac{(m-1)}{4f^2}\Big(\frac{\partial F}{\partial s}\Big)^2\Big] \\ &- \frac{4(m-1)}{f^3}\Big[\beta''\frac{\partial \ln F}{\partial t} + (\beta')^2\frac{\partial^2 \ln F}{\partial t^2} + \frac{(m-2)}{4}(\beta')^2\Big(\frac{\partial \ln F}{\partial t}\Big)^2\Big]\|u\|^2 \\ &- \frac{m(m-1)}{f^3}\beta'\frac{\partial \ln F}{\partial t}\end{aligned}$$

Proof. Let $(E_1,\ldots,E_m)$ be local orthonormal frame on M, then

$$\Big(E_1^H,\ldots,E_m^H,\frac{1}{\sqrt{f}}E_1^V,\ldots,\frac{1}{\sqrt{f}}E_m^V\Big)$$

are a local orthonormal frame on TM. Localy we obtain

$$\begin{aligned}\sum_{i,j=1}^{m}\|R(u,E_i)E_j\|^2 &= \sum_{i,j=1}^{m} g\big(R(u,E_i)E_j, R(u,E_i)E_j\big) \\ &= \sum_{i,j,k,l,s=1}^{m} u_k u_l g\big(R(E_k,E_i)E_j,E_s\big)g\big(R(E_l,E_i)E_j,E_s\big) \\ &= \sum_{i,j,k,l,s=1}^{m} u_k u_l g\big(R(E_j,E_s)E_k,E_i\big)g\big(R(E_j,E_s)E_l,E_i\big) \\ &= \sum_{i,j,s=1}^{m} g\big(R(E_j,E_s)u, g(R(E_j,E_s)u,E_i)E_i\big) \\ &= \sum_{i,j=1}^{m}\|R(E_i,E_j)u\|^2 \qquad (14)\end{aligned}$$

From Theorem 4 and definition of scalar curvature, we have

$$\widetilde{\sigma} = \sum_{i,j=1}^{m}\widetilde{K}(E_i^H,E_j^H) + \frac{2}{f}\sum_{i,j=1}^{m}\widetilde{K}(E_i^H,E_j^V) + \frac{1}{f^2}\sum_{i\neq j=1}^{m}\widetilde{K}(E_i^V,E_j^V)$$

$$\begin{aligned}\widetilde{\sigma} &= \sigma - \sum_{i,j=1}^{m} \frac{3f}{4}\|R(E_i,E_j)u\|^2 + \frac{2}{f}\sum_{i,j=1}^{m}\Big[\frac{f}{4}\|R(u,E_j)E_i\|^2 + \frac{1}{4}|E_i(\alpha)|^2\Big(\frac{\partial \ln F}{\partial s}\Big)^2 \\ &- \frac{1}{2f}|E_i(\alpha)|^2\frac{\partial^2 F}{\partial s^2} - \frac{1}{2f}\frac{\partial F}{\partial s}g(\nabla_{E_i}\operatorname{grad}_M \alpha, E_i)\Big] \\ &- \frac{1}{f^2}\sum_{i\neq j=1}^{m}\Big\{\frac{2}{f}\Big[\beta''\frac{\partial \ln F}{\partial t} + (\beta')^2\frac{\partial^2 \ln F}{\partial t^2}\Big]\Big[|g(E_i,u)|^2 + |g(E_j,u)|^2\Big] \\ &+ \frac{\beta'}{f}\frac{\partial \ln F}{\partial t} + \frac{(\beta')^2}{f}\Big(\frac{\partial \ln F}{\partial t}\Big)^2\Big[\|u\|^2 - |g(E_i,u)|^2 - |g(E_j,u)|^2\Big] \\ &+ \frac{1}{4}\Big(\frac{\partial \ln F}{\partial s}\Big)^2\|\operatorname{grad}_M \alpha\|^2\Big\}\end{aligned} \tag{15}$$

$$\begin{aligned}\widetilde{\sigma} &= \sigma - \sum_{i,j=1}^{m} \frac{3f}{4}\|R(E_i,E_j)u\|^2 + \frac{1}{2}\sum_{i,j=1}^{m}\|R(u,E_j)E_i\|^2 + \frac{m}{2f}\|\operatorname{grad}(\alpha)\|^2\Big(\frac{\partial \ln F}{\partial s}\Big)^2 \\ &- \frac{m}{f^2}\|\operatorname{grad}(\alpha)\|^2\frac{\partial^2 F}{\partial s^2} - \frac{m}{f^2}\frac{\partial F}{\partial s}\operatorname{trace}[g(\nabla_{\operatorname{grad}_M}\alpha,*)] \\ &- \frac{1}{f^2}\Big\{\frac{4(m-1)}{f}\Big[\beta''\frac{\partial \ln F}{\partial t} + (\beta')^2\frac{\partial^2 \ln F}{\partial t^2}\Big]\|u\|^2 + m(m-1)\frac{\beta'}{f}\frac{\partial \ln F}{\partial t} \\ &+ \frac{(\beta')^2}{f}\Big(\frac{\partial \ln F}{\partial t}\Big)^2(m-1)(m-2)\|u\|^2 + \frac{m(m-1)}{4}\Big(\frac{\partial \ln F}{\partial s}\Big)^2\|\operatorname{grad}_M \alpha\|^2\Big\}\end{aligned}$$

$$\begin{aligned}\widetilde{\sigma} &= \sigma - \sum_{i,j=1}^{m} \frac{3f}{4}\|R(E_i,E_j)u\|^2 + \frac{1}{2}\sum_{i,j=1}^{m}\|R(u,E_j)E_i\|^2 + \frac{m}{2f}\|\operatorname{grad}(\alpha)\|^2\Big(\frac{\partial \ln F}{\partial s}\Big)^2 \\ &- \frac{m}{f^2}\|\operatorname{grad}(\alpha)\|^2\frac{\partial^2 F}{\partial s^2} - \frac{m}{f^2}\frac{\partial F}{\partial s}\operatorname{trace}[g(\nabla_{\operatorname{grad}_M}\alpha,*)] \\ &- \frac{4(m-1)}{f^3}\Big[\beta''\frac{\partial \ln F}{\partial t} + (\beta')^2\frac{\partial^2 \ln F}{\partial t^2}\Big]\|u\|^2 - \frac{m(m-1)}{f^3}\beta'\frac{\partial \ln F}{\partial t} \\ &- \frac{(m-1)(m-2)}{f^3}(\beta')^2\Big(\frac{\partial \ln F}{\partial t}\Big)^2\|u\|^2 - \frac{m(m-1)}{4f^2}\Big(\frac{\partial \ln F}{\partial s}\Big)^2\|\operatorname{grad}_M \alpha\|^2\end{aligned}$$

Using formula (14) we deduce

$$\begin{aligned}\widetilde{\sigma} &= \sigma + \frac{2-3f}{4}\sum_{i,j=1}^{m}\|R(E_i,E_j)u\|^2 + \frac{m}{2f}\|\operatorname{grad}(\alpha)\|^2\Big(\frac{\partial \ln F}{\partial s}\Big)^2 \\ &- \frac{m}{f^2}\|\operatorname{grad}(\alpha)\|^2\frac{\partial^2 F}{\partial s^2} - \frac{m}{f^2}\frac{\partial F}{\partial s}\operatorname{trace}[g(\nabla_{\operatorname{grad}_M}\alpha,*)] \\ &- \frac{4(m-1)}{f^3}\Big[\beta''\frac{\partial \ln F}{\partial t} + (\beta')^2\frac{\partial^2 \ln F}{\partial t^2}\Big]\|u\|^2 - \frac{m(m-1)}{f^3}\beta'\frac{\partial \ln F}{\partial t} \\ &- \frac{(m-1)(m-2)}{f^3}(\beta')^2\Big(\frac{\partial \ln F}{\partial t}\Big)^2\|u\|^2 - \frac{m(m-1)}{4f^2}\Big(\frac{\partial \ln F}{\partial s}\Big)^2\|\operatorname{grad}_M \alpha\|^2\end{aligned}$$

$$\begin{aligned}\widetilde{\sigma} = \sigma &+ \frac{2-3f}{4}\sum_{i,j=1}^{m}\|R(E_i,E_j)u\|^2 - \frac{m}{f^2}\frac{\partial F}{\partial s}\,\mathrm{trace}[g(\nabla_{\mathrm{grad}_M}\alpha,*)]\\
&+\frac{m}{f^2}\|\,\mathrm{grad}(\alpha)\|^2\Big[\frac{1}{2f}(\frac{\partial F}{\partial s})^2 - \frac{\partial^2 F}{\partial s^2} - \frac{(m-1)}{4f^2}\Big(\frac{\partial F}{\partial s}\Big)^2\Big]\\
&-\frac{4(m-1)}{f^3}\Big[\beta''\frac{\partial \ln F}{\partial t} + (\beta')^2\frac{\partial^2 \ln F}{\partial t^2} + \frac{(m-2)}{4}(\beta')^2\Big(\frac{\partial \ln F}{\partial t}\Big)^2\Big]\|u\|^2\\
&-\frac{m(m-1)}{f^3}\beta'\frac{\partial \ln F}{\partial t}\end{aligned}$$

□

Corollary 1. *Let (M,g) be a Riemannian manifold of constant sectional curvature λ and (TM, g_f^S) its tangent bundle equipped with the Mus-Sasaki metric. If $\widetilde{\sigma}$ denote the scalar curvature of TM, then for any local orthonormal frame $(E_1,\dots,E_m)$ on M, we have*

$$\begin{aligned}\widetilde{\sigma} = (m-1)\lambda\Big[m &+ \frac{(2-3f)}{2}\lambda\|u\|^2\Big] - \frac{m}{f^2}\frac{\partial F}{\partial s}\,\mathrm{trace}[g(\nabla_{\mathrm{grad}_M}\alpha,*)]\\
&+\frac{m}{f^2}\|\,\mathrm{grad}(\alpha)\|^2\Big[\frac{1}{2f}\Big(\frac{\partial F}{\partial s}\Big)^2 - \frac{\partial^2 F}{\partial s^2} - \frac{(m-1)}{4f^2}\Big(\frac{\partial F}{\partial s}\Big)^2\Big]\\
&-\frac{4(m-1)}{f^3}\Big[\beta''\frac{\partial \ln F}{\partial t} + (\beta')^2\frac{\partial^2 \ln F}{\partial t^2} + \frac{(m-2)}{4}(\beta')^2\Big(\frac{\partial \ln F}{\partial t}\Big)^2\Big]\|u\|^2\\
&-\frac{m(m-1)}{f^3}\beta'\frac{\partial \ln F}{\partial t}\end{aligned}\tag{16}$$

Proof. Taking account that $\sigma = m(m-1)\lambda$ and for any vector fields $X, Y, Z \in TM$

$$R(X,Y)Z = \lambda(g(Y,Z)Y - g(X,Z)Y)$$

then we obtain

$$\begin{aligned}\sum_{i,j=1}^{m}\|R(E_i,E_j)u\|^2 &= \lambda^2\sum_{\substack{i\neq j\\ i,j=1}}^{m}\|g(E_j,u)E_i) - g(E_i,u)E_j)\|^2\\
&= \lambda^2\sum_{\substack{i\neq j\\ i,j=1}}^{m}\big(\|g(E_j,u)\|^2 + g(E_i,u)\|^2\big)\\
&= 2(m-1)\lambda^2\|u\|^2\end{aligned}\tag{17}$$

From Theorem 6 and formula (17) we deduce

$$\begin{aligned}\widetilde{\sigma} = (m-1)\lambda\Big[m &+ \frac{(2-3f)}{2}\lambda\|u\|^2\Big] - \frac{m}{f^2}\frac{\partial F}{\partial s}\,\mathrm{trace}[g(\nabla_{\mathrm{grad}_M}\alpha,*)]\\
&+\frac{m}{f^2}\|\,\mathrm{grad}(\alpha)\|^2\Big[\frac{1}{2f}\Big(\frac{\partial F}{\partial s}\Big)^2 - \frac{\partial^2 F}{\partial s^2} - \frac{(m-1)}{4f^2}\Big(\frac{\partial F}{\partial s}\Big)^2\Big]\\
&-\frac{4(m-1)}{f^3}\Big[\beta''\frac{\partial \ln F}{\partial t} + (\beta')^2\frac{\partial^2 \ln F}{\partial t^2} + \frac{(m-2)}{4}(\beta')^2\Big(\frac{\partial \ln F}{\partial t}\Big)^2\Big]\|u\|^2\\
&-\frac{m(m-1)}{f^3}\beta'\frac{\partial \ln F}{\partial t}\end{aligned}\tag{18}$$

□

Theorem 7. *Let (M, g) be a Riemannian manifold and (TM, g_f^S) its tangent bundle equipped with the Mus-Sasaki metric. If $f = \text{const}$ then (TM, g_f^S) has a constant scalar curvature if and only $f = \frac{3}{2}$ and (M, g) has a constant scalar curvature, or (M, g) is flat.*

Proof. From Theorem 6 we have

$$\widetilde{\sigma} = \sigma + \frac{(2-3f)}{4} \sum_{i,j=1}^{m} \|R(E_i, E_j)u\|^2$$

then $\widetilde{\sigma}$ is constant if and only $\sigma = \text{const}$ and $f = \frac{3}{2}$ or $R = 0$. □

Corollary 2. *Let (M, g) be a Riemannian manifold and (TM, g_f^S) its tangent bundle equipped with the Mus-Sasaki metric. If $f = \frac{3}{2}$ then $\widetilde{\sigma} = \sigma$.*

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