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ON MINIMAL SPECTRUM OF MULTIPLICATION LATTICE MODULES

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Abstract. We study the minimal prime elements of multiplication lattice module M over a C-lattice L. Moreover, we topologize the spectrum $\pi(M)$ of minimal prime elements of M and study several properties of it. The compactness of $\pi(M)$ is characterized in several ways. Also, we investigate the interplay between the topological properties of $\pi(M)$ and algebraic properties of M.

Keywords: prime element; mimimal prime element; Zariski topology

MSC 2010: 06D10, 06E10, 06E99, 06F99

1. Introduction

The notion of minimal prime elements of a lattice module is a generalization of minimal prime elements of a multiplicative lattice. The prime and minimal prime elements of multiplicative lattice were introduced and studied by Thakare, Manjarekar and Maeda [12], Thakare and Manjarekar [11], and the minimal prime ideals of 0-distributive lattices by Pawar and Thakare [9]. Keimel [7] unified the study of minimal prime ideals for various structures, e.g. commutative rings, distributive lattices, lattice ordered groups, f-rings. In this paper, we have carried out investigations leading to the study of generalizations of notions in commutative rings and multiplicative lattices along the lines of Dilworth (see [6]).

A complete lattice L with the least element 0 and the greatest element 1 is said to be a *multiplicative lattice* if a binary operation "·" called multiplication on L satisfying the following conditions is defined:

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(1) $a \cdot b = b \cdot a$ for all $a, b \in L$,

(2) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in L$,

(3) $a \cdot \bigvee_{\alpha} b_\alpha = \bigvee_{\alpha} (a \cdot b_\alpha)$ for all $a, b_\alpha \in L$,

(4) $a \cdot 1 = a$ for all $a \in L$.

Henceforth, $a \cdot b$ will be simply denoted by ab.

An element $p \neq 1$ of a multiplicative lattice L is said to be *prime* if $ab \leqslant p$ implies either $a \leqslant p$ or $b \leqslant p$. A prime element $p \in L$ is said to be a *minimal prime* over an element $a \in L$ if $a \leqslant p$ and whenever there is a prime element $q \in L$ with $a < q \leqslant p$, then $q = p$. In L, a minimal prime element over 0 will be called a *minimal prime element of* L. For $a \in L$, its radical is denoted by $\sqrt{a}$ and defined as $\sqrt{a} = \bigvee\{x \in L \colon x^n \leqslant a$ for some $n \in \mathbb{Z}^+\}$. An element $a \in L$ is called *semiprime* or *radical* if $\sqrt{a} = a$.

An element $a \in L$ is said to be *compact* if $a \leqslant \bigvee X$, $X \subseteq L$ implies that there exists a finite number of elements $x_1, x_2, \ldots, x_n \in X$ such that $a \leqslant \bigvee\limits_{i=1}^{n} x_i$. We denote the set of all compact elements of a multiplicative lattice L by L_*. In a multiplicative lattice L, an element $a \in L$ is said to be *nilpotent* if $a^n = 0$ for some $n \in \mathbb{Z}^+$ and is said to be *reduced* if the only nilpotent element of L is 0.

An element $e \in L$ is said to be *meet principal* or *join principal* if it satisfies the identity $a \wedge be = ((a : e) \wedge b)e$ or $(ae \vee b) : e = (b : e) \vee a$, respectively, for $a, b \in L$. Also, e is said to be *principal* if it is both join and meet principal. A multiplicative lattice L is said to be *principally generated* (PG) if every element of L is a join of principal elements of L. A multiplicative lattice L is said to be *compactly generated* (CG) if every element of L is the join of compact elements of L. According to Alarcon et al. [1], if L is a compactly generated multiplicative lattice with 1 compact, then maximal elements exist in L and every maximal element is a prime element. Further, in a compactly generated multiplicative lattice, if every finite product of compact elements is a compact element, then prime elements and minimal primes over $a \in L$ exist (see [1]).

By a *C-lattice* we mean a multiplicative lattice L with the greatest element 1, which is compact as well as multiplicative identity, that is, generated under joins by a multiplicatively closed subset C of compact elements of L.

A complete lattice M with the smallest element 0_M and the greatest element 1_M is said to be a *lattice module* over the multiplicative lattice L or *L-module* if there is a multiplication between elements of M and L, denoted by aN for $a \in L$ and $N \in M$, which satisfies the following properties:

(1) $(ab)N = a(bN)$;

(2) $\bigvee_{\alpha} a_\alpha \bigvee_{\beta} N_\beta = \bigvee_{\alpha,\beta} (a_\alpha N_\beta)$;

(3) $1_L N = N$;

(4) $0_L N = 0_M$ for $a, b, a_\alpha \in L$ and for $N, N_\beta \in M$.

Let M be a lattice module over a multiplicative lattice L. For $N \in M$ and $b \in L$, denote $(N : b) = \bigvee\{X \in M\colon aX \leqslant N\}$. If $a, b \in L$, we write $(a : b) = \bigvee\{x \in L\colon bx \leqslant a\}$. If $A, B \in M$, then $(A : B) = \bigvee\{x \in L\colon xB \leqslant A\}$.

An element $A \in M$ is called *weak meet principal* if $(B : A)A = B \wedge A$ for all $B \in M$; A is called *weak join principal* if $bA : A = b \vee (0 : A)$ for all $b \in L$; and A is *weak principal* if A is both weak meet principal and weak join principal. Lattice module M over a multiplicative lattice L is called a *multiplication lattice module* if for every element $N \in M$ there exists an element $a \in L$ such that $N = a1_M$.

An element $N \neq 1_M$ in M is said to be *prime* if $aX \leqslant N$ implies $X \leqslant N$ or $a1_M \leqslant N$, i.e. $a \leqslant (N : 1_M)$ for every $a \in L$ and $X \in M$. An element $N \neq 1_M$ of M is called a *maximal element* if for every element B of M such that $N \leqslant B$, either $N = B$ or $B = 1_M$. Let M be an L-module. An element N in M is called *compact* if $N \leqslant \bigvee\limits_{\alpha\in I} A_\alpha$ (I is an indexed set) implies $N \leqslant A_{\alpha_1} \vee A_{\alpha_2} \vee \ldots \vee A_{\alpha_n}$ for some subset $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of I.

In this paper, a lattice module M will be a multiplication lattice module, which is compactly generated with the largest element 1_M being compact and L will be a C-lattice.

For general background and terminology of multiplicative lattice and multiplication lattice module, the reader may consult [1], [2], [4]–[6], [12], [11].

2. The Zariski topology

In [3], the Zariski topology over the prime spectrum $\mathrm{Spec}(M)$ of a lattice module M over a C-lattice L has been studied by Ballal and Kharat. In [10], Phadatare et al. introduced and studied the concept of quasi-prime elements as a generalization of prime elements and also the Zariski topology on the quasi-prime spectrum of a lattice module M over a C-lattice L.

In this paper most of the results in [12] and [11] are generalized.

Definition 2.1. Let M be a lattice module over a multiplicative lattice L. An element $P \in M$ is called a *minimal prime over an element* $N \in M$ if $N \leqslant P$ and there is no other prime element Q of M such that $N \leqslant Q < P$.

Lemma 2.2. *Let M be a multiplication lattice module over a C-lattice L and $(0_M : 1_M)$ be a radical element. Then for $x \in L$, $(0_M : x) = (0_M : x^n)$ for every integer $n \geqslant 1$.*

P r o o f. Note that $(0_M : x) = \bigvee\{N \in M\colon\ xN \leqslant 0_M\}$ and as $x^n \leqslant x$, we have $(0_M : x) \leqslant (0_M : x^n)$ for every integer $n \geqslant 1$. Let $N_1 = (0_M : x^n)$. Since M is a multiplication lattice module, $N_1 = a1_M$ for some $a \in L$. So $x^n a^n 1_M \leqslant x^n a 1_M = 0_M$. Hence $xa \leqslant \sqrt{(0_M : 1_M)} = (0_M : 1_M)$. So $xa1_M \leqslant 0_M$, i.e. $N_1 \leqslant (0_M : x)$ and consequently $(0_M : x) = (0_M : x^n)$ for each integer $n \geqslant 1$. $\square$

Theorem 2.3 ([8])**.** *Let M be a multiplication lattice module over a C-lattice L and $a \in L$ be proper. A prime element $P \in M$ with $a1_M \leqslant P$ is minimal if and only if for $x \in L_*$ with $x1_M \leqslant P$ there is an element $y \in L_*$ such that $y1_M \nleqslant P$ and $x^n y1_M \leqslant a1_M = N$ for some positive integer n.*

The following result characterizes a prime element to be a minimal prime.

Theorem 2.4. *Let M be a multiplication lattice module over a C-lattice L and $(0_M : 1_M)$ be a radical element. A prime element $P \in M$ is a minimal prime if and only if for $x \in L_*$, P contains precisely one of $x1_M$ and $(0_M : x)$.*

P r o o f. Suppose that the condition is true for prime element $P \in M$. Let $x \in L_*$ be such that $x1_M \leqslant P$ and $(0_M : x) \nleqslant P$. Then there exists $y \in L_*$ such that $y1_M \leqslant (0_M : x)$ but $y1_M \nleqslant P$. Thus, $xy1_M \leqslant 0_M$ and hence $x^n y1_M \leqslant 0_M$ for every integer $n \geqslant 1$. This shows that for each $x \in L_*$ with $x1_M \leqslant P$ there exists an element $y \in L_*$ such that $y1_M \nleqslant P$ and $x^n y1_M \leqslant 0_M$. By Theorem 2.3, it follows that P is minimal.

Conversely, suppose that a prime element $P \in M$ is minimal and also that $x1_M \leqslant P$ for $x \in L_*$. Then by Theorem 2.3, there exists $y \in L_*$ such that $y1_M \nleqslant P$ and $x^n y1_M = 0_M$ for some positive integer n. Consequently, $y1_M \leqslant (0_M : x^n)$. By Lemma 2.2, we have $(0_M : x^n) = (0_M : x)$ and hence $y1_M \leqslant (0_M : x)$. This implies that $(0_M : x) \nleqslant P$.

Now, if $x1_M \nleqslant P$ and $(0_M : x) \nleqslant P$, then there exists $y \in L_*$ such that $y1_M \leqslant (0_M : x)$ but $y1_M \nleqslant P$. Hence, we have $xy1_M \leqslant 0_M$ and so $xy1_M \leqslant P$. But $x1_M \nleqslant P$ and $y1_M \nleqslant P$ together contradicts the fact that P is a prime. This shows that P contains precisely one of $x1_M$ and $(0_M : x)$. $\square$

Let $\sigma(M)$ be the set of prime elements of a lattice module M. For an element $N \in M$ we set $V(N) = \{P \in \sigma(M)\colon\ N \leqslant P\}$. Taking the sets $\{V(N)\colon\ N \in M\}$ as a base for closed sets, $\sigma(M)$ becomes a topological space and this topology is called the *Zariski topology* (see [3]).

The restriction of the Zariski topology to the set of minimal prime elements $\pi(M)$ makes it a topological space and it is called the minimal prime spectrum of M.

The following results about a minimal prime spectrum are immediate.

Corollary 2.5. *Let M be a multiplication lattice module over a reduced C-lattice L. For $a \in L$, $V(0_M : a) = \pi(M) - V(a1_M)$. In particular, $V(a1_M)$ and $V(0_M : a)$ are disjoint open and closed sets.*

Corollary 2.6. *Let M be a multiplication lattice module over a reduced C-lattice L with 1_M being compact. Then $\pi(M)$ is a Hausdorff space with a base of open and closed sets.*

Definition 2.7 ([11]). A subset S of a multiplicative lattice L is said to be multiplicatively closed if $x, y \in S$ implies $xy \in S$, and is said to be sub-multiplicatively closed if $x, y \in X$ implies $a \leqslant xy$ for some $a \in S$.

In order to characterize prime elements of lattice modules in terms of multiplicatively closed subset of L, we need the following lemma.

Lemma 2.8 ([4]). *Let M be a multiplication lattice module over a* PG *C-lattice L and $N \in M$ with $N < 1_M$. Then the following conditions are equivalent.*

(1) *N is a prime element in M.*
(2) *$(N : 1_M)$ is a prime element in L.*
(3) *There exists a prime element p in L with $(0_M : 1_M) \leqslant p$ such that $N = p1_M$.*

For $N \in M$ we define $C(N) = \{x \in L\colon\ x \nleqslant (N : 1_M)\}$.

Lemma 2.9. *Let M be a multiplication lattice module over a* PG *C-lattice L. An element $P \in M$ is a prime if and only if $C(P)$ is a multiplicatively closed subset of L.*

P r o o f. Suppose that $P \in M$ is a prime and $x, y \in C(P)$. Then $x \nleqslant (P : 1_M)$ and $y \nleqslant (P : 1_M)$. Since $P \in M$ is a prime, by Lemma 2.8 we have that $(P : 1_M) \in L$ is a prime. As $x \nleqslant (P : 1_M)$, $y \nleqslant (P : 1_M)$ and $(P : 1_M)$ is a prime, $xy \nleqslant (P : 1_M)$, i.e. $xy \in C(P)$ and hence $C(P)$ is multiplicatively closed.

Conversely, suppose that $C(P)$ is a multiplicatively closed subset of L and $xy1_M \leqslant P$ for $x, y \in L$. Then $xy \leqslant (P : 1_M)$ and so $xy \notin C(P)$. If $x \nleqslant (P : 1_M)$ and $y \nleqslant (P : 1_M)$, then $x \in C(P)$, $y \in C(P)$ and this contradicts the fact that $C(P)$ is multiplicatively closed. Therefore $x \leqslant (P : 1_M)$ or $y \leqslant (P : 1_M)$, i.e. $x1_M \leqslant P$ or $y1_M \leqslant p$. Consequently, P is a prime. $\square$

Lemma 2.10 ([11]). *Let a be an element of a C-lattice L and S be a multiplicatively closed subset of L satisfying the property $s \nleqslant a$ for all $s \in S$. Then there is a multiplicatively closed subset S' of L containing S which is maximal with respect to the property $s' \nleqslant a$ for all $s' \in S'$.*

Lemma 2.11 ([11]). (*Separation lemma*) *Let S be a sub-multiplicatively closed subset of a C-lattice L. Suppose that $a \in L$ and $t \nleqslant a$ for every $t \in S$. Then there exists a prime element $p \in L$ such that $a \leqslant p$ and it is maximal with respect to $t \nleqslant p$ for each $t \in S$.*

An element a in a complete lattice L is said to be *completely join prime* if $a \leqslant \bigvee S$, $S \subseteq L$ implies $a \leqslant s$ for some $s \in S$.

Lemma 2.12. *Let M be a multiplication lattice module over a* PG *C-lattice L and suppose every element of L is a completely join prime. A prime element $P \in M$ with $a1_M \leqslant P$ is minimal if and only if $C(P)$ is a maximal multiplicatively closed subset of L with $x \nleqslant a$ for all $x \in C(P)$ and $a \in L$.*

Proof. Suppose that $C(P)$ is a maximal multiplicatively closed subset of L with $x \nleqslant a$ for all $x \in C(P)$. By Lemma 2.11 there is a prime element $(Q : 1_M) \geqslant a$ that is maximal with respect to the property that $x \nleqslant (Q : 1_M)$ for all $x \in C(P)$. Hence, by Lemma 2.9, $C(Q)$ is a multiplicatively closed subset of L. As $a \leqslant (Q : 1_M)$, we have $x \nleqslant a$ for any $x \in C(Q)$. But $C(P)$ is a maximal multiplicatively closed subset of L with the property that $x \nleqslant a$ for all $x \in C(P)$, hence we must have $C(Q) \subseteq C(P)$. Now, if $y \in C(P)$, then $y \nleqslant (Q : 1_M)$ and hence $y \in C(Q)$. Consequently, we have $C(P) = C(Q)$. Now, let $z \leqslant (P : 1_M)$, i.e. $z \in C(P)$. Then $z \notin C(Q)$ and it implies that $z \leqslant (Q : 1_M)$ and it further implies $(P : 1_M) \leqslant (Q : 1_M)$. Similarly, we have $(Q : 1_M) \leqslant (P : 1_M)$ and hence $(P : 1_M) = (Q : 1_M)$. It follows that $P = Q$.

Now we show that P is a minimal prime. Suppose that $P' \in M$ is a prime with $a \leqslant (P' : 1_M) < (P : 1_M)$. Then by Lemma 2.9, $C(P')$ is a multiplicatively closed subset of L with $x \nleqslant a$ for all $x \in C(P')$ and $C(P) \subseteq C(P')$. This contradicts the maximality of $C(P)$. Hence, P is a minimal prime element of M with $a1_M \leqslant P$.

Conversely, suppose that $P \in M$ is a minimal prime with $a1_M \leqslant P$. Then by Lemma 2.9, $C(P)$ is a multiplicatively closed subset of L with $x \nleqslant a$ for all $x \in C(P)$. By Lemma 2.10, there is a maximal multiplicatively closed subset S which contains $C(P)$ and $x \nleqslant a$ for all $x \in S$. We show that $S = C(P')$, where $P' = p1_M$ and $p = \bigvee(L - S)$. Let $y \in C(P') = \{z \in L\colon\ z \nleqslant \bigvee(L - S)\}$. This gives $y \nleqslant \bigvee(L - S)$, i.e. $y \in S$ and $C(P') \subseteq S$. On the other hand, if $s \in S$, then $s \notin L - S$ and $s \nleqslant \bigvee(L - S)$. As each element of L is a completely join prime, we have $s \in C(P')$ and therefore $C(P) = C(P')$.

By the first part, as S is a maximal multiplicatively closed subset of L with respect to $x \nleqslant a$ for all $x \in S$, we conclude that P' is a minimal prime with $a1_M \leqslant P'$. Clearly, $C(P) \subseteq S = C(P')$ gives that $P' \leqslant P$ and since P is minimal, we must have $P = P'$. Hence, $C(P) = S = C(P')$ is the required maximal multiplicatively closed subset of L with $x \nleqslant a$ for all $x \in M$ and $a \in L$. $\square$

For $N \in M$ define $\sqrt[M]{N} = \bigvee\{x \in L\colon\ x^n 1_M \leqslant N\} 1_M$.

Theorem 2.13. *Let L be a* PG *C-lattice in which every element is completely join prime and let M be a multiplication lattice module over L. For $N \in M$, the radical $\sqrt[M]{N} = \bigwedge\{P\colon\ P$ is a minimal prime element of M with $N \leqslant P\}$.*

Proof. Observe that for a prime element $P \in M$ with $N \leqslant P$ we have $\sqrt[M]{N} \leqslant P$. Therefore $\sqrt[M]{N} \leqslant \bigwedge\{P\colon\ P$ is a minimal prime element of M with $N \leqslant P\}$.

Now, let $x \in L_*$ be such that $x1_M \nleqslant \sqrt[M]{N}$ and let $S = \{x^i\colon\ x^i \nleqslant (N : 1_M)$ and i is an integer$\}$. Observe that S is a multiplicatively closed subset of L. By Lemma 2.10, there is a maximal multiplicatively closed set S' such that $y \nleqslant (N : 1_M)$ for $y \in S'$. Let $p' = \bigvee(L - S')$. Then $S' = C(p'1_M) = C(P')$. By Lemma 2.12, P' is a minimal prime element of M with $N \leqslant P'$. Clearly, $x \in C(P')$ and as such $x \nleqslant (P : 1_M)$. This gives that $\bigwedge\{P\colon\ P$ is a minimal prime element of M with $N \leqslant P\} \leqslant \sqrt[M]{N}$. Consequently, $\sqrt{N} = \bigwedge\{P\colon\ P$ is a minimal prime element of M with $N \leqslant P\}$. $\square$

Corollary 2.14. *Let M be a lattice module over a reduced* PG *C-lattice L and $N \in M$. Then for a prime element $P \in M$ with $N \leqslant P$ there exists a minimal prime element $Q \in M$ such that $N \leqslant Q \leqslant P$.*

Proof. Suppose $P \in M$ is a prime element with $N \leqslant P$. Then by Lemma 2.9, $C(P)$ is a multiplicatively closed subset of L with $x \nleqslant (N : 1_M)$ for all $x \in C(P)$. By Lemma 2.10, there is a maximal multiplicatively closed set S such that $y \nleqslant (N : 1_M)$ for all $y \in S$. Also, $C(Q) = S$, where $Q = p1_M = \bigvee(L - S)1_M$ is a minimal prime element of M with $N \leqslant Q$ and $C(P) \subseteq C(Q) = S$ implies that $Q \leqslant P$. $\square$

Lemma 2.15 ([12])**.** *Let L be a C-lattice. Then each nonzero element of L is contained in a maximal multiplicatively closed subset of L not containing zero.*

For $N \in M$ we set $U(N) = \{P \in \pi(M)\colon\ N \nleqslant P\}$.

Theorem 2.16. *Let L be a* PG *C-lattice in which every element is completely join prime and let M be a multiplication lattice module over L. Then $(0_M : a) = \bigwedge U(a1_M) = \{P \in \pi(M)\colon\ a1_M \nleqslant P\}$, $a \in L$.*

Proof. Suppose $P \in M$ is a minimal prime. Then by Theorem 2.4 we have $(0_M : a) \leqslant P$ when $a1_M \nleqslant P$ and therefore $(0_M : a) \leqslant \bigwedge\{P \in \pi(M)\colon\ a1_M \nleqslant P\} = Q$. If $(0_M : a) < Q$, then there exists $x \in L_*$ such that $x1_M \leqslant Q$ and $x1_M \nleqslant (0_M : a)$. Clearly, $ax1_M \nleqslant 0_M$ and so $ax \neq 0$. By Lemma 2.15, ax is contained in some maximal multiplicatively closed subset S of L not containing 0. As proved in Lemma 2.12, $S = C(P)$, where $P = p1_M$ and $p = \bigvee(L - S)$ is a minimal prime element of L. Now $ax \in S$ implies $ax \nleqslant (P : 1_M)$ and hence $ax1_M \nleqslant P$.

Since P is a minimal prime and $a1_M \nleqslant P$, we have $x1_M \nleqslant P$. Therefore $x1_M \nleqslant Q$, a contradiction and consequently, $(0_M : a) = \bigwedge\{P \in \pi(M)\colon\ a1_M \nleqslant P\}$. $\square$

Theorem 2.17. *Let L be a* PG *C-lattice in which every element is a completely join prime and let M be a multiplication lattice module over L. Then $a1_M = (0_M : (0_M : a1_M))$ if and only if $a1_M = \bigwedge\{P \in \pi(M) : a1_M \leqslant P\}$, $a \in L$.*

P r o o f. Suppose $a1_M = (0_M : (0_M : a1_M))$, $a \in L$. By Theorem 2.4 we have $\bigwedge\{P \in \pi(M)\colon\ (0_M : a) \nleqslant P\} = \bigwedge\{P \in \pi(M)\colon\ a1_M \leqslant P\}$. But $(0_M : (0_M : a1_M)) = \bigwedge\{P \in \pi(M)\colon\ (0_M : a) \nleqslant P\}$ gives that $a1_M = \bigwedge\{P \in \pi(M)\colon\ a1_M \leqslant P\}$.

Conversely, suppose that $a1_M = \bigwedge\{P \in \pi(M)\colon\ a1_M \leqslant P\}$. By Theorem 2.16 we have $(0_M : (0_M : a1_M)) = \bigwedge\{P \in \pi(M)\colon\ (0_M : a) \nleqslant P\}$. Now, by Theorem 2.4 we have $\bigwedge\{P \in \pi(M)\colon\ (0_M : a) \nleqslant P\} = \bigwedge\{P \in \pi(M)\colon\ a1_M \leqslant P\}$ and by assumption, $a1_M = (0_M : (0_M : a1_M))$. $\square$

Theorem 2.18. *Let M be a multiplication lattice module over a* PG *C-lattice L. Then $(0_M : a) = \bigwedge\{V(0_M : a)\}$, $a \in L$.*

P r o o f. Note that $(0_M : a) \leqslant \bigwedge\{V(0_M : a)\}$, $a \in L$ follows immediately. Now, let $x \in L_*$ be such that $x1_M \nleqslant (0_M : a)$. Then $ax1_M \nleqslant 0_M$ and so $ax \neq 0$. Therefore ax is contained in some maximal multiplicatively closed subset S of L. Then $S = V(P) = V(p1_M)$, where $p = \bigvee(L - S)$ and p is a minimal prime element of L. Now $ax \in C(P)$ implies $ax \nleqslant (P : 1_M)$ and hence $ax1_M \nleqslant P$. Since P is a minimal prime, we have $x1_M \nleqslant P$ and $a1_M \nleqslant P$. By Theorem 2.4 we have $(0_M : a) \leqslant P$ and hence $P \in V(0_M : a)$. As $x1_M \nleqslant P$, we have $x1_M \nleqslant \bigwedge(V(0_M : a))$. Thus, $x1_M \nleqslant (0_M : a)$ implies $x1_M \nleqslant \bigwedge(V(0_M : a))$, i.e. $\bigwedge(V(0_M : a)) \leqslant (0_M : a)$. $\square$

We now show that the minimal prime spectrum $\pi(M)$ is a completely regular Hausdorff space, i.e. a Tychonoff space.

Theorem 2.19. *Let M be a multiplication lattice module over a* PG *C-lattice L. Then the topology on $\pi(M)$ for which the collection $\{U(a1_M)\colon\ a \in L\}$ is a base for open sets is Tychonoff.*

P r o o f. Suppose that $P_1, P_2 \in \pi(M)$ with $P_1 \neq P_2$. Clearly $P_1 \nleqslant P_2$ and $P_2 \nleqslant P_1$. Let $x \in L_*$ with $x1_M \leqslant P_1$ be such that $x1_M \nleqslant P_2$. By Theorem 2.3, there is $y \in L_*$ with $y1_M \nleqslant P_1$ and $x^n y1_M = 0_M$ for some integer n. If $y1_M \nleqslant P_2$, then this together with $x1_M \nleqslant P_2$ gives $x^n y1_M \nleqslant P_2$, which is a contradiction to the fact that $0_M \leqslant P_2$. Therefore $y1_M \leqslant P_2$. Clearly, $P_1 \in U(y1_M)$, $P_2 \in U(x1_M)$ and

$U(x1_M) \cap U(y1_M) = \{P \in \pi(M)\colon\ x1_M \nleqslant P,\ y1_M \nleqslant P\} = U(xy1_M) = U(x^n y1_M) = U(0_M) = \varphi$. Consequently, $\pi(M)$ is a Hausdorff space and hence singletons are closed.

Now, let $Q \in \pi(M)$ and F be a closed subset of $\pi(M)$ such that $Q \notin F$. Then $Q \in \pi(M) - F$ and $\pi(M) - F$ is open in $\pi(M)$. Then there is an open set $U(s1_M)$ for some $s \in L$ such that $Q \in U(s1_M) \subseteq \pi(M) - F$. Define a function f on $\pi(M)$ as $f(Q) = 0_M$ if $Q \in U(s1_M)$ and $f(Q) = 1_M$ otherwise. Then $f(Q) = 0_M$ and $f(F) = 1$. Note that f is continuous and hence $\pi(M)$ is completely regular. Consequently, $\pi(M)$ is a completely regular Hausdorff space, i.e. a Tychonoff space. □

Corollary 2.20. *$\pi(M)$ is totally disconnected and zero dimensional space.*

Theorem 2.21. *Let M be a multiplication lattice module over a* PG *C-lattice L. Let $x, y \in L$. Then $U(x1_M) \subseteq U(y1_M)$ if and only if $0_M : (0_M : x1_M) \leqslant 0_M : (0_M : y1_M)$. In addition, $U(x1_M) = U(y1_M)$ if and only if $(0_M : x) \leqslant (0_M : y)$.*

P r o o f. Suppose that $U(x1_M) \subseteq U(y1_M)$ for $x, y \in L$. By Theorem 2.16 we have $(0_M : y) \leqslant (0_M : x)$ and hence $(0_M : y) \nleqslant P$ which implies $(0_M : x) \nleqslant P$ and so $\{P \in \pi(M)\colon\ (0_M : y) \nleqslant P\} \subseteq \{P \in \pi(M)\colon\ (0_M : x) \nleqslant P\}$. By Theorem 2.4 we have $0_M : (0_M : x1_M) \leqslant 0_M : (0_M : y1_M)$.

Conversely, suppose that $0_M : (0_M : x1_M) \leqslant 0_M : (0_M : y1_M)$. Therefore $\{P \in \pi(M)\colon\ (0_M : y) \nleqslant P\} \subseteq \{P \in \pi(M)\colon\ (0_M : x) \nleqslant P\}$ and so $\{P \in \pi(M)\colon\ y1_M \leqslant P\} \subseteq \{P \in \pi(M)\colon\ x1_M \leqslant P\}$ by Theorem 2.4. This gives $\{P \in \pi(M)\colon\ x1_M \nleqslant P\} \subseteq \{P \in \pi(M)\colon\ y1_M \nleqslant P\}$ and therefore $U(x1_M) \subseteq U(y1_M)$.

For the second part, suppose that $U(x1_M) = U(y1_M)$. Then $U(x1_M) \subseteq U(y1_M)$ implies $0_M : (0_M : x1_M) \leqslant 0_M : (0_M : y1_M)$ and $U(y1_M) \subseteq U(x1_M)$ implies $0_M : (0_M : y1_M) \leqslant 0_M : (0_M : x1_M)$. Hence, $0_M : (0_M : y1_M) = 0_M : (0_M : x1_M)$ and $0_M : (0_M : (0_M : y1_M)) \leqslant 0_M : (0_M : (0_M : x1_M))$. Consequently, $(0_M : x) = (0_M : y)$.

Conversely, suppose that $(0_M : x) = (0_M : y)$. Then $0_M : (0_M : x1_M) = 0_M : (0_M : y1_M)$, i.e. $0_M : (0_M : x1_M) \leqslant 0_M : (0_M : y1_M)$ and $0_M : (0_M : y1_M) \leqslant 0_M : (0_M : x1_M)$ and the result follows by the first part. □

Theorem 2.22. *Let M be a multiplication lattice module over a* PG *C-lattice L. Let I be an indexing set and $S = \{x_r\colon\ r \in I\}$ be a set of points in L such that the collection of sets $\{U(x_r 1_M)\colon\ r \in I\}$ has the finite intersection property. Then the intersection of all $\{U(x_r 1_M)\colon\ r \in I\}$ is nonempty.*

P r o o f. We have $\bigcap\limits_{i=1}^{n} U(x_i) = U(y1_M)$, where $y = x_1 x_2 \ldots x_n$. Note that the multiplication of finite number of nonzero x_r, $r \in I$ is nonzero. The collection of all nonzero x_r, $r \in I$ together with finite multiplication of $x_r \in S$ is multiplicatively closed subset of L not containing 0. By Lemma 2.10, there is a maximal multiplicatively closed subset S' of L containing S and not containing 0. We have $S' = C(P) = C(p1_M)$, where $p = \bigvee(L - S')$ and p is a minimal prime element of L. Clearly, $P \in U(x_r 1_M)$ for all $x_r \in S'$. As $S \subseteq S'$, we have $P \in U(x_r 1_M)$ for all $x_r \in S$. Thus, $P \in \bigcap\limits_{r\in I} U(x_r 1_M)$, which implies that $\bigcap\limits_{r\in I} U(x_r 1_M) \neq \varphi$. $\square$

If the family $\{V(x1_M)\colon x \in L\}$ is considered as an open basis for $\pi(M)$, the resulting topology is called the dual topology and denoted by τ^{d}. We denote the topology for which $\{U(x1_M)\colon x \in L\}$ is an open basis by τ.

Theorem 2.23. *Let M be a multiplication lattice module over a* PG *C-lattice L. The topology τ on $\pi(M)$ for which $\{U(x1_M)\colon x \in L\}$ is a basis for open sets is finer than the topology τ^{d} on $\pi(M)$ for which $\{V(x1_M)\colon x \in L\}$ is a basis for open sets and moreover $\tau = \tau^{\mathrm{d}}$.*

Proof. We know that $\{V(x1_M)\colon x \in L\}$ is a basis for open sets for the topology on $\pi(M)$ denoted by τ^{d}. Clearly, $V(x1_M) = \pi(M) - U(x1_M)$ for all $x \in L$. Note that for $x \in L$, $U(x1_M)$ is closed in $\pi(M)$. Hence, $V(x1_M)$ is open in the topology τ for $\pi(M)$, i.e. τ is finer than τ^{d}.

Now, for any $x \in L$ we have $U(x1_M) = V(0_M : x)$. Thus, every basic open set in τ is open in τ^{d} and so we conclude that $\tau = \tau^{\mathrm{d}}$. $\square$

Theorem 2.24. *Let M be a multiplication lattice module over a* PG *C-lattice L. The following statements are equivalent in M.*

(1) *$\pi(M)$ is compact.*
(2) *The poset $\{U(x1_M)\colon x \in L\}$, under set inclusion, is a Boolean lattice.*
(3) *For $x \in L$ there exist $N_1 = y_1 1_M, N_2 = y_2 1_M, \ldots, N_n = y_n 1_M \in M$ with $y_i 1_M = N_i \leqslant (0_M : x)$ for $i = 1, 2, \ldots, n$ and $(0_M : x) \wedge \bigwedge\limits_{i=1}^{n} (0_M : y_i) = 0_M$.*
(4) *For $x \in L$ there exist $N_1 = y_1 1_M, N_2 = y_2 1_M, \ldots, N_n = y_n 1_M \in M$ such that $0_M : (0_M : x1_M) = \bigwedge\limits_{i=1}^{n} (0_M : y_i)$.*
(5) *$\tau = \tau^{\mathrm{d}}$.*
(6) *$\{U(x1_M)\colon x \in L\}$ is a subbasis for open sets of $\pi(M)$ with respect to the topology τ.*
(7) *$\{V(x1_M)\colon x \in L\}$ is a subbasis for open sets of $\pi(M)$ with respect to the topology τ^{d}.*

Proof. (1) $\Rightarrow$ (2): Clearly the set $\{U(x1_M)\colon x \in L\}$ is partially ordered under set inclusion.

Now, we first show that

(i) $U(x1_M) \cup U(y1_M) = U(x1_M \vee y1_M)$;

(ii) $U(x1_M) \cap U(y1_M) = U(xy1_M)$.

Let $P \in U(x1_M) \cup U(y1_M)$, then $P \in U(x1_M)$ or $P \in U(y1_M)$ and so $x1_M \nleqslant P$ or $y1_M \nleqslant P$. Therefore $x1_M \vee y1_M \nleqslant P$ and this implies $P \in U(x1_M \vee y1_M)$. Now, let $Q \in U(x1_M \vee y1_M)$, then $x1_M \vee y1_M \nleqslant Q$ and this implies that $x1_M \nleqslant Q$ or $y1_M \nleqslant Q$. Therefore $Q \in U(x1_M) \cup U(y1_M)$. Consequently, $U(x1_M) \cup U(y1_M) = U(x1_M \vee y1_M)$. Similarly, $U(x1_M) \cap U(y1_M) = U(xy1_M)$.

From this we conclude that $(\{U(x1_M)\colon\ x \in L\}, \cup, \cap)$ is a lattice.

Now, $U(0.1_M) = U(0_M) = \varphi$ and $U(1.1_M) = U(1_M) = \pi(M)$. This shows that $(\{U(x1_M)\colon\ x \in L\}, \cup, \cap)$ is a bounded lattice. Again, observe that $U(x1_M) \cup (U(y1_M) \cap U(z1_M)) = (U(x1_M) \cup U(y1_M)) \cap (U(x1_M) \cup U(z1_M))$ and $U(x1_M) \cap (U(y1_M) \cup U(z1_M)) = (U(x1_M) \cap U(y1_M)) \cup (U(x1_M) \cap U(z1_M))$. This shows that $(\{U(x1_M)\colon\ x \in L\}, \cup, \cap)$ is a distributive lattice.

Finally, we show that $(\{U(x1_M)\colon\ x \in L\}, \cup, \cap)$ is complemented. Note that for $x \in L$ we have $V(x1_M) \cap V(0_M : x) = \varphi$. Then $V(x1_M) \cap \{V(N)\colon\ N \leqslant (0_M : x)\} = \varphi$. Since $\pi(M)$ is compact, there exist $N_1, N_2, \ldots, N_n \leqslant (0_M : x)$ such that $V(x1_M) \cap \{V(N_i)\colon\ N_i \leqslant (0_M : x),\ i = 1, 2, \ldots, n\} = \varphi$. By taking complements in $\pi(M)$, we get $\pi(M) = U(x1_M) \cup U(N_1) \cup \ldots \cup U(N_n)$. Since each $N_i \leqslant (0_M : x)$ for $i = 1, 2, \ldots, n$, we have $U(x1_M) \cap \bigcup\limits_{i=1}^{n} U(N_i) = \varphi$. For, if $P \in U(x1_M) \cap \bigcup\limits_{i=1}^{n} U(N_i)$, then $x1_M \nleqslant P$, which implies $(0_M : x) \leqslant P$. Therefore $N_i \leqslant P$ for $i = 1, 2, \ldots, n$, a contradiction as $P \in \bigcup\limits_{i=1}^{n} U(N_i)$ and so $N_k \nleqslant P$ for some k, $1 \leqslant k \leqslant n$. Thus, we have $\pi(M) = U(x1_M) \cup \bigcup\limits_{i=1}^{n} U(N_i)$ and $U(x1_M) \cap \bigcup\limits_{i=1}^{n} U(N_i) = \varphi$. Consequently, $(\{U(x1_M)\colon\ x \in L\}, \cup, \cap)$ is a Boolean lattice.

(2) $\Rightarrow$ (3): Suppose that the finite union of $\{U(x1_M)\colon\ x \in L\}$ forms a Boolean lattice and suppose that the complement of $U(x1_M)$ is $\bigcup\limits_{i=1}^{n} U(N_i)$. As $U(x1_M) \cap \bigcup\limits_{i=1}^{n} U(N_i) = \varphi$, we get $U(x1_M) \cap U(N_i) = \varphi$, $i = 1, 2, \ldots, n$. Therefore $\{P \in \pi(M)\colon\ xN_i \nleqslant P\} = \varphi$, $i = 1, 2, \ldots, n$, i.e. $U(xN_i) = \varphi$ for $i = 1, 2, \ldots, n$, which implies $xN_i = 0_M$ for $i = 1, 2, \ldots, n$. Thus $N_i \leqslant (0_M : x)$ for $i = 1, 2, \ldots, n$. Also, $\pi(M) = U(x1_M) \cup \bigcup\limits_{i=1}^{n} U(N_i)$ gives $\bigwedge(\pi(M)) = \bigwedge\Big(U(x1_M) \cup \bigcup\limits_{i=1}^{n} U(N_i)\Big)$, i.e. $0_M = \bigwedge(\pi(M)) = \bigwedge\Big(U(x1_M) \vee \bigvee\limits_{i=1}^{n} N_i\Big)$. Note that $\bigwedge\Big(U(x1_M) \vee \bigvee\limits_{i=1}^{n} N_i\Big) = \bigwedge(U(x1_M)) \wedge \bigwedge\limits_{i=1}^{n} (U(N_i))$. Then by Theorem 2.16 we have $(0_M : x) \wedge \bigwedge\limits_{i=1}^{n} (0_M : y_i) = 0_M$.

(3) $\Rightarrow$ (4): Suppose that (3) holds. Then for any $x \in L$ there exist $N_1 = y_1 1_M$, $N_2 = y_2 1_M$, ..., $N_n = y_n 1_M \in M$ with $y_i 1_M = N_i \leqslant (0_M : x)$ for $i = 1, 2, \ldots, n$

and $(0_M : x) \wedge \bigwedge_{i=1}^{n} (0_M : y_i) = 0_M$. This implies $(0_M : x1_M) \bigwedge_{i=1}^{n} (0_M : y_i) = 0_M$, i.e. $\bigwedge_{i=1}^{n} (0_M : y_i) \leqslant (0_M : (0_M : x1_M))$. Also note that $(0_M : (0_M : x1_M)) \leqslant (0_M : y_i)$ for $i = 1, 2, \ldots, n$. Hence $(0_M : (0_M : x1_M)) \leqslant \bigwedge_{i=1}^{n} (0_M : y_i)$ and consequently, $(0_M : (0_M : x1_M)) = \bigwedge_{i=1}^{n} (0_M : y_i)$.

(4) $\Rightarrow$ (5): Let x be an element of L. By (4), there exist $N_1 = y_1 1_M, N_2 = y_2 1_M, \ldots, N_n = y_n 1_M \in M$ such that $(0_M : (0_M : x1_M)) = \bigwedge_{i=1}^{n} (0_M : y_i)$. Hence we have

$$V(0_M : (0_M : x1_M)) = V\left(\bigwedge_{i=1}^{n} (0_M : y_i)\right) = \bigcup_{i=1}^{n} V(0_M : y_i) = \bigcup_{i=1}^{n} U(y_i 1_M) = V(x1_M).$$

Taking complements in $\pi(M)$, we have $\pi(M) - V(x1_M) = \pi(M) - \bigcup_{i=1}^{n} U(y_i 1_M)$, i.e. $U(x1_M) = \bigcap_{i=1}^{n} V(y_i 1_M)$. It follows that $U(x1_M)$ is a finite intersection of open sets in dual topology τ^{d}. Hence, $U(x1_M)$ is open in τ^{d}, which implies τ^{d} is finer than τ, and τ is finer than τ^{d} follows by Theorem 2.23.

(5) $\Rightarrow$ (1): Suppose that $\tau = \tau^{\mathrm{d}}$. Then $\{U(x1_M) \colon x \in L\}$ is also a base for closed sets in $\pi(M)$. Let $\{U(y1_M) \colon y \in K\}$ be a family of closed sets with finite intersection property in $\pi(M)$, where $K \subseteq L$. Then $\bigcap_{i=1}^{n} U(y_i 1_M) = U(y_1 y_2 \ldots y_n 1_M) \neq \varphi$ and so $y_1 y_2 \ldots y_n 1_M \neq 0_M$ for any finite number of elements $y_1, y_2, \ldots, y_n \in K$. All the nonzero elements in K together with the finite multiplication of elements in K form a multiplicatively closed set not containing 0. This multiplicatively closed set is again contained in some maximal multiplicatively closed set S not containing 0. As proved in Lemma 2.12, $S = C(P) = C(p1_M)$, where $p = \bigvee(L - S)$ is a minimal prime element of L. Note that $K \subseteq C(P)$ and therefore $P \in U(y1_M)$ for all $y \in K$. Thus, $p \in \bigcap\{U(y1_M) \colon y \in K\} \neq \varphi$ and so $\pi(L)$ is compact.

(5) $\Rightarrow$ (6): The implication follows immediately as $\{V(x1_M) \colon x \in L\}$ is a basis for open sets in τ^{d}.

(6) $\Rightarrow$ (5): Let $\{U(x1_M) \colon x \in L\}$ be any basis for open sets in τ. Then we have $U(x1_M) = \bigcap_{i=1}^{n} V(x_i)$ as $\{V(x1_M) \colon x \in L\}$ is a subbasis for open sets in $\pi(M)$ with respect to τ. This implies that $\{U(x1_M) \colon x \in L\}$ is open in τ^{d} and hence $\tau \subseteq \tau^{\mathrm{d}}$ and the result follows by Theorem 2.23.

(6) $\Rightarrow$ (7): If $\{V(x1_M) \colon x \in L\}$ is a subbasis for open sets in τ, then $\{\pi(M) - V(x1_M) \colon x \in L\} = \{U(x1_M) \colon x \in L\}$ forms a subbasis for open sets in τ^{d} and conversely. $\square$

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