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Communications in Mathematics, Vol. 26 (2018), No. 2, 83–97

Persistent URL: http://dml.cz/dmlcz/147652

Communications in Mathematics 26 (2018) 83–97

DOI: 10.2478/cm-2018-0006

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Abstract. We consider and solve extremal problems in various bounded weakly pseudoconvex domains in $\mathbb{C}^n$ based on recent results on boundedness of Bergman projection with positive Bergman kernel in Bergman spaces A^p_α in such type domains. We provide some new sharp theorems for distance function in Bergman spaces in bounded weakly pseudoconvex domains with natural additional condition on Bergman representation formula.

1 Introduction and preliminaries

This paper deals with certain new applications of recent deep results on Bergman projection with positive Bergman kernel in Bergman type spaces in general Ω domains like smoothly bounded pseudoconvex domains of finite type m in $\mathbb{C}^n$ with Levi form which has at least $n-2$ positive eigenvalues at each point of the boundary $\partial\Omega$ and related domains to extremal problems related with distance function. This paper can be also considered as direct continuation of our previous recent papers on this topic (see [2], [14], [15], [16], [17]).

To make the exposition easier and convenient for readers in this section we provide basic preliminaries on general pseudoconvex domains we consider in this paper taken from [1]–[9], [10], [18], [19].

As the simplest model case we provide below a unit disk case with a complete proof, then to pass to more difficult cases. In particular, among the other things,

2010 MSC: 42B15, 42B30
Key words: Bergman spaces, distance estimates, pseudoconvex domains, analytic functions
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we provide a new sharp theorem for dist function for Hilbert A^2_α weighted Bergman spaces for bounded pseudoconvex domains whose boundary points are all of finite type and with locally diagonalizable Levi form.

Throughout this paper constants are denoted by C and C_i, $i \in \mathbb{N}$ or by C with other indexes. They are positive and may not be the same at each occurrence.

We will use the notation $A \lesssim B$, for functions A and B of several variables, to denote that $A \leq CB$ for a constant independent of certain parameters which will be clear in the context. The sentence $A \approx B$ will mean $A \lesssim B$ and $B \lesssim A$.

The exposition in this paper is specific. We don't move into all the details of all objects referring the reader to concrete papers, but use some vital results from those papers as tools to prove our theorems.

Now we collect some information on the Bergman projection with positive Bergman kernel in various domains in higher dimension. We also provide preliminaries on various types of weakly pseudoconvex domains and discuss the action of the Bergman projection with positive Bergman kernel in Bergman spaces and some related issues.

We provide some known facts on this function for completeness related with so called Levi polynomial. The most natural way to define more general classes of general domains than strongly pseudoconvex (weakly pseudoconvex domains) in higher dimension is to put some condition on Levi form of that domain, namely various natural conditions on that form. We add some basic facts and definitions of Levi form now.

Let D be an open set. A domain $D \subset \mathbb{C}^n$ is said to be pseudoconvex if the function $\varphi(z) = -\log \operatorname{dist}(z, bD)$ is plurisubharmonic on D. Note that φ is a continuous function which tends to ∞ as $z \to bD$. We denote below everywhere by $H(D)$ the space of all analytic functions in D. We denote by dv or $d\lambda$ the Lebesgues measure on D.

Let now D be a bounded domain in $\mathbb{C}^n$ which is with C^2 boundary and is strongly pseudoconvex. Thus there is a defining function $\rho \in C^\infty(\mathbb{C}^n)$ for $D = \{\rho < 0\}$ and $|\nabla \rho| > 0$ on bD, with ρ strictly plurisubharmonic. Define $P_w(z)$, the Levi polynomial at w by

$$P_w(z) = \sum_j \frac{\partial \rho(w)}{\partial w_j}(z_j - w_j) + \frac{1}{2}\sum_{j,k} \frac{\partial^2 \rho(w)}{\partial w_j w_k}(z_j - w_j)(z_k - w_k),$$

which is a quadratic (holomorphic) polynomial in z. A basic property of P_w is that

$$\rho(w) + 2\operatorname{Re} P_w(z) + L_w(z-w) = \mathcal{T}_w \tag{1}$$

is the second-order Taylor expansion of $\rho(z)$ about w.

Here

$$L_w(z-w) = \sum_{j,k} \frac{\partial^2 \rho(w)}{\partial w_j \overline{w}_k}(z_j - w_j)(\bar{z}_k - \overline{w}_k)$$

is the Levi form and

$$L_w(z-w) \geq c|z-w|^2, \quad c > 0,$$

by the strict plurisubharmonicity we assumed. So we can write the above as

$$2\,\mathrm{Re}(-\rho(w) - P_w(z)) = -\rho(z) - \rho(w) + L_w\left(z - w\right) + o(|z - w|^2) \tag{2}$$

as $|z - w| \to 0$. Now to extend the above when $z - w$ is not small, we define the function $g(w, z)$ by

$$g(w, z) = -P_w(z)\chi + |z - w|^2(1 - \chi) - \rho(w). \tag{3}$$

Here $\chi = \chi(|z-w|^2)$ is a C^∞ function which is 1 when $|z-w| \leq \frac{\mu}{2}$ and vanishes when $|z-w| \geq \mu$. We take μ to be a small constant, fixed so that (2) and the strict positive of L_w guarantee that we have

$$2\,\mathrm{Re}\ g(w, z) \geq \begin{cases} -\rho(w) - \rho(z) + c|w - z|^2, & \text{if } |w - z| \leq \mu \\ c, & \text{if } |w - z| \geq \mu \end{cases} \tag{4}$$

for some constant $c > 0$.

We need some basic facts from [4] on weakly pseudoconvex domains with diagonalizable Levi form. We refer the reader to [4] for definitions of complex tangent bundle $T^{p,q}$ of a domain.

Definition 1. Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$ with smooth C^∞ boundary. Let p be a point on the boundary of Ω. We say that the Levi form is locally diagonalizable at p if there exist a neighbourhood V of p and a smooth basis $\mathcal{B}$ of sections of the complex tangent bundle $T^{0,1}$ in $V \cap \partial\Omega$ which diagonalizes the Levi form. When this property holds at every point of the boundary, we say that Ω has a locally diagonalizable Levi form.

Definition 2. A function $K(z, w) \in L^\infty(\Omega \times \Omega)$ is called a $\mathcal{B}$-type kernel if there exist two constants C and C' such that, for $(z, w) \in U_k \times U_k$,

$$|K(z, w)| \leq C \prod_{i=1}^{n} \mathcal{F}_i^{(k)}(z, \delta_k),$$

where $\delta_k = |\rho(z)| + |\rho(w)| + \gamma_k(z, w)$ and for $(z, w) \notin \bigcup_{k=1}^{N} U_k \times U_k$, $|K(z, w)| \leq C'$.

We refer the reader to [4] for definitions of U_k sets and $\mathcal{F}_i^k$ functions and $\gamma_k(z, w)$ functions.

Remark 1. Note that the notion of $\mathcal{B}$-type kernel does not depend neither on the choice of the basis diagonalizing the Levi form nor on the choice of m and the U_k.

The following assertion is vital for our next section.

Proposition 1. *(see [4]) Let $K(z, w)$ be a $\mathcal{B}$-type kernel. Let $|P|$ be the operator associated to $|K(z, w)|$ (Bergman type projection with positive Bergman type kernel). Then $|P|$ maps continuously $L^p(\Omega)$ into itself for $1 < p < +\infty$.*

Note in [4] the Forelly-Rudin type estimate can also be found for this domains. Namely

$$\int_{\Omega} |K(z,w)||r(w)|^{-\varepsilon} dv(w) \lesssim |r(z)|^{-\varepsilon},\ z \in \Omega,$$

where $\varepsilon \in (0,1)$, for all $\mathcal{B}$ type kernels.

We assume that in this type domains in this paper the Bergman kernel is $\mathcal{B}$ type kernel. Then the Bergman projection with positive kernel is bounded. This follows directly from Schur test and estimates of Forelly-Rudin type, (see also [4]).

We also need some facts for pseudoconvex domains whose boundary Levi form has at most one degenerate eigenvalue.

In [7], the author shows that the Bergman kernel function, associated to pseudoconvex domains of finite type with the property that the Levi form of the boundary has at most one degenerate eigenvalue, is a standard kernel of Calderün-Zygmund type with respect to the Lebesgue measure. As an application, author shows that the Bergman projection with positive Bergman kernel on these domains preserves some of the Lebesgue classes.

Let $\Omega \subset \mathbb{C}^n$ be a bounded domain. The Bergman projection P on Ω is the orthogonal projection.

$$P : L^2(\Omega) \to H(\Omega) \cap L^2(\Omega) = A^2(\Omega),$$

where $H(\Omega)$ denotes the set of holomorphic functions on Ω. There is a corresponding kernel function $K_{\Omega}(z,w)$, the Bergman kernel function, such that

$$Pf(z) = \int_{\Omega} K_{\Omega}(z,w) f(w) dw,\ z \in \Omega.$$

Theorem A. *(see [7]) Let Ω be a pseudoconvex domain of finite type with the property that the Levi form of the boundary has at most one degenerate value. Let P be the Bergman projection or the Bergman projection with positive Bergman kernel associated to the domain Ω. Then P maps $L^p(\Omega)$ to $L^p(\Omega)$, boundedly, for all $1 < p < \infty$.*

We need for proof the Forelly-Rudin type estimate for bounded symmetric domain (see [9] and references there). The weighted Bergman projection is bounded in these domains, see [9], [10], [18], [19]. We refer the reader to [9] for the Forelly-Rudin type estimate in these domains.

This lemma leads to a boundedness of the Bergman projection with positive Bergman kernel in these types of domains (see [9], [10], [18], [19]).

Lemma A. *(see [8]) Let Ω be a bounded symmetric domain, $t > -1$ and $c \in \mathbb{R}$ such that $c > \frac{a(r-1)}{2}$. Then*

$$I_{c,t}(z) = \int_{\Omega} \frac{h(w,w)^t}{|h(z,w)|^{N+t+c}} dv(w) \sim h(z,z)^{-c}.$$

Let $D \subset \mathbb{C}$ be a bounded, strongly Levi-pseudoconvex domain with minimally smooth boundary. In [11] authors proved $L^p(D)$-regularity for the Bergman projection P, and for the operator $|P|$ whose kernel is the absolute value of the Bergman kernel with p in the range $(1, +\infty)$. As an application, in [11] authors shows that the space of holomorphic functions in a neighborhood of $\overline{D}$ is dense in $\vartheta L^p(D)$.

Paper [11] is the first in a series of papers dealing with the L^p-theory of reproducing operators such as the Cauchy-Fantappiè integrals and the Szegő and Bergman projections for domains in $\mathbb{C}^n$ whose boundaries have minimal smoothness. In the paper [11] authors study the Bergman projection with positive Bergman kernel on domains in $\mathbb{C}^n$ that are strongly pseudoconvex and have C^2 boundaries.

Let further Ω be smoothly bounded pseudoconvex domain of finite type m in $\mathbb{C}^n$, and the Levi form of $b\Omega$ has at least $n-2$ positive eigenvalues at each $z \in b\Omega$. The Bergman projection, as usual, is the orthogonal projection

$$P\colon L^2(\Omega) \to H^2(\Omega),$$

with corresponding kernel function $K(z, w)$ called, as usual, the Bergman kernel function. We set

$$L^p_k(\Omega) = \{f \in L^p(\Omega) : D^\alpha f \in L^p,\ |\alpha| \le k\},$$

and let $\Lambda_\alpha(\Omega)$ be the Lipschitz space of order $\alpha > 0$. If Ω is strongly pseudoconvex, Phong and Stein (see [13]) proved continuity results of P in the spaces mentioned above. For weakly pseudoconvex domains in $\mathbb{C}^n$, however, much less is known. When Ω is a smoothly bounded convex domain of finite type in $\mathbb{C}^n$, McNeal and Stein (see [12]) proved the continuity results in $L^p_k(\Omega)$, $1 < p < \infty$, $k \ge 0$, and in $\Lambda_\alpha(\Omega)$, $\alpha > 0$.

In [1] authors assumed that Ω is a smoothly bounded pseudoconvex domain of finite type m in $\mathbb{C}^n$ and the Levi-form of $b\Omega$ has at least $n-2$ positive eigenvalues at each $z \in b\Omega$, and in this paper [1] the authors proved sharp Sobolev space and Lipschitz space estimates for the Bergman projection operator P, associated to Ω. We refer to [1] for definitions of functional spaces BMO and Lipschitz class.

In [1] authors proved the following theorem:

Theorem B. *(see [1])*

1. *The Bergman projection operator associated to Ω, P, maps $L^p_k(\Omega)$ to $L^p_k(\Omega)$, boundedly, for $1 < p < \infty$ and $k \in \mathbb{N}$.*
2. *P maps $\Lambda_\alpha(\Omega)$ to $\Lambda_\alpha(\Omega)$, boundedly, for all $0 < \alpha < \infty$.*
3. *P maps $L^\infty(\Omega)$ to $BMO(\Omega)$, boundedly.*

To prove this theorem, the authors used a detailed analysis of the local geometry of $b\Omega$, and use estimates of the Bergman kernel function and its derivatives near $b\Omega$.

We refer the reader for definitions of important τ_i and M functions to [1], [7] to formulate next proposition.

Definition 3. A smooth function, $K(z, w)$, defined on $\Omega \times \Omega$ is said to be a kernel of B-type if there is an independent constant $C > 0$ so that for every $z, w \in \Omega$,

$$|K(z, w)| \le C \prod_{i=1}^{n} \tau_i(z, \delta)^{-2},$$

where $\delta = |r(z)|+|r(w)|+M(z,w)$ and where r is a defining function of our domain (see [1], [7]).

Note that the Bergman kernel function $K(z,w)$ is a kernel of $\mathcal{B}$-type, (see [1], [7] for this fact).

Let again Ω be smoothly bounded pseudoconvex domain of finite type m in $\mathbb{C}^n$, and the Levi form of $b\Omega$ has at least $n-2$ positive eigenvalues at each $z \in b\Omega$.

Theorem C. *(see [1]) Let $K(z,w)$ be a kernel of $\mathcal{B}$-type on Ω. If $|P|$ is the operator associated to $|K(z,w)|$ (Bergman projection with positive Bergman kernel), then $|P| : L^p(\Omega) \to L^p(\Omega)$, $1 < p < \infty$, boundedly.*

The proof is based on Proposition 3.1 of [1].

Proof. (Sketch) Let $0 < \varepsilon < 1$ be arbitrary and let $U_0, U_1, \dots, U_N$ be the open sets as in the proof of Proposition 3.1 of [1], then we have (Forelly-Rudin type estimate)

$$\int_{U_k} |K(z,w)||r(w)|^{-\varepsilon} dv(w) \lesssim \int_{\mathbb{C}^n} |r(w)|^{-\varepsilon} \prod_{i=1}^{n} \tau_i(z,\delta)^{-2} dv(w),$$

where δ is defined as in [1]. Since $0 < \varepsilon < 1$, we may apply integrations similar to those in the proof of Proposition 3.1 of [1] to find that

$$\int_{\Omega} |K(z,w)||r(w)|^{-\varepsilon} dv(w) \lesssim |r(z)|^{-\varepsilon}.$$

The authors also obtain analogous estimates when the integration variable is z instead of w. If q is the conjugate exponent of p and $f \in L^p(\Omega)$, then Hölder's inequality implies

$$\begin{aligned} \big|(|P|f)(z)\big|^p &\le \left(\int_{\Omega} |K(z,w)||f(w)|^p |r(w)|^{\varepsilon p/q} dv(w)\right)\left(\int_{\Omega} |K(z,w)||r(w)|^{-\varepsilon} dv(w)\right)^{p/q} \\ &\lesssim \left(\int_{\Omega} |K(z,w)||f(w)|^p |r(w)|^{-\varepsilon p/q} dv(w)\right)|r(z)|^{-\varepsilon p/q}. \end{aligned}$$

Thus it follows that

$$\begin{aligned} \int_{\Omega} \big|(|P|f)(z)\big|^p dz &\lesssim \int_{\Omega}\int_{\Omega} |K(z,w)||f(w)|^p |r(w)|^{\varepsilon p/q} |r(z)|^{-\varepsilon p/q} dv(w) dz \\ &= \int_{\Omega}\left(\int_{\Omega} |K(z,w)||r(z)|^{-\varepsilon p/q} dz\right)|r(w)|^{\varepsilon p/q}|f(w)|^p dv(w) \\ &\lesssim \int_{\Omega} |f(w)|^p dv(w), \end{aligned}$$

if $\varepsilon < q/p$. □

2 Main results

In this section we first formulate a result in the unit disk (see [15]) and then repeat arguments we provided in proof of that theorem in various situations in higher dimension to get results in weakly pseudoconvex domains. The core of all proofs is the boundedness of the Bergman projection with positive Bergman kernel. These type results we provided above in various weakly pseudoconvex domains. The next important tool for our proofs is the Forelly-Rudin type estimates in these domains which we also discussed in previous section (Forelly-Rudin type estimates in the unit disk).

We add some preliminaries in the unit disk which will be needed for our main results.

Let $\mathbf{U}$ be, as usual, the unit disk on the complex plane, dm_2 be the normalized Lebesgue measure on $\mathbf{U}$. Let $H(\mathbf{U})$ be the space of all analytic functions on the unit disk $\mathbf{U}$. For $f \in H(\mathbf{U})$ and $f(z) = \sum_k a_k z^k$, define the fractional derivative of the function f as usual in the following manner

$$D^\alpha f(z) = \sum_{k=0}^{\infty}(k+1)^\alpha a_k z^k, \ \alpha \in \mathbb{R}.$$

We will write $Df(z)$ if $\alpha = 1$. Obviously, for all $\alpha \in \mathbb{R}$, $D^\alpha f \in H(\mathbf{U})$ if $f \in H(\mathbf{U})$.

For $k > s, \ 0 < p, q \le \infty$, the weighted analytic Besov space $A^{q,p}_s(\mathbf{U})$ is the class of analytic functions satisfying

$$\|f\|^q_{A^{q,p}_s} = \int_0^1 \left(\int_{\mathbf{T}} |D^k f(r\xi)|^p d\xi\right)^{\frac{q}{p}} (1-r)^{(k-s)q-1} dr < \infty,$$

where $\mathbf{T} = \{\xi : |\xi| = 1\}$ be circle and $d\xi$ be the Lebesgue measure on the circle $\mathbf{T}$ and with standard modification for $p = \infty$ or $q = \infty$.

We denote by $A^q_s(\mathbf{U})$ the $A^{q,q}_s(\mathbf{U})$ analytic Besov spaces in the unit disk for all real numbers s. Note also that for $s < 0$ we have that these spaces are $A^q_{-sq-1}(\mathbf{U})$ Bergman spaces according to definition above for unit ball and we will use this notation below for all negative s numbers in Besov spaces.

It is well-known that $A^q_{-sq-1}(\mathbf{U}) \subset A^\infty_{-t}(\mathbf{U})$, $t = s - \frac{1}{q}$, $t < 0$, $s < 0$.

Let further

$$\Omega^k_{\varepsilon,-t} = \{z \in \mathbf{U} : |D^k f(z)|(1-|z|^2)^\varepsilon \ge -t\},$$

$\varepsilon \ge 0$, $t < 0$, $\Omega^0_{\varepsilon,-t} = \Omega_{\varepsilon,-t}$.

In the following sharp theorem we calculated distances from a weighted Bloch class to Bergman spaces for $q \le 1$.

Theorem D. *(see [15]) Let $0 < q \le 1 <$, $s < 0$, $t \le s - \frac{1}{q}$, $\beta > \frac{1-sq}{q} - 2$ and $\beta > -1 - t$. Let $f \in A^\infty_{-t}$. Then the following are equivalent:*

1. $l_1 = \operatorname{dist}_{A^\infty_{-t}}(f, A^q_{-sq-1})$;

2. $l_2 = \inf\left\{\varepsilon > 0 \colon \int_{\mathbf{U}} \left(\int_{\Omega_{\varepsilon,-t}(f)} \frac{(1-|w|)^{\beta+t}}{|1-\overline{z}w|^{2+\beta}} dm_2(w)\right)^q (1-|z|)^{-sq-1} dm_2(z) < \infty\right\}$.

Proof. First we show that $l_1 \le Cl_2$. For $\beta > -1-t$, we have

$$f(z) = C(\beta)\left(\int_{\mathbf{U}\setminus\Omega_{\varepsilon,-t}} \frac{f(w)(1-|w|)^{\beta}}{(1-\overline{w}z)^{\beta+2}}dm_2(w) + \int_{\Omega_{\varepsilon,-t}} \frac{f(w)(1-|w|)^{\beta}}{(1-\overline{w}z)^{\beta+2}}dm_2(w)\right)$$
$$= f_1(z) + f_2(z),$$

where $C(\beta)$ is a well-known Bergman representation constant.

For $t < 0$,

$$|f_1(z)| \le C\int_{\mathbf{U}\setminus\Omega_{\varepsilon,-t}} \frac{|f(w)|(1-|w|)^{\beta}}{|1-\overline{w}z|^{\beta+2}}dm_2(w)$$
$$\le C\varepsilon \int_{\mathbf{U}} \frac{(1-|w|)^{\beta+t}}{|1-\overline{w}z|^{\beta+2}}dm_2(w)$$
$$\le C\varepsilon\frac{1}{(1-|z|)^{-t}}.$$

So $\sup_{z\in\mathbf{U}} |f_1(z)|(1-|z|)^{-t} < C\varepsilon$.

For $s < 0$, $t < 0$, we have

$$\int_{\mathbf{U}} |f_2(z)|^q(1-|z|)^{-sq-1}dm_2(z)$$
$$\le C\int_{\mathbf{U}}\left(\int_{\Omega_{\varepsilon,-t}} \frac{(1-|w|)^{\beta+t}}{|1-\overline{w}z|^{\beta+2}}dm_2(w)\right)^q(1-|z|)^{-sq-1}dm_2(z) \le C.$$

So we finally have

$$\operatorname{dist}_{A^{\infty}_{-t}}(f, A^q_{-sq-1}) \le C\|f-f_2\|_{A^{\infty}_{-t}} = C\|f_1\|_{A^{\infty}_{-t}} \le C\varepsilon.$$

It remains to prove that $l_2 \le l_1$. Let us assume that $l_1 < l_2$. Then we can find two numbers ε, ε_1 such that $\varepsilon > \varepsilon_1 > 0$, and a function $f_{\varepsilon_1} \in A^q_{-sq-1}$, $\|f-f_{\varepsilon_1}\|_{A^{\infty}_{-t}} \le \varepsilon_1$, and

$$\int_{\mathbf{U}}\left(\int_{\Omega_{\varepsilon,-t}} \frac{(1-|w|)^{\beta+t}}{|1-\overline{z}w|^{\beta+2}}dm_2(w)\right)^q(1-|z|)^{-sq-1}dm_2(z) = \infty.$$

Hence as above we easily get from $\|f-f_{\varepsilon_1}\|_{A^{\infty}_{-t}} \le \varepsilon_1$ that

$$(\varepsilon-\varepsilon_1)\chi_{\Omega_{\varepsilon,-t}(f)}(z)(1-|z|)^t \le C|f_{\varepsilon_1}(z)|,$$

and hence

$$M = \int_{\mathbf{U}}\left(\int_{\mathbf{U}} \frac{\chi_{\Omega_{\varepsilon,-t}(f)}(z)(1-|w|)^{\beta+t}}{|1-\overline{w}z|^{\beta+2}}dm_2(w)\right)^q(1-|z|)^{-sq-1}dm_2(z)$$
$$\le C\int_{\mathbf{U}}\left(\int_{\mathbf{U}} \frac{|f_{\varepsilon_1}(w)|(1-|w|)^{\beta}}{|1-\overline{w}z|^{\beta+2}}dm_2(w)\right)^q(1-|z|)^{-sq-1}dm_2(z).$$

Since for $q \leq 1$,

$$\left(\int_{\mathbf{U}} \frac{|f_{\varepsilon_1}(w)|(1-|w|)^{\alpha}}{|1-\overline{w}z|^{t}} dm_2(w)\right)^{q} \leq C \int_{\mathbf{U}} \frac{|f_{\varepsilon_1}(w)|^{q}(1-|w|)^{\alpha q+q-2}}{|1-\overline{w}z|^{tq}} dm_2(w), \quad (5)$$

where $\alpha > \frac{1-q}{q}$, $t > 0$, $f_{\varepsilon_1} \in H(\mathbf{U})$, $z \in \mathbf{U}$ and the Forelly-Rudin type estimate

$$\int_{\mathbf{U}} \frac{(1-|z|)^{-sq-1}}{|1-\overline{w}z|^{q(\beta+2)}} dm_2(z) \leq \frac{C}{(1-|w|)^{q(\beta+2)+sq-1}}, \quad (6)$$

where $s < 0$, $\beta > \frac{1-sq}{q} - 2$, $w \in \mathbf{U}$, (Forelly-Rudin type estimates in the unit disk). We get

$$M \leq C \int_{\mathbf{U}} |f_{\varepsilon_1}(z)|^{q}(1-|z|)^{-sq-1} dm_2(z).$$

So, we arrive at a contradiction. The theorem is proved. □

The following theorem is a version of Theorem D for the case $q > 1$. The core of the proof of Theorem E as we see from the proof of Theorem D is the boundedness of the Bergman projection with positive Bergman kernel and the Forelly-Rudin type estimates. This simple observation will be heavily used by us below.

Theorem E. *(see [15]) Let $q > 1$ $s < 0$, $t \leq s - \frac{1}{q}$, $\beta > \frac{-1-sq}{q}$ and $\beta > -1-t$. Let $f \in A^{\infty}_{-t}$. Then the following are equivalent:*

1. $\widehat{l}_1 = \operatorname{dist}_{A^{\infty}_{-t}}(f, A^{q}_{-sq-1})$;
2. $\widehat{l}_2 = \inf\left\{\varepsilon > 0 : \int_{\mathbf{U}} \left(\int_{\Omega_{\varepsilon,-t}(f)} \frac{(1-|w|)^{\beta+t}}{|1-\overline{z}w|^{2+\beta}} dm_2(w)\right)^{q} (1-|z|)^{-sq-1} dm_2(z) < \infty\right\}$.

The proof of Theorem E is the same actually as the proof of Theorem D. The only difference is the boundedness of Bergman type projection operator but with the positive Bergman kernel. This fact will be heavily used by us below.

Indeed the close inspection of the proof of Theorem D shows the proof of Theorem E is the same as the proof of Theorem D but here we will use (7) (see below) instead of (5). For $\varepsilon > 0$, $q > 1$, $\beta > 0$, $\alpha > \frac{-1}{q}$,

$$\left(\int_{\mathbf{U}} \frac{|f(z)|(1-|z|)^{\alpha}}{|1-\overline{w}z|^{\beta+2}} dm_2(z)\right)^{q} \leq C \int_{\mathbf{U}} \frac{|f(z)|^{q}(1-|z|)^{\alpha q}}{|1-\overline{w}z|^{\beta q-\varepsilon q+2}} dm_2(z)(1-|w|)^{-\varepsilon q}, w \in \mathbf{U}, \quad (7)$$

which follows immediately from Hölder's inequality and (6).

Now we will add a new results. Since all proofs are short repetitions of arguments which are needed in higher dimension, they will be partially omitted. Let D be a bounded domains (domain) in $\mathbb{C}^n$. We put for $f \in H(D)$

$$\Omega_{\varepsilon,f} = \left\{z \in D : |f(z)| \left(\operatorname{dist}(z, \partial D)\right)^{\frac{n+1}{p}} \geq \varepsilon\right\}.$$

For a bounded domain D with C^2 boundary we define Bergman spaces A^p_α and A^∞_β as follows

$$A^p_\alpha(D) = \Big\{ f \in H(D) : \int_D |f(z)|^p \operatorname{dist}(z, \partial D)^\alpha dv(z) < \infty \Big\}, \quad \alpha > -1,$$
$$A^\infty_\beta(D) = \Big\{ f \in H(D) : \sup_z |f(z)| \operatorname{dist}(z, \partial D)^\beta < \infty \Big\}, \quad \beta \geq 0,$$

because of known embedding between these two spaces (see [6]) for such type domains dist problem can be posed and more generally we arrive at problem of estimates of $\operatorname{dist}_X(f, Y)$ if X, Y are quazinormed spaces, subspaces of $H(D)$, $f \in X$ and Y is embedded in X.

We deal with bounded domains with C^2 boundary at least below so the dist problem can be always posed. The core of all our proofs are so called Forelly-Rudin type estimates and boundedness of Bergman type projections with positive Bergman kernel for various types of domains discussed in previous section. The rest follows from the proof of the unit disk case which we provided above. The short proofs in more complicated domains hence will be partially omitted.

We assume that in our Theorems 1, 2 the following condition holds: $\rho(z)$ is equivalent to $\operatorname{dist}(z, \partial D)$, $z \in D$, where ρ is a defining function of D domain. We also assume that for each F, $F \in A^\infty_{(n+1)/p}$ the Bergman reproducing formula with kernel K is valid that is

$$F(w) = \int_D K(z, w) F(z) dv(z), \quad w \in D.$$

Note as we can check (see [3]) these both conditions are valid for bounded strongly pseudoconvex domains with smooth boundary.

Namely for example for bounded strongly pseudoconvex domains and for defining function of a domain $\rho(z)$ it is known $\rho(z)$ is equivalent to $\operatorname{dist}(z, \partial D)$, (see [4], [7]).

Moreover the second condition is valid in homogeneous Siegel domains, in tubular domains of second type, and bounded symmetric domains (see [14], [15], [16], [17]).

Similar condition on defining function of a general pseudoconvex domain can be seen in [5].

Theorem 1. *Let D be a bounded pseudoconvex domains with C^2 boundary whose boundary points are all of finite type and with locally diagonizable Levi form, $\frac{n+1}{p} < 1$, $1 < p < \infty$ and let $f \in A^\infty_{(n+1)/p}(D)$. Then*

$$\operatorname{dist}_{A^\infty_{(n+1)/p}}(f, A^p_0(D))$$
$$\asymp \inf \Big\{ \varepsilon > 0 : \int_D \Big(\int_D \chi_{\Omega_{\varepsilon,f}}(z) |K(z,w)| \operatorname{dist}(z, \partial D)^{-\frac{n+1}{p}} d\lambda(z) \Big)^p d\lambda(w) < \infty \Big\}.$$

Proof. Let us show first that

$$
\begin{aligned}
&l_1(f) = \operatorname{dist}_{A^\infty_{(n+1)/p}}(f, A^p_0) \\
&\le C \inf\left\{\varepsilon > 0 : \int_D \left(\int_D \chi_{\Omega_{\varepsilon,f}}(z)|K(z,w)| \operatorname{dist}(z,\partial D)^{-\frac{n+1}{p}} d\lambda(z)\right)^p d\lambda(w) < \infty\right\} \\
&\qquad = Cl_2(f).
\end{aligned}
$$

We repeat unit disk arguments now. According to our assumptions we have

$$F(w) = \int_D K(z,w)F(z)d\lambda(z), \quad w \in D$$

(for bounded symmetric domains this is valid automatically).

Then following unit disk case we have

$$
\begin{aligned}
F(w) &= F_1(w) + F_2(w), && w \in D \\
F_2(w) &= \int_{D\setminus\Omega_{\varepsilon,f}} K(z,w)F(z)d\lambda(z), && w \in D \\
F_1(w) &= \int_{\Omega_{\varepsilon,f}} K(z,w)F(z)d\lambda(z), && w \in D.
\end{aligned}
$$

Note again using the Forelly-Rudin estimate

$$|F_2(w)| \le C\varepsilon\left(\rho(w)^{-\tau}\right), \quad \tau = \frac{n+1}{p}, \quad w \in D,$$

and also obviously, since $\operatorname{dist}(z,\partial D) \asymp \rho(z)$,

$$\int_D |F_1(w)|^p d\lambda(w) \le C\int_D \left(\int_D \chi_{\Omega_{\varepsilon,f}}(z)|K(z,w)| \operatorname{dist}(z,\partial D)^{-\frac{n+1}{p}} d\lambda(z)\right)^p d\lambda(w) \le C,$$

where $1 < p < \infty$, $\frac{n+1}{p} < 1$.

We used that (see previous section)

$$\int_D |K(z,w)|\,\rho(z)^{-\tau} d\lambda(z) \le \rho(w)^{-\tau}, \quad w \in D, \quad \tau \in (0,1).$$

The rest follows similarly as above in case of the unit disk. Let us show now the reverse that is $l_2(f) \le l_1(f)$. We again repeat arguments of the unit disk case using the Bergman projection theorem.

For some ε, $\varepsilon_1 > 0$, $\varepsilon > \varepsilon_1$, $f_{\varepsilon_1} \in A^p_0(D)$ and assuming the reverse that $l_1 < l_2$, we have now

$$\|(f - f_{\varepsilon_1})\|_{A^\infty_{(n+1)/p}} \le \varepsilon_1 \tag{8}$$

and

$$M(f) = \int_D \left(\int_D \chi_{\Omega_{\varepsilon,f}}(z)|K(z,w)| \operatorname{dist}(z,\partial D)^{-\frac{n+1}{p}} d\lambda(z)\right)^p d\lambda(w) = \infty.$$

Hence from (8) we have

$$(\varepsilon-\varepsilon_1)\chi_{\Omega_{\varepsilon,f}}(z)\rho(z)^{-\tau}\le C\,|f_{\varepsilon_1}(z)|\,,\ z\in D,\ \tau=\frac{n+1}{p}.$$

From here we have that obviously

$$M(f)\le C\int_D\left(\int_D|f_{\varepsilon_1}(z)||K(z,w)|d\lambda(z)\right)^p d\lambda(w)\le C\|f_{\varepsilon_1}\|_{A_0^p},$$

for $1<p<\infty$.

Since the Bergman projection with positive kernel is bounded (see previous section) we got contradiction.

The proof is complete. □

Note, the proof of the Theorem 1 is based fully on estimates from [4] and remarks we made above. This result is sharp under certain natural additional conditions.

We always assume below that Ω is bounded domain with C^2 boundary (at least), so dist problem can be posed.

Based on our discussion and results of Ahn-Cho we can now formulate the following result.

Theorem 2. *Let Ω be C^∞ smoothly bounded with defining function r pseudoconvex domain of finite type m in $\mathbb{C}^n$ and the Levi form on $\partial\Omega$ has at least $n-2$ positive eigenvalues at each $z\in\partial\Omega$. Let $\frac{n+1}{p}<1$, $1<p<\infty$ and let $f\in A^\infty_{(n+1)/p}(D)$. Then*

$$\begin{aligned}&\operatorname{dist}_{A^\infty_{(n+1)/p}}(f,A_0^p(D))\\ &\asymp\inf\left\{\varepsilon>0:\int_D\left(\int_D\chi_{\Omega_{\varepsilon,f}}(z)|K(z,w)|\operatorname{dist}(z,\partial D)^{-\frac{n+1}{p}}d\lambda(z)\right)^p d\lambda(w)<\infty\right\}.\end{aligned}$$

Proof. Let us show first that

$$\begin{aligned}&l_1(f)=\operatorname{dist}_{A^\infty_{(n+1)/p}}(f,A_0^p)\\ &\le C\inf\left\{\varepsilon>0:\int_D\left(\int_D\chi_{\Omega_{\varepsilon,f}}(z)|K(z,w)|\operatorname{dist}(z,\partial D)^{-\frac{n+1}{p}}d\lambda(z)\right)^p d\lambda(w)<\infty\right\}\\ &=Cl_2(f).\end{aligned}$$

We repeat unit disk arguments again. According to our assumptions we have

$$F(w)=\int_D K(z,w)F(z)d\lambda(z),\quad w\in D$$

(for bounded symmetric domains this is valid automatically).

Then following unit disk case we have

$$\begin{aligned}F(w)&=F_1(w)+F_2(w), && w\in D\\ F_2(w)&=\int_{D\setminus\Omega_{\varepsilon,f}}K(z,w)F(z)d\lambda(z), && w\in D\\ F_1(w)&=\int_{\Omega_{\varepsilon,f}}K(z,w)F(z)d\lambda(z), && w\in D.\end{aligned}$$

Note again using the Forelly-Rudin estimate

$$|F_2(w)| \le C\varepsilon\left(\rho(w)^{-\tau}\right), \quad \tau = \frac{n+1}{p}, \quad w \in D,$$

and also obviously, since $\operatorname{dist}(z, \partial D) \asymp \rho(z)$,

$$\int_D |F_1(w)|^p d\lambda(w) \le C \int_D \left(\int_D \chi_{\Omega_{\varepsilon,f}}(z) |K(z,w)| \operatorname{dist}(z,\partial D)^{-\frac{n+1}{p}} d\lambda(z) \right)^p d\lambda(w) \le C,$$

where $1 < p < \infty$, $\frac{n+1}{p} < 1$.

We used that

$$\int_D |K(z,w)|\, \rho(z)^{-\tau} d\lambda(z) \le \rho(w)^{-\tau}, \quad w \in D, \quad \tau \in (0,1).$$

The rest follows similarly as above in case of the unit disk. Let us show now the reverse that is $l_2(f) \le l_1(f)$. We again repeat arguments of the unit disk case using the Bergman projection theorem.

For some ε, $\varepsilon_1 > 0$, $\varepsilon > \varepsilon_1$, $f_{\varepsilon_1} \in A_0^p(D)$ and assuming the reverse that $l_1 < l_2$, we have now

$$\|(f - f_{\varepsilon_1})\|_{A^\infty_{(n+1)/p}} \le \varepsilon_1 \tag{9}$$

and

$$M(f) = \int_D \left(\int_D \chi_{\Omega_{\varepsilon,f}}(z) |K(z,w)| \operatorname{dist}(z,\partial D)^{-\frac{n+1}{p}} d\lambda(z) \right)^p d\lambda(w) = \infty.$$

Hence from (8) we have

$$(\varepsilon - \varepsilon_1)\chi_{\Omega_{\varepsilon,f}}(z)\rho(z)^{-\tau} \le C\,|f_{\varepsilon_1}(z)|\,, \quad z \in D, \quad \tau = \frac{n+1}{p}.$$

From here we have that obviously

$$M(f) \le C \int_D \left(\int_D |f_{\varepsilon_1}(z)||K(z,w)| d\lambda(z) \right)^p d\lambda(w) \le C \|f_{\varepsilon_1}\|_{A_0^p},$$

for $1 < p < \infty$.

Since the Bergman projection with positive kernel is bounded (see previous section) we got contradiction.

The proof is complete. □

From [9] taking into account our discussion above and repeating arguments of the unit disk case we have another sharp theorem (without additional assumption needed for other two theorems):

Theorem 3. *Let Ω be bounded symmetric domain with C^2 boundary, $1 < p < \infty$. Then we have*

$$\begin{aligned}
&\operatorname{dist}_{A^{\infty}_{(n+1+\alpha)/p}}(f, A^p_\alpha(\Omega)) \\
&\quad \asymp \inf\Big\{\varepsilon > 0 : \int_\Omega \Big(\int_\Omega \chi_{\Omega_{\varepsilon,f}}(z) \frac{1}{|h(z,w)|^{\beta+N}} \big(h(z,z)\big)^{-t+\beta} d\delta(z)\Big)^p \\
&\qquad\qquad \times \big(h(w,w)\big)^\alpha d\delta(w) < \infty\Big\},
\end{aligned}$$

for all $\beta > \beta_0$, for all $t = \frac{n+1+\alpha}{p} < 1$ and for some large enough β_0.

Similar results with similar proofs are valid based on discussion above for pseudoconvex domains of finite type with the property that the Levi form of the boundary has almost one degenerate eigenvalue. We leave this to interested readers.

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Received: 01 November, 2016
Accepted for publication: 20 November, 2017
Communicated by: Karl Dilcher