POLYNOMIALS, SIGN PATTERNS AND DESCARTES' RULE OF SIGNS

Vladimir Petrov Kostov, Nice

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Abstract. By Descartes' rule of signs, a real degree d polynomial P with all nonvanishing coefficients with c sign changes and p sign preservations in the sequence of its coefficients ($c + p = d$) has $\mathrm{pos} \leqslant c$ positive and $\mathrm{neg} \leqslant p$ negative roots, where $\mathrm{pos} \equiv c \pmod 2$ and $\mathrm{neg} \equiv p \pmod 2$. For $1 \leqslant d \leqslant 3$, for every possible choice of the sequence of signs of coefficients of P (called sign pattern) and for every pair $(\mathrm{pos}, \mathrm{neg})$ satisfying these conditions there exists a polynomial P with exactly pos positive and exactly neg negative roots (all of them simple). For $d \geqslant 4$ this is not so. It was observed that for $4 \leqslant d \leqslant 8$, in all nonrealizable cases either $\mathrm{pos} = 0$ or $\mathrm{neg} = 0$. It was conjectured that this is the case for any $d \geqslant 4$. We show a counterexample to this conjecture for $d = 11$. Namely, we prove that for the sign pattern $(+,-,-,-,-,-,+,+,+,+,+,-)$ and the pair $(1, 8)$ there exists no polynomial with 1 positive, 8 negative simple roots and a complex conjugate pair.

Keywords: real polynomial in one variable; sign pattern; Descartes' rule of signs

MSC 2010: 26C10, 30C15

1. Introduction

The classical Descartes' rule of signs says that the real polynomial $P(x, a) := x^d + a_{d-1}x^{d-1} + \ldots + a_0$ does not have more positive roots than the number c of sign changes in the sequence of its coefficients. This rule has been announced by René Descartes (1596–1650) in his work La Géométrie published in 1637. When the roots are counted with multiplicity, then the number of positive roots has the same parity as c. (As indicated in [1], 18th century authors used to count roots with multiplicity while omitting the parity conclusion; later this conclusion was attributed (see [2]) to a paper of Gauss of 1828 (see [6]), although it is absent there, but was published by Fourier in 1820, see page 294 in [5].) When applied to $P(-x)$, these results give an upper bound on the number of negative roots of P. It is proved in [1] that all

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possible cases (i.e. of c, $c-2$, $c-4$, $\ldots$ positive roots) are realizable by suitably chosen polynomials P with c sign changes. Notice that here we do not impose restrictions on the number of negative roots.

In what follows we consider polynomials P without zero coefficients. Denoting by p the number of sign preservations in the sequence of coefficients of P, and by pos_P and neg_P the number of positive and negative roots of P, respectively, one can write:

$$(1.1) \qquad \mathrm{pos}_P \leqslant c, \quad \mathrm{pos}_P \equiv c \pmod 2, \quad \mathrm{neg}_P \leqslant p, \quad \mathrm{neg}_P \equiv p \pmod 2.$$

We call a finite sequence σ of $\pm$ signs a *sign pattern*; we assume that the leading sign of σ is $+$. For a given sign pattern of length $d+1$ with c sign changes and p sign preservations, we call (c,p) its *Descartes pair*, $c+p=d$. For a given sign pattern σ with Descartes pair (c,p) we call $(\mathrm{pos},\mathrm{neg})$ an *admissible pair* for σ if conditions (1.1), with $\mathrm{pos}_P = \mathrm{pos}$ and $\mathrm{neg}_P = \mathrm{neg}$, are satisfied.

One could ask the question whether given a sign pattern σ of length $d+1$ and an admissible pair $(\mathrm{pos},\mathrm{neg})$ one can find a degree d real monic polynomial the signs of whose coefficients define the sign pattern σ and which has exactly pos simple positive and exactly neg simple negative roots. In such a case we say that the couple $(\sigma,(\mathrm{pos},\mathrm{neg}))$ is *realizable.*

It turns out that for $d=1,2$ and 3 the answer is positive, but for $d=4$ the answer is negative; this is due to Grabiner, see [7]. Namely, for the sign pattern $\sigma^* := (+,+,-,+,+)$ (with Descartes pair $(2,2)$), the pair $(2,0)$ is admissible, see (1.1), but the couple $(\sigma^*,(2,0))$ is not realizable. The proof of this is easy—for a monic polynomial $P_4 := x^4 + a_3x^3 + \ldots + a_0$ with signs of the coefficients defined by σ^* and having exactly two positive roots $u<v$, one has $a_j>0$ for $j\neq 2$, $a_2<0$ and $P_4(\frac{1}{2}(u+v))<0$. Hence, $P_4(-\frac{1}{2}(u+v))<0$ because $a_j(\frac{1}{2}(u+v))^j = a_j(-\frac{1}{2}(u+v))^j$, $j=0,2,4$ and $0<a_j(\frac{1}{2}(u+v))^j = -a_j(-\frac{1}{2}(u+v))^j$, $j=1,3$. As $P_4(0)=a_0>0$, there are two negative roots $\xi < -\frac{1}{2}(u+v) < \eta$ as well.

Modulo the standard $(\mathbb{Z}_2\times\mathbb{Z}_2)$-action described below, Grabiner's example is the only nonrealizable couple (sign pattern, admissible pair) for $d=4$. The $(\mathbb{Z}_2\times\mathbb{Z}_2)$-action is defined on such couples by two generators. Denote by $\sigma(j)$ the jth component of the sign pattern σ. The first of the generators replaces the sign pattern σ by σ^r, where σ^r stands for the reverted (i.e. read from the back) sign pattern multiplied by $\sigma(0)$, and keeps the same pair $(\mathrm{pos},\mathrm{neg})$. This generator corresponds to the fact that the polynomials $P(x)$ and $x^dP(1/x)/P(0)$ are both monic and have the same number of positive and negative roots. The second generator exchanges pos with neg and changes the signs of σ corresponding to the monomials of odd (or even) powers if d is even (or odd); the rest of the signs are preserved. We denote the new sign pattern by σ_m. This generator corresponds to the fact that the roots of the polynomials (both monic) $P(x)$ and $(-1)^d P(-x)$ are mutually opposite, and if σ is the sign pattern of P, then σ_m is the one of $(-1)^d P(-x)$. For a given sign pattern σ and an admissible pair $(\mathrm{pos}, \mathrm{neg})$, the couples $(\sigma, (\mathrm{pos}, \mathrm{neg}))$, $(\sigma^r, (\mathrm{pos}, \mathrm{neg}))$, $(\sigma_m, (\mathrm{neg}, \mathrm{pos}))$ and $((\sigma_m)^r, (\mathrm{neg}, \mathrm{pos}))$ are simultaneously realizable or not. (One has $(\sigma_m)^r = (\sigma^r)_m$.)

All cases of couples (sign pattern, admissible pair) for $d = 5$ and $d = 6$ which are not realizable are described in [1]. For $d = 7$, this is done in [3] and for $d = 8$ in [3] and [8]. For $d = 5$ there is a single nonrealizable case (up to the $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-action). The sign pattern is $(+, +, -, +, -, -,)$ and the admissible pair is $(3, 0)$. For $n = 6$, $n = 7$ and $n = 8$ there are 4, 6, and 19, respectively, nonrealizable cases. In all of them one of the numbers pos or neg is 0. It is conjectured in [3] that this is the case for any d.

In the present paper we show that the conjecture fails for $d = 11$.

Notation 1. For $d = 11$ we denote by σ^0 the following sign pattern (we give on the first and third lines below the sign patterns σ^0 and σ^0_m, respectively, while the line in the middle indicates the positions of the monomials of odd powers):

$$\begin{array}{rl} \sigma^0 = (& + \; - \; - \; - \; - \; - \; + \; + \; + \; + \; + \; - \;) \\ & \;\;\; 11 \quad 9 \quad 7 \quad 5 \quad 3 \quad 1 \\ \sigma^0_m = (& + \; + \; - \; + \; - \; + \; + \; - \; + \; - \; + \; + \;) \end{array}$$

In a sense σ^0 is centre-antisymmetric—it consists of one plus, five minuses, five pluses and one minus.

Theorem 1. *The sign pattern σ^0 is not realizable with the admissible pair* $(1, 8)$.

The next section contains comments concerning the above result and realizability of sign patterns and admissible pairs in general. Section 3 contains some technical lemmas which allow to simplify the proof of Theorem 1. The method of the proof is explained in Section 4. Section 5 contains the proofs of lemmas used in Section 4.

2. Comments

Theorem 1 shows that the problem of classifying all nonrealizable cases (sign pattern, admissible pair) for any degree d is a difficult one. At present, an exhaustive conjectural answer is not known. One could try to find sufficient conditions for realizability expressed, say, in terms of the ratios between d, c and p. In papers [3] and [9] series of nonrealizable cases were found (defined either for every degree d or for every odd or even degree sufficiently large). In all of them either pos $=0$ or neg $=0$. The construction of such series with pos $\neq 0 \neq$ neg and the proof of their nonrealizability seems to be sufficiently hard for $d \geqslant 9$.

One of the series of nonrealizable cases considered in [3] and [4] concerns sign patterns with exactly two sign changes, consisting of m pluses followed by n minuses followed by q pluses, $m+n+q=d+1$. Set

$$\kappa := \frac{d-m-1}{m}\frac{d-q-1}{q}.$$

Lemma 1. *For $\kappa \geqslant 4$, such a sign pattern is not realizable with the admissible pair $(0, d-2)$. The sign pattern is realizable with any admissible pair of the form $(2, v)$ except for the case $v=0$, $n=1$, m and q being even.*

Lemma 1 coincides with Proposition 6 of [3]. One of the tools for constructing new realizable cases is the following concatenation lemma (also proved in [3]):

Lemma 2. *Suppose that the monic polynomials P_j of degrees d_j and with sign patterns of the form $(+, \sigma_j)$, $j=1,2$ (where σ_j contains the last d_j components of the corresponding sign pattern) realize the pairs $(\mathrm{pos}_j, \mathrm{neg}_j)$. Then*

(1) *if the last position of σ_1 is $+$, then for any $\varepsilon > 0$ small enough, the polynomial $\varepsilon^{d_2}P_1(x)P_2(x/\varepsilon)$ realizes the sign pattern $(+, \sigma_1, \sigma_2)$ and the pair $(\mathrm{pos}_1+\mathrm{pos}_2, \mathrm{neg}_1+\mathrm{neg}_2)$;*

(2) *if the last position of σ_1 is $-$, then for any $\varepsilon > 0$ small enough, the polynomial $\varepsilon^{d_2}P_1(x)P_2(x/\varepsilon)$ realizes the sign pattern $(+, \sigma_1, -\sigma_2)$ and the pair $(\mathrm{pos}_1+\mathrm{pos}_2, \mathrm{neg}_1+\mathrm{neg}_2)$ (here $-\sigma_2$ is obtained from σ_2 by changing each $+$ into $-$ and vice versa).*

It is clear that if Theorem 1 were true, then one should not be able to deduce the realizability of the sign pattern σ^0 with the admissible pair $(1,8)$ with the help of Lemma 2. Now we show that this is indeed impossible. It suffices to check the cases $\deg P_1 \geqslant 6$, $\deg P_2 \leqslant 5$ due to the centre-antisymmetry of σ^0 and the possibility to use the $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-action.

In all these cases the sign pattern of the polynomial P_1 has exactly two sign changes (it comprises the first sign $+$, the five minuses that follow and the next between one and five pluses). These cases are (we use the notation from Lemma 1) $m=1$, $n=5$, $q=1,\dots,5$. The values of κ are 16, 10, 8, 7 and $\frac{32}{5}$, respectively, all of them are greater than 4. By Descartes' rule, the polynomial P_1 can have either 0 or 2 positive roots. Should it have 2, then its concatenation with P_2 should have at least 2 positive roots (by Lemma 2) which is impossible. So P_1 has no positive roots. The sign patterns defined by P_1 and P_2 have $4+(q-1)$ and $5-q$ sign preservations, respectively. By Lemma 1 the polynomial P_1 has less than or equal to $(2+(q-1))$ negative roots, and as P_2 has less than or equal to $(5-q)$ ones, the concatenation of P_1 and P_2 has less than or equal to 6 negative roots. Therefore a polynomial realizing the couple $(\sigma^0,(1,8))$ (if it exists) cannot be represented as a concatenation of two polynomials P_1 and P_2.

Still this does not exclude the existence of such a polynomial. In [3], certain examples of polynomials realizing given sign patterns and admissible pairs had to be constructed directly. Before passing to the proof of Theorem 1 we explain the role that the concatenation lemma could play in solving the problem of realizability of sign patterns with admissible pairs.

If in the process of solving this problem one arrives at a situation when there exists $d_0 \in \mathbb{N}$ such that for $d \geqslant d_0$ the realizability of all realizable cases can be deduced from some general statements and from the concatenation lemma, then it would be sufficient to find the exhaustive list of realizable cases for $d < d_0$ and the problem would be solved. One could interpret as a general statement Lemma 1 or the fact, that for even d, a sign pattern consisting of $d+1$ pluses is realizable with the pair $(0,0)$, see [3], etc. The (non)existence of such a degree d_0 is not self-evident, and if it exists, it is not a priori clear how many new general statements of (non)realizability have to be proved.

3. Preliminaries

Notation 2. We denote by S the subset of $\mathbb{R}^{11}$ such that if $a \in S$, then the signs of the coefficients of the polynomial $P(x,a) = x^{11} + a_{10}x^{10} + \ldots + a_0$ define the sign pattern σ^0 and the polynomial P realizes the pair $(1,8)$.

By T we denote the subset of S for which $a_{10} = -1$. For a polynomial from S one can obtain the conditions $a_{11} = 1$, $a_{10} = -1$ by rescaling and multiplication by a nonzero constant (a_{11} stands for the leading coefficient).

Lemma 3. *For $a \in \overline{S}$ one has $a_j \neq 0$ for $j = 9,8,7,4,3,2$, and one does not have $a_6 = 0$ and $a_5 = 0$ simultaneously.*

Indeed, for $a_j = 0$ (where j is one of the indices 9, 8, 7, 4, 3, 2) there are less than 8 sign changes in the sign pattern σ^0_m, hence by Descartes' rule of signs the polynomial $P(\cdot,a)$ has less than 8 negative roots counted with multiplicity. The same is true for $a_5 = a_6 = 0$.

Lemma 4. *For $a \in \overline{S}$ one has $a_0 \neq 0$.*

R e m a r k 1. A priori the set $\overline{S}$ can contain polynomials with all roots real and nonzero. The positive ones can either be a triple root or a double root and a simple root (but not three simple roots).

P r o o f of Lemma 4. Consider first the case $a_j \neq 0$, $j \neq 0$, $a_0 = 0$. Hence, the polynomial P has a root at 0, either 0 or 2 positive roots and 8 negative roots. Suppose that P has no positive roots. Then the degree 10 polynomial P/x defines a sign pattern with exactly two sign changes and 8 negative roots. There exists no such polynomial. Indeed, if it has distinct negative roots and no positive roots, then this would contradict Lemma 1 (in notation of Lemma 1, one has $\kappa = \frac{32}{5} > 4$). If it has 8 negative roots counted with multiplicity, then one can make them distinct by a series of perturbations which do not change the signs of the coefficients of the polynomial, which increase the number of distinct negative roots while keeping their total multiplicity equal to 8 and which do not introduce new positive roots.

More exactly, suppose that P has a negative root $-b$ of multiplicity r, $1 < r \leqslant 8$. Set $P \mapsto P + \varepsilon P_1$, where $\varepsilon \in (\mathbb{R}, 0)$, $\varepsilon > 0$ and if $P = (x+b)^r x Q_1 Q_2$, where Q_1, Q_2 are polynomials, Q_2 having a complex conjugate pair of roots, Q_1 having $8 - r$ negative roots counted with multiplicity, then $P_1 = (x+b)^{r-1} x Q_1$ (this decreases the multiplicity of the root $-b$ by l and introduces a new simple negative root).

If the polynomial P/x has two positive roots, then, in fact, this must be a positive double root g because $a \in \overline{S}$. In this case the perturbations are with P_1 of the form $(x+b)^{r-1} x Q_1 (x-g)^2$; after having thus obtained P with 8 negative simple roots and a double root at g, one makes another perturbation $P \mapsto P \pm \varepsilon x$ (the sign of ε depends on whether P has a minimum or maximum at g) after which the degree 10 polynomial P/x has 8 negative simple roots and no other real root which is a contradiction with Lemma 1.

Suppose now that $a_j \neq 0$, $j \geqslant 2$ and $a_1 = a_0 = 0$. In the same way one considers the degree 9 polynomial P/x^2 to obtain a contradiction with Lemma 1. In this case one has $\kappa = 7$.

Suppose now that exactly one of the coefficients a_5 or a_6 is 0 (we assume this is a_5, for a_6 the reasoning is analogous) and either $a_1 \neq 0$, $a_0 = 0$ or $a_1 = a_0 = 0$ (all other coefficients a_j being nonzero). Then in the perturbations we set $P_1 = (x+b)^{r-1} x (x+h_1)(x+h_2) Q_1$, where the real numbers h_i are distinct, different from any of the roots of P and chosen in such a way that the coefficient δ of x^5 of P_1 is 0. Such choice is possible because all coefficients of the polynomial $(x+b)^{r-1} Q_1$ are positive, hence δ is of the form $A + (h_1 + h_2) B + C h_1 h_2$, where $A > 0$, $B > 0$ and $C > 0$. □

From now on we consider mainly T (and not S) in order not to take into account the possibility for a_{10} to vanish at some points of $\overline{S}$.

R e m a r k 2. Lemmas 3 and 4 imply that for a polynomial in $\overline{T}$ exactly one of the following possibilities exists:

(1) all its coefficients are nonvanishing;

(2) exactly one of them is vanishing, and this coefficient is either a_1 or a_5 or a_6;

(3) exactly two of them are vanishing, and these are either a_1 and a_5 or a_1 and a_6.

Lemma 5. *There exists no real degree* 11 *polynomial the signs of whose coefficients define the sign pattern* σ^0 *and which has a single positive simple root, negative roots of total multiplicity* 8 *and a complex conjugate pair with nonpositive real part.*

P r o o f. Suppose that such a monic polynomial exists. One can represent it in the form $P = P_1P_2P_3$, where $\deg P_1 = 8$, all roots of P_1 are negative, hence

$P_1 = \sum_{j=0}^{8} \alpha_j x^j$, $\alpha_j > 0$, $\alpha_8 = 1$;

$P_2 = x - w$, $w > 0$;

$P_3 = x^2 + \beta_1 x + \beta_0$, $\beta_j \geqslant 0$, $\beta_1^2 - 4\beta_0 < 0$.

By Descartes' rule of signs, the polynomial $P_1P_2 = \sum_{j=0}^{9} \gamma_j x^j$, $\gamma_9 = 1$, has exactly one sign change in the sequence of its coefficients. It is clear that as $0 > a_{10} = \gamma_8 + \beta_1$, and as $\beta_1 \geqslant 0$, one must have $\gamma_8 < 0$. But then $\gamma_j < 0$ for $j = 0, \ldots, 8$. For $j = 4, \ldots, 8$, one has $a_j = \gamma_{j-2} + \beta_1\gamma_{j-1} + \beta_0\gamma_j < 0$ which means that the signs of a_j do not form the sign pattern σ^0. $\square$

R e m a r k 3. Lemma 5 implies that the set $\overline{T}$ can contain only polynomials with negative roots of total multiplicity 8 and positive roots of total multiplicity 1 or 3 (i.e. either one simple, or one simple and one double, or one triple positive root), and no root at 0 (Lemma 4). Indeed, when approaching the boundary of T, the complex conjugate pair can coalesce into a double positive (but never nonpositive) root; the latter might eventually coincide with the simple positive root.

4. The method of the proof

Consider $\mathbb{R}^{10}$ as the space of the coefficients of the polynomial $P(x,a)|_{a_{10}=-1}$. Suppose that there exists a monic polynomial $P(x,a^*)$ with signs of its coefficients defined by the sign pattern σ^0 (with $a_{10} = -1$) with 8 distinct negative, a simple positive and two complex conjugate roots. Then for a close to $a^* \in \mathbb{R}^{10}$, all polynomials $P(x,a)$ share with $P(x,a^*)$ these properties. Therefore the interior of the set T is nonempty. In what follows we denote by Γ the connected component of T which a^* belongs to. Denote by $-\delta$ the value of a_9 for $a = a^*$ (recall that this value is negative).

Lemma 6. *There exists a compact set $K \subset \overline{\Gamma}$ containing all points of $\overline{\Gamma}$ with $a_9 \in [-\delta, 0)$. Hence, there exists $\delta_0 > 0$ such that for every point of $\overline{\Gamma}$ one has $a_9 \leqslant -\delta_0$, and for at least one point of K and for no point of $\overline{\Gamma} \setminus K$, the equality $a_9 = -\delta_0$ holds.*

P r o o f. Suppose that there exists an unbounded sequence $\{a^n\}$ of values $a \in \overline{\Gamma}$ with $a_9^n \in [-\delta, 0)$. Hence, one can perform rescalings $x \mapsto \beta_n x$, $\beta_n > 0$ such that the largest of the moduli of the coefficients of the monic polynomials $Q_n := (\beta_n)^{-11} P(\beta_n x, a^n)$ equals 1. These polynomials belong to $\overline{S}$, not necessarily to $\overline{T}$ because a_{10} after the rescalings, in general, is not equal to -1. The coefficient of x^9 in Q_n equals $a_9^n (\beta_n)^{-2}$. The sequence $\{a^n\}$ being unbounded, there exists a subsequence β_{n_k} tending to ∞. This means that the sequence of monic polynomials $Q_{n_k} \in \overline{S}$ with bounded coefficients has a polynomial in $\overline{S}$ with $a_9 = 0$ as one of its limit points which contradicts Lemma 3.

Hence, the tuple of coefficients a_j of $P(x, a) \in \overline{\Gamma}$ with $a_9 \in [-\delta, 0)$ remains bounded (hence, the same holds true for the moduli of the roots of P) from which the existence of K and δ_0 follows. $\square$

The above lemma implies the existence of a polynomial $P_0 \in \overline{\Gamma}$ with $a_9 = -\delta_0$. We say that P_0 is a_9-*maximal*. Our aim is to show that no polynomial of $\overline{\Gamma}$ is a_9-maximal which is the contradiction that will be used in the proof of Theorem 1.

Definition 1. A real univariate polynomial is *hyperbolic* if it has only real (not necessarily simple) roots. We denote by $H \subset \overline{\Gamma}$ the set of hyperbolic polynomials in $\overline{\Gamma}$. Hence, these are monic degree 11 polynomials having positive and negative roots of respective total multiplicities 3 and 8 (vanishing roots are impossible by Lemma 3). By $U \subset \overline{\Gamma}$ we denote the set of polynomials in $\overline{\Gamma}$ having a complex conjugate pair, a simple positive root and negative roots of total multiplicity 8. Thus, $\overline{\Gamma} = H \cup U$ and $H \cap U = \emptyset$. We denote by U_0, U_2, $U_{2,2}$, U_3 and U_4 the subsets of U for which the polynomial $P \in U$ has 8 simple negative roots, one double and 6 simple negative roots, at least two negative roots of multiplicity greater than or equal to 2, one triple and 5 simple negative roots and a negative root of multiplicity greater than or equal to 4, respectively.

The following lemma on hyperbolic polynomials will be used further in the proofs.

Lemma 7. *Suppose that V is a hyperbolic polynomial of degree $d \geqslant 2$ with no root at 0. Then:*

(1) *V does not have two or more consecutive vanishing coefficients.*

(2) *If V has a vanishing coefficient, then the signs of its surrounding two coefficients are opposite.*

(3) *The number of positive (or negative) roots of V is equal to the number of sign changes in the sequence of its coefficients (or coefficients of $V(-x)$).*

The proofs of the lemmas of this section except Lemma 6 are given in Section 5 (Lemmas 7–12), in Section 6 (Lemma 13) and in Section 7 (Lemmas 14–16).

Lemma 8.
(1) *No polynomial of $U_{2,2} \cup U_4$ is a_9-maximal.*
(2) *For each polynomial of U_3 there exists a polynomial of U_0 with the same values of a_9, a_6, a_5 and a_1.*
(3) *For each polynomial of $U_0 \cup U_2$ there exists a polynomial of $H \cup U_{2,2}$ with the same values of a_9, a_6, a_5 and a_1.*

Lemma 8 implies that if there exists an a_9-maximal polynomial in $\overline{\Gamma}$, then there exists such a polynomial in H. So from now on, we aim at proving that H contains no such polynomial, hence H and $\overline{\Gamma}$ are empty.

Lemma 9. *There exists no polynomial in H having exactly two distinct real roots.*

Lemma 10. *The set H contains no polynomial having one triple positive root and negative roots of total multiplicity 8.*

Lemma 10 and Remark 1 imply that a polynomial in H (if any) satisfies the following condition:

Condition A. It has a double and a simple positive roots and negative roots of total multiplicity 8.

Lemma 11. *There exists no polynomial $P \in H$ having exactly three distinct real roots and satisfying the conditions $\{a_1 = 0, a_5 = 0\}$ or $\{a_1 = 0, a_6 = 0\}$.*

It follows from the lemma and from Lemma 3 that a polynomial $P \in H$ having exactly three distinct real roots (hence a double and a simple positive and an 8-fold negative one) can satisfy at most one of the conditions $a_1 = 0$, $a_5 = 0$ and $a_6 = 0$.

Lemma 12. *No polynomial in H having exactly three distinct real roots is a_9-maximal.*

Thus, an a_9-maximal polynomial in H (if any) must satisfy Condition A and have at least four distinct real roots.

Lemma 13. *The set H contains no polynomial having a double and a simple positive roots and exactly two distinct negative roots of total multiplicity 8, and satisfying either the conditions $\{a_1 = a_5 = 0\}$ or $\{a_1 = a_6 = 0\}$.*

At this point we know that an a_9-maximal polynomial of H satisfies Condition A and one of the two following conditions:

Condition B. It has exactly four distinct real roots and satisfies exactly one or none of the equalities $a_1 = 0$, $a_5 = 0$ or $a_6 = 0$.

Condition C. It has at least five distinct real roots.

Lemma 14. *The set H contains no a_9-maximal polynomial satisfying Conditions* A *and* B.

Therefore an a_9-maximal polynomial in H (if any) must satisfy Conditions A and C.

Lemma 15. *The set H contains no a_9-maximal polynomial having exactly five distinct real roots.*

Lemma 16. *The set H contains no a_9-maximal polynomial having at least six distinct real roots.*

Hence, the set H contains no a_9-maximal polynomial at all. It follows from Lemma 8 that there is no such polynomial in $\overline{\Gamma}$. Hence $\overline{\Gamma} = \emptyset$.

5. Proofs of Lemmas 7–12

Proof of Lemma 7. *Part* (1): Suppose that a hyperbolic polynomial V with two or more vanishing coefficients exists. If V is degree d hyperbolic, then $V^{(k)}$ is also hyperbolic for $1 \leqslant k < d$. Therefore we can assume that V is of the form $x^l L + c$, where $\deg L = d - l$, $l \geqslant 3$, $L(0) \neq 0$ and $c = V(0) \neq 0$. If V is hyperbolic and $V(0) \neq 0$, then so is also $W := x^d V(1/x) = cx^d + x^{d-l} L(1/x)$ and also $W^{(d-l)}$, which is of the form $ax^l + b$, $a \neq 0 \neq b$. However, given that $l \geqslant 3$, this polynomial is not hyperbolic.

Part (2): For the proof of part (2) we use exactly the same reasoning, but with $l = 2$. The polynomial $ax^2 + b$, $a \neq 0 \neq b$ is hyperbolic if and only if $ab < 0$.

Part (3): To prove part (3) we consider the sequence of coefficients of $V := \sum_{j=0}^{d} v_j x^j$, $v_0 \neq 0 \neq v_d$. Set $\Phi := \sharp\{k\colon\ v_k \neq 0 \neq v_{k-1},\ v_k v_{k-1} < 0\}$, $\Psi := \sharp\{k\colon\ v_k \neq 0 \neq v_{k-1},$ $v_k v_{k-1} > 0\}$ and $\Lambda := \sharp\{k\colon\ v_k = 0\}$. Then $\Phi + \Psi + 2\Lambda = d$. By Descartes' rule of signs the number of positive and negative roots of V is $\mathrm{pos}_V \leqslant \Phi + \Lambda$ and $\mathrm{neg}_V \leqslant \Psi + \Lambda$, respectively. As $\mathrm{pos}_V + \mathrm{neg}_V = d$, one must have $\mathrm{pos}_V = \Phi + \Lambda$ and $\mathrm{neg}_V = \Psi + \Lambda$. It remains to notice that $\Phi + \Lambda$ is the number of sign changes in the sequence of coefficients of V (and $\Psi + \Lambda$ of $V(-x)$), see part (2) of the lemma. $\square$

P r o o f of Lemma 8. *Part* (1): A polynomial of $U_{2,2}$ or U_4 is, respectively, representable in the form:

$$P^\dagger := (x+u)^2(x+v)^2 S\Delta \quad \text{and} \quad P^* := (x+u)^4 S\Delta,$$

where $\Delta := (x^2 - \xi x + \eta)(x - w)$ and $S := x^4 + Ax^3 + Bx^2 + Cx + D$. All coefficients u, u, v, w, ξ, η, A, B, C, D are positive and $\xi^2 - 4\eta < 0$ (see Lemma 5); for A, B, C and D this follows from the fact that all roots of $P^\dagger/\Delta$ and P^*/Δ are negative. (The roots of $x^4 + Ax^3 + Bx^2 + Cx + D$ are not necessarily different from $-u$ and $-v$.) We consider the two Jacobian matrices

$$J_1 := \Big(\frac{\partial(a_{10}, a_9, a_1, a_5)}{\partial(\xi, \eta, w, u)}\Big) \quad \text{and} \quad J_2 := \Big(\frac{\partial(a_{10}, a_9, a_1, a_6)}{\partial(\xi, \eta, w, u)}\Big).$$

In the case of $P^\dagger$ their determinants equal

$$\begin{aligned}
\det J_1 = \Pi(&CDv + 2CDu + C^2uv + 2BDv^2 + 4BDuv \\
&+ 2BDu^2 + 2BCuv^2 + BCu^2v + ADv^3 + 2ADuv^2 \\
&+ 3ADu^2v + Cu^2v^3 + ACuv^3 + 2ACu^2v^2), \\
\det J_2 = \Pi(&BDv + 2BDu + Dv^3 + 2Duv^2 + 3Du^2v + BCuv + 2ADv^2 \\
&+ 4ADuv + 2ADu^2 + Cuv^3 + 2u^2v^2C + 2ACuv^2 + ACu^2v),
\end{aligned}$$

where $\Pi := -2v(w+u)(-\eta - w^2 + w\xi)(\xi u + \eta + u^2)$.

These determinants are nonzero. Indeed, each of the factors is either a sum of positive terms or equals $-\eta - w^2 + w\xi < -\frac{1}{4}\xi^2 - w^2 + w\xi = -(\frac{1}{2}\xi - w)^2 \leqslant 0$. Thus, one can choose values of (ξ, η, w, v) close to the initial one (u, A, B, C and D remain fixed) to obtain any values of (a_{10}, a_9, a_1, a_5) or (a_{10}, a_9, a_1, a_6) close to the initial one. In particular, $a_{10} = -1$, $a_1 = a_5 = 0$ or $a_{10} = -1$, $a_1 = a_6 = 0$ while a_9 can have values larger than the initial one. Hence, this is not an a_9-maximal polynomial. (If the change of the value of (ξ, η, w, v) is small enough, the values of the coefficients a_j, $j = 0, 2, 3, 4, 6$ or $5, 7$ and 8 can change, but their signs remain the same.) The same reasoning is valid for P^* as well in which case one has

$$\begin{aligned}
\det J_1 &= M(3CD + C^2u + 8BDu + 3BCu^2 + 6ADu^2 + u^4C + 3ACu^3), \\
\det J_2 &= M(3BD + 6u^2D + BCu + 8ADu + 3u^3C + 3ACu^2),
\end{aligned}$$

with $M := -4u^2(w+u)(-\eta - w^2 + w\xi)(\xi u + \eta + u^2)$.

Part (2): We observe that if the triple root of $P \in U_3$ is at $-u < 0$, then in the case when P is increasing (or decreasing) in a neighbourhood of $-u$, the polynomial

$P - \varepsilon x^2(x+u)$ (or $P + \varepsilon x^2(x+u)$), where $\varepsilon > 0$ is small enough, has three simple roots close to $-u$; it belongs to $\overline{\Gamma}$, its coefficients a_j, $2 \neq j \neq 3$, are the same as the ones of P, the signs of a_2 and a_3 are also the same.

Part (3): For the proof of part (3), we observe first that

- ▷ for $x < 0$ the polynomial P has four maxima and four minima and
- ▷ for $x > 0$ one of the following three things holds true: either $P' > 0$, or there is a double positive root γ of P', or P' has two positive roots $\gamma_1 < \gamma_2$ (they are both either smaller than or greater than the positive root of P).

Suppose first that $P \in U_0$. Consider the family of polynomials $P - t$, $t \geqslant 0$. Denote by t_0 the smallest value of t for which one of the three things happens: either $P - t$ has a double negative root v (hence a local maximum), or $P - t$ has a triple positive root γ, or $P - t$ has a double and a simple positive roots (the double one is at γ_1 or γ_2). In the second and third cases one has $P - t_0 \in H$. In the first case, if $P - t_0$ has another double negative root, then $P - t_0 \in U_{2,2}$ and we are done. If not, then consider the family of polynomials

$$P_s := P - t_0 - s(x^2 - v^2)^2(x^2 + v^2)^2 = P - t_0 - s(x^8 - 2v^4x^4 + v^8), \quad s \geqslant 0.$$

The polynomial $-(x^8 - 2v^4x^4 + v^8)$ has double real roots at $\pm v$ and a double complex conjugate pair. It has the same signs of the coefficients of x^8, x^4 and 1 as $P - t_0$ and P. The rest of the coefficients of $P - t_0$ and P_s are the same. As s increases, the value of P_s for every $x \neq \pm v$ decreases. So for some $s = s_0 > 0$ for the first time one has either $P_s \in U_{2,2}$ (another local maximum of P_s becomes a double negative root) or $P_s \in H$ (P_s has positive roots of total multiplicity 3, but not three simple ones). This proves part (3) for $P \in U_0$.

If $P \in U_2$ and the double negative root is a local minimum, then the proof of part (3) is just the same. If this is a local maximum, then one skips the construction of the family $P - t$ and starts constructing the family P_s directly. □

P r o o f of Lemma 9. Suppose that such a polynomial exists. Then it must be of the form $P := (x+u)^8(x-w)^3$, $u > 0$, $w > 0$. The conditions $a_{10} = -1$ and $a_1 > 0$ read:

$$8u - 3w = -1 \quad \text{and} \quad u^7w^2(3u - 8w) > 0.$$

In the plane of the variables (u, w), the domain $\{u > 0,\ w > 0,\ 3u - 8w > 0\}$ does not intersect the line $8u - 3w = -1$, which proves the lemma. □

P r o o f of Lemma 10. Represent the polynomial in the form $P = (x+u_1)\ldots\times(x+u_8)(x-\xi)^3$, where $u_j > 0$ and $\xi > 0$. The numbers u_j are not necessarily distinct. The coefficient a_{10} then equals $u_1 + \ldots + u_8 - 3\xi$. The condition $a_{10} = -1$

implies $\xi = \xi_* := \frac{1}{3}(u_1 + \ldots + u_8 + 1)$. Denote by $\widetilde{a}_1$ the coefficient a_1 expressed as a function of $(u_1, \ldots, u_8, \xi)$. Using computer algebra (say, MAPLE) one finds $27\widetilde{a}_1|_{\xi=\xi_*}$:

$$27\widetilde{a}_1|_{\xi=\xi_*} = -(-u_1 \ldots u_8 + X + Y)(u_1 + \ldots + u_8 + 1)^2,$$

where $Y := u_1 \ldots u_8(1/u_1 + \ldots + 1/u_8)$ and $X := u_1 \ldots u_8 \sum_{1\leqslant i,\, j\leqslant 8,\, i\neq j} u_i/u_j$ (the sum X contains 56 terms). We show that $a_1 < 0$, which by contradiction proves the lemma. The factor $(u_1+\ldots+u_8+1)^2$ is positive. The factor $\Xi := -u_1 \ldots u_8 + X + Y$ contains a single monomial with a negative coefficient, namely, $-u_1 \ldots u_8$. Consider the sum

$$\begin{aligned} -u_1 \ldots u_8 + u_1^2 u_3 u_4 u_5 u_6 u_7 u_8 + u_2^2 u_3 u_4 u_5 u_6 u_7 u_8 \\ = u_3 u_4 u_5 u_6 u_7 u_8((u_1 - u_2)^2 + u_1 u_2) > 0 \end{aligned}$$

(the second and third monomials are in X). Hence, Ξ is representable as a sum of positive quantities, so $\Xi > 0$ and $a_1 < 0$. $\square$

P r o o f of Lemma 11. Suppose that such a polynomial exists. Then it must be of the form $(x + u)^8(x - w)^2(x - \xi)$, where $u > 0$, $w > 0$, $\xi > 0$, $w \neq \xi$. One checks numerically (say, using MAPLE) for each of the two systems of algebraic equations $a_{10} = -1$, $a_1 = 0$, $a_5 = 0$ and $a_{10} = -1$, $a_1 = 0$, $a_6 = 0$, that each real solution (u, w, ξ) or (u, v, w) contains a nonpositive component. $\square$

P r o o f of Lemma 12. Making use of Condition A formulated after Lemma 10, we consider only polynomials of the form $(x + u)^8(x - w)^2(x - \xi)$. Consider the Jacobian matrix

$$J_1^* := \Big(\frac{\partial(a_{10}, a_9, a_1)}{\partial(u, w, \xi)}\Big).$$

Its determinant equals $6u^6(u + w)(u - 7w)(\xi - w)(k + u)$. All factors except $u - 7w$ are nonzero. Thus, for $u \neq 7w$ one has $\det J_1 \neq 0$, so one can fix the values of a_{10} and a_1 and vary the one of a_9 arbitrarily close to the initial one by choosing suitable values of u, w and ξ. Hence, the polynomial is not a_9-maximal. For $u = 7w$, one has $a_3 = -117649w^7(35w + 8\xi) < 0$, which is impossible. Hence, there exist no a_9-maximal polynomials which satisfy only the condition $a_1 = 0$ or none of the conditions $a_1 = 0$, $a_5 = 0$ or $a_6 = 0$. To see that there exist no such polynomials satisfying only the condition $a_5 = 0$ or $a_6 = 0$ one can consider the matrices $J_5^* := (\partial(a_{10}, a_9, a_5)/\partial(u, w, \xi))$ and $J_6^* := (\partial(a_{10}, a_9, a_6)/\partial(u, w, \xi))$. Their determinants equal, respectively,

$$112u^2(u + w)(5u - 3w)(\xi - w)(\xi + u) \quad \text{and} \quad 112u(u + w)(3u - w)(\xi - w)(\xi + u).$$

They are nonzero respectively for $5u \neq 3w$ and $3u \neq w$, in which cases in the same way we conclude that the polynomial is not a_9-maximal. If $u = \frac{3}{5}w$, then $a_1 = -\frac{2187}{390625}w^9(-3w + 34\xi)$ and $a_{10} = -\xi + \frac{14}{5}w$. As $a_1 > 0$ and $a_{10} < 0$, one has $w > \frac{34}{3}\xi$ and $\xi > \frac{14}{5}w > \frac{34}{3}\frac{14}{5}\xi$, which is a contradiction. If $w = 3u$, then $a_6 = 14u^4(10u + \xi) > 0$, which is again a contradiction. □

6. Proof of Lemma 13

The multiplicities of the negative roots of P define the following a priori possible cases:

(i) (7,1), (ii) (6,2), (iii) (5,3), (iv) (4,4).

In all of them the proof is carried out simultaneously for the two possibilities $\{a_1 = a_5 = 0\}$ and $\{a_1 = a_6 = 0\}$. In order to simplify the proof we fix one of the roots to be equal to -1 (this can be achieved by a change $x \mapsto \beta x$, $\beta > 0$, followed by $P \mapsto \beta^{-11}P$). This allows to deal with one parameter less. By doing so we can no longer require that $a_{10} = -1$, but only that $a_{10} < 0$.

Case (i): We use the following parametrisation: $P = (x+1)^7(sx+1)(tx-1)^2 \times (wx-1)$, $s > 0$, $t > 0$, $w > 0$, $t \neq w$, i.e. the negative roots of P are at -1 and $-1/s$ and the positive ones at $1/t$ and $1/w$.

The condition $a_1 = w + 2t - s - 7 = 0$ yields $s = w + 2t - 7$. For $s = w + 2t - 7$ one has

$$a_3 = a_{32}w^2 + a_{31}w + a_{30}, \quad a_4 = a_{42}w^2 + a_{41}w + a_{40},$$

where

$$a_{32} = -2t + 7, \quad a_{31} = -(2t-7)^2, \quad a_{30} = -2t^3 + 28t^2 - 98t + 112,$$
$$a_{42} = t^2 - 14t + 21, \quad a_{41} = 2t^3 - 35t^2 + 140t - 147,$$
$$a_{40} = -14t^3 + 112t^2 - 294t + 21.$$

The coefficient a_{30} has a single real root $9.436\ldots$, hence $a_{30} < 0$ for $t > 9.436\ldots$. On the other hand,

$$a_{32}w^2 + a_{31}w = w(-2t+7)(w+2t-7) = w(-2t+7)s,$$

which is negative for $t > 9.436\ldots$. Thus, the inequality $a_3 > 0$ fails for $t > 9.436\ldots$. Observing that $a_{41} = (2t-7)a_{42}$, one can write

$$a_4 = (w + 2t - 7)wa_{42} + a_{40} = swa_{42} + a_{40}.$$

The real roots of a_{42} (or a_{40}) equal $1.708\ldots$ and $12.291\ldots$ (or $1.136\ldots$). Hence, for $t \in [1.708\ldots, 12.291\ldots]$ the inequality $a_4 > 0$ fails. It remains to consider the possibility $t \in (0, 1.708\ldots)$.

It is to be checked directly that for $s = w + 2t - 7$ one has

$$\frac{a_{10}}{t} = (7t-2)w(w+2t-7) + t(7-2t) = (7t-2)ws + t(7-2t),$$

which is nonnegative (hence $a_{10} < 0$ fails) for $t \in [\frac{2}{7}, \frac{7}{2}]$. Similarly,

$$a_6 = a_6^* w(w+2t-7) + a_6^\dagger = a_6^* ws + a_6^\dagger,$$

where

$$a_6^* = 21t^2 - 70t + 35, \quad a_6^\dagger = -70t^3 + 350t^2 - 490t + 140.$$

The real roots of a_6^* (or $a_6^\dagger$) equal $0.612\ldots > \frac{2}{7} = 0.285\ldots$ and $2.720\ldots$ (or $0.381\ldots > \frac{2}{7}$, 2 and $2.618\ldots$), hence for $t \in \left(0, \frac{2}{7}\right)$ one has $a_6^* > 0$ and $a_6^\dagger > 0$, i.e. $a_6 > 0$ and the equality $a_6 = 0$ or the inequality $a_6 < 0$ is impossible. This finishes the proof of Case (i). $\square$

Case (ii): We parametrise P as follows: $P = (x+1)^6(Tx^2 + Sx - 1)^2(wx - 1)$, $T > 0$, $w > 0$. In this case we presume S to be real, not necessarily positive. The factor $(Tx^2 + Sx - 1)^2$ contains the double positive and negative roots of P.

From $a_1 = w + 2S - 6 = 0$ one finds $S = \frac{1}{2}(6 - w)$. For $S = \frac{1}{2}(6 - w)$ one has

$$\frac{a_{10}}{T} = (6w-1)T + 6w - w^2, \quad a_7 = a_{72}T^2 + a_{71}T + a_{70},$$

where

$$a_{72} = 15w - 20, \quad a_{71} = -20w^2 + 105w - 78,$$
$$4a_{70} = 15w^3 - 162w^2 + 468w - 192.$$

Suppose first that $w > \frac{1}{6}$. The inequality $a_{10} < 0$ is equivalent to $T < (w^2 - 6w)/(6w-1)$. As $T > 0$, this implies $w > 6$.

For $T = (w^2 - 6w)/(6w-1)$ one obtains $a_7 = 3C/4(6w-1)^2$, where the numerator $C := 40w^5 - 444w^4 + 1345w^3 - 502w^2 + 300w - 64$ has a single real root $0.253\ldots$. Hence, for $t > 6$ one has $C > 0$ and $a_7|_{T=(w^2-6w)/(6w-1)} > 0$. On the other hand, $a_{70} = a_7|_{T=0}$ has roots $0.489\ldots$, $4.504\ldots$ and $5.805\ldots$, so for $w > 6$ one has $a_7|_{T=0} > 0$. For $w > 6$ fixed and for $T \in [0, (w^2 - 6w)/(6w-1)]$, the value of the derivative

$$\frac{\partial a_7}{\partial T} = (30w - 40)T - 20w^2 + 105w - 78$$

is maximal for $T = (w^2 - 6w)/(6w-1)$; this value equals

$$-\frac{90w^3 - 430w^2 + 333w - 78}{6w-1},$$

which is negative because the only real root of the numerator is $3.882\ldots$. Thus, $\partial a_7/\partial T < 0$ and a_7 is minimal for $T = (w^2 - 6w)/(6w - 1)$. Hence, the inequality $a_7 < 0$ fails for $w > \frac{1}{6}$. For $w = \frac{1}{6}$ one has $a_{10} = \frac{35}{36}T > 0$.

So suppose that $w \in \left(0, \frac{1}{6}\right)$. In this case the condition $a_{10} < 0$ implies $T > (w^2 - 6w)/(6w - 1)$. For $T = (w^2 - 6w)/(6w - 1)$ one gets

$$a_4 = \frac{3D}{4(6w-1)^2}, \quad \text{where } D := 64w^5 - 300w^4 + 502w^3 - 1345w^2 + 444w - 40$$

has a single real root $3.939\ldots$. Hence, for $w \in \left(0, \frac{1}{6}\right)$ one has $D < 0$ and $a_4|_{T=(w^2-6w)/(6w-1)} < 0$. The derivative $\partial a_4/\partial T = -w^2 - 2T - 6$ being negative one has $a_4 < 0$ for $w \in \left(0, \frac{1}{6}\right)$, i.e. the inequality $a_4 > 0$ fails. This finishes the proof of Case (ii). $\square$

Case (iii): We use the following parametrisation: $P = (x+1)^5(xs+1)^3(xt-1)^2 \times (xw-1)$. From $a_1 = w+2t-5-3s = 0$ one gets $s = \frac{1}{3}(w+2t-5)$. For $s = \frac{1}{3}(w+2t-5)$ one has $27a_{10} = tS(w + 2t - 5)^2$, where

$$S := 10wt^2 - 2t^2 + 5w^2t - 21wt + 5t - 2w^2 + 10w. \tag{6.1}$$

The factor S can be represented as a polynomial in w or in t; for each of the cases we give its discriminant (and the latter's real roots) as well:

$$\begin{aligned} S &= (5t-2)w^2 + (10 - 21t + 10t^2)w + 5t - 2t^2, \\ D_1 &= 5(t-2)(2t-1)(10t^2 - 13t + 10), \ 0.5, \ 2 \\ S &= (10w-2)t^2 + (5w^2 - 21w + 5)t - 2w^2 + 10w, \\ D_2 &= 5(w^2 - 5w + 1)(5w^2 - w + 5), \ 0.208\ldots, \ 4.791\ldots \end{aligned}$$

Hence, for $t \in [0.5, 2]$ or for $w \in [0.208\ldots, 4.791\ldots]$ one has $D_1 \leqslant 0$ and $D_2 \leqslant 0$, respectively, hence $S \geqslant 0$ and the inequality $a_{10} < 0$ fails. The partial derivative

$$\frac{\partial S}{\partial t} = 5w^2 - 21w + 20wt - 4t + 5 = 5w(w - 4.2) + (20w - 4)t + 5$$

is positive for $t > 2$ and $w > 4.791\ldots$. Hence, $S > 0$ for $t > 2$ and $w > 4.791\ldots$. For $(t, w) \in (0, 0.5) \times (0, 0.208\ldots)$ one has $w + 2t - 5 < 0$, i.e. $s < 0$. Thus, Case (iii) is impossible outside the two semi-strips

$$\Sigma_1 := \{(t, w) \in (0, 0.5) \times (4.791\ldots, \infty)\} \text{ and } \Sigma_2 := \{(t, w) \in (2, \infty) \times (0, 0.208\ldots)\}.$$

Lemma 17. *The inequality $a_4 > 0$ fails on Σ_2.*

P r o o f. Indeed,

$$27a_4 = w^4 + s_3 w^3 + s_2 w^2 + s_1 w + s_0,$$

where

$$\begin{aligned} s_3 &= -10t + 25, \quad s_2 = -30t^2 + 60t - 120, \\ s_1 &= -22t^3 + 75t^2 - 120t + 175, \\ s_0 &= -20t^4 + 110t^3 - 300t^2 + 350t - 410. \end{aligned}$$

For $(t, w) \in \Sigma_2$ one has

$$w^4 + s_3 w^3 \leqslant (0.208\ldots)^4 + (-10 \times 2 + 25) \times (0.208\ldots)^3 < 0.05.$$

The trinomial s_2 is negative (because its discriminant is negative), so $s_2 w^2 < 0$. The quantity s_0 is decreasing for $t \geqslant 2$ (because the only real root of its derivative equals 1), so in Σ_2 one has $s_0 < s_0|_{t=2} = -350$. Finally, the quantity s_1 is decreasing (its derivative has no real roots), hence in Σ_2 the term $s_1 w$ is less than $s_1|_{t=2} w \leqslant 59 \times 0.208\ldots < 13$. Thus $a_4 < 0.05 - 350 + 13 < 0$ in Σ_2. $\square$

We define the sets

$$\begin{aligned} \Sigma_3 &:= \{(t, w) \in [0, 0.5] \times [6.75\ldots, \infty)\}, \\ \Sigma_4 &:= \{(t, w) \in [0.25, 0.5] \times [4.791\ldots, 6.75]\}, \\ \Sigma_5 &:= \{(t, w) \in [0, 0.25] \times [5, 6.75]\}, \\ \Sigma_6 &:= \{(t, w) \in [0, 0.25] \times [4.791\ldots, 5]\}. \end{aligned}$$

One can observe that $\Sigma_1 \subset (\Sigma_3 \cup \Sigma_4 \cup \Sigma_5 \cup \Sigma_6)$. For $w = 6.75$ one has

$$27a_6 = 14t^5 + 511.75t^4 - 44.09375t^3 - 6341.949214t^2 - 4336.44531t + 3760.50781.$$

Its real roots are $-36.303\ldots$, $-3.058\ldots$, $-1.324\ldots$, $0.503\ldots$ and $3.629\ldots$. Hence, for $t \in (0, 0.5)$ and $w = 6.75$ one has $a_6 > 0$. One can represent $27\partial a_6/\partial w$ in the form $(4w - 5 + 2t)g$, where

$$g := 4t^4 + 4t^3 w + t^2 w^2 - 35t^2 - 20wt^2 + 90t - 10w^2 t + 20wt - 5 - 40w + 10w^2.$$

Hence, $g|_{w=6.75} = 4t^4 + 27t^3 - 124.4375t^2 - 230.625t + 180.625$ with real roots $-9.360\ldots$, $-1.982\ldots$, $0.610\ldots$ and $3.982\ldots$, so $g|_{w=6.75} > 0$ for $t \in (0, \frac{1}{2})$.

Lemma 18. *The derivative $\partial g/\partial w = (2t^2 - 20t + 20)w + 4t^3 - 20t^2 + 20t - 40$ is positive on Σ_3.*

Hence, this is the case of $\partial a_6/\partial w$ and a_6 as well, so the inequality $a_6 < 0$ or the equality $a_6 = 0$ fails of Σ_3.

P r o o f. On Σ_3 one has

$$(2t^2 - 20t + 20)w > (-20t + 20)w > 10 \times 6.75 = 67.5$$

and

$$4t^3 - 20t^2 + 20t - 40 > 4t^3 - 40 > -40,$$

so $\partial a_6/\partial w > 0$. □

Lemma 19. *One has $a_{10} \geqslant 0$ on Σ_4.*

P r o o f. One has $a_{10} = \frac{1}{27}t(w+2t-5)^2 S$, see (6.1), hence $S|_{t=0.25} = -0.75w^2 + 5.375w + 1.125$, which is positive for $w \in [4.791\ldots, 6.75]$. The lemma follows from $\partial S/\partial t = (20w-4)t + 5w^2 - 21w + 5$ being positive for $(t, w) \in \Sigma_4$. □

Lemma 20. *One has $a_6 > 0$ in Σ_5.*

P r o o f. We use the following expression for $27a_6$:

$$\begin{aligned} 27a_6 &= h_4w^4 + h_3w^3 + h_2w^2 + h_1w + h_0, \\ h_4 &= t^2 - 10t + 10, \\ h_3 &= 6t^3 - 35t^2 + 50t - 70, \\ h_2 &= 12t^4 - 30t^3 + 90t + 90, \\ h_1 &= 8t^5 - 20t^4 - 70t^3 + 355t^2 - 460t + 25, \\ h_0 &= -40t^5 + 100t^4 - 50t^3 - 50t^2 + 50t + 260. \end{aligned}$$

Hence, the values for $w = 5$ of the derivatives $27\partial^s a_6/\partial w^s$ are the following polynomials:

$$\begin{aligned} \frac{27\partial^0 a_6}{\partial w^0} &= 300t^4 - 400t^3 - 2025t^2 + 135, \\ \frac{27\partial^1 a_6}{\partial w^1} &= 8t^5 + 100t^4 + 80t^3 - 1770t^2 - 810t + 675, \\ \frac{27\partial^2 a_6}{\partial w^2} &= 24t^4 + 120t^3 - 750t^2 - 1320t + 1080, \\ \frac{27\partial^3 a_6}{\partial w^3} &= 36t^3 - 90t^2 - 900t + 780, \\ \frac{27\partial^4 a_6}{\partial w^4} &= 24t^2 - 240t + 240. \end{aligned}$$

All of them are positive for $t \in [0, 0.25]$, from which and from the Taylor series of a_6 w.r.t. the variable w the lemma follows. □

Lemma 21. *One has* $a_{10} \geqslant 0$ *on* Σ_6.

P r o o f. Recall that the quantity S is defined by (6.1). The values for $t = 0$ of the derivatives $\partial^s S/\partial t^s$ are:

$$\frac{\partial^0 S}{\partial t^0} = -2w^2 + 10w, \quad \frac{\partial^1 S}{\partial t^1} = 5w^2 - 21w + 5, \quad \frac{\partial^2 S}{\partial t^2} = 20w - 4.$$

They are all nonnegative for $w \in [4.791, 5]$ from which and from the Taylor series of S w.r.t. the variable t one gets $S \geqslant 0$ in Σ_6 and the lemma follows. □

This finishes the proof of Case (iii). □

Case (iv): $P = (x+1)^4(sx+1)^4(tx-1)^2(wx-1)$. The condition $a_1 = w + 2t - 4s - 4 = 0$ implies $s = \frac{1}{4}(w + 2t - 4)$. For $s = \frac{1}{4}(w + 2t - 4)$ one has $256a_{10} = t(w + 2t - 4)^3 H^*$, where

$$H^* := 8wt^2 - 2t^2 + 4w^2t - 5wt + 4t + 8w - 2w^2. \tag{6.2}$$

Lemma 22. *The inequality* $H^* \geqslant 0$ (*hence* $a_{10} \geqslant 0$) *holds in each of the two cases* $t \in [\frac{1}{2}, 2]$ *and* $w \in [\frac{1}{4}, 4]$. *It holds also for* $(t, w) \in [2, \infty) \times [4, \infty)$, *for* $(t, w) \in (0, \frac{1}{2}] \times (0, \frac{1}{4}]$ *and for* $(t, w) \in [0.3, \frac{1}{2}] \times [4, 6.71]$.

R e m a r k 4. In other words, for $t > 0$, $w > 0$, the inequality $a_{10} < 0$ fails outside the domain $\Omega_1 \cup \Omega_2 \cup \Omega_3$, where

$$\Omega_1 := (2, \infty) \times \left(0, \tfrac{1}{4}\right), \quad \Omega_2 := \left(0, \tfrac{1}{2}\right) \times (6.71, \infty), \quad \Omega_3 := (0, 0.3) \times (4, 6.71].$$

We set $\Omega_3 = \Omega_3^- \cup \Omega_3^+$, where

$$\Omega_3^- := (0, 0.3) \times (4, 5], \quad \Omega_3^+ := (0, 0.3) \times (5, 6.71].$$

P r o o f of Lemma 22. We represent H^* in two ways:

$$H^* = H_{2w}w^2 + H_{1w}w + H_{0w},$$
$$H_{2w} = 4t - 2, \quad H_{1w} = 8t^2 - 5t + 8, \quad H_{0w} = -2t^2 + 4t$$

and

$$H^* = H_{2t}t^2 + H_{1t}t + H_{0t},$$
$$H_{2t} = 8w - 2, \quad H_{1t} = 4w^2 - 5w + 4, \quad H_{0t} = -2w^2 + 8w.$$

The first statement of the lemma follows from $H_{jw} \geqslant 0$, $j = 1, 2, 3$ for $t \in [\frac{1}{2}, 2]$ and $H_{jt} \geqslant 0$, $j = 1, 2, 3$ for $w \in [\frac{1}{4}, 4]$. The quantity H^* is a degree 2 polynomial in t.

For $t = 2$ and $w \in [4, \infty)$ one has

$$H^* = 30w + 6w^2 > 0, \quad \frac{\partial^2 H^*}{\partial t^2} = 16w - 4 \geqslant 0$$

and

$$\frac{\partial H^*}{\partial t} = 16wt - 4t + 4w^2 - 5w + 4 = (16w - 4)t + w(4w - 5) + 4 > 0,$$

so by representing H^* as a Taylor series in the variable t we see again that $H^* > 0$ for $(t, w) \in [2, \infty) \times [4, \infty)$. Next, for $(t, w) \in (0, \frac{1}{2}] \times (0, \frac{1}{4}]$ one can write

$$H^* = t(4 - 2t - 5w) + 2w(4 - w) + 8wt^2 + 4w^2t > 0.$$

Finally, as $\partial H^*/\partial t = (16w - 4)t + 4w^2 - 5w + 4$, where the polynomial $4w^2 - 5w + 4$ has no real roots, one has $\partial H^*/\partial t > 0$ in $[0.3, \frac{1}{2}] \times [4, 6.71]$. On the other hand, for $t = 0.3$ the polynomial H^* equals $w(7.22 - 0.8w) + 1.02$, which is positive for $w \in [4, 6.71]$. Hence $H^* > 0$ in $[0.3, \frac{1}{2}] \times [4, 6.71]$. □

Lemma 23. *The inequality $a_5 \geqslant 0$ fails for $(t, w) \in [2, \infty) \times (0, \frac{1}{4}] \supset \Omega_1$.*

P r o o f. The quantity $a_5^* := 256a_5$ equals

$$\begin{aligned} 1536t + 768w - 1536t^2 - 384w^2 - 1536wt + 768w^2t + 1280wt^2 \\ - 32w^3t - 416w^2t^2 - 384wt^3 - 16t^3w^2 + 16t^4w - 72t^2w^3 \\ - 22tw^4 - 128w^3 + 512t^3 + 44w^4 - 64t^4 - 96t^5 + w^5. \end{aligned}$$

The values v_j for $t = 2$ of its partial derivatives $\partial^j a_5^*/\partial t^j$, $j = 0, \ldots, 5$ equal

$$\begin{aligned} v_0 &= -3072 - 640w^2 - 480w^3 + w^5, \\ v_1 &= -8192 - 512w - 1088w^2 - 320w^3 - 22w^4, \\ v_2 &= -15360 - 1280w - 1024w^2 - 144w^3, \\ v_3 &= -23040 - 1536w - 96w^2, \\ v_4 &= -24576 + 384w, \\ v_5 &= -11520, \end{aligned}$$

respectively. They are all negative for $w \in (0, \frac{1}{4}]$. Hence, all coefficients of the Taylor series w.r.t. t of the coefficient a_5 for $t = 2$, $w \in (0, \frac{1}{4}]$, are negative and so is a_5 for $(t, w) \in [2, \infty)$. □

Lemma 24. *The inequality $a_6 \leqslant 0$ fails for $(t, w) \in (0, \frac{1}{2}] \times [6.71, \infty) \supset \Omega_2$ and for $(t, w) \in (0, 0.3] \times [5, \infty) \supset \Omega_3^+$.*

Thus, after Lemmas 22, 23 and 24 it remains to prove that for $(t,w) \in \Omega_3^-$ the sign(s) of some (of the) coefficient(s) a_j is/are not the one(s) prescribed by the sign pattern.

P r o o f of Lemma 24. One has

$$\begin{aligned}256a_6 = {} & 1024 - 768w - 1536t - 576w^2t + 1920t^2 + 864w^2 - 352w^3 - 1280t^3 \\ & + 800t^4 - 256t^5 + 26w^4 + 4w^5 - 16t^6 + 384wt - 384wt^2 + 400w^3t \\ & + 720w^2t^2 + 448wt^3 - 352t^3w^2 - 256t^4w + 40t^3w^3 + 104t^4w^2 \\ & + 64t^5w - 272t^2w^3 - t^2w^4 - 56tw^4 - 2tw^5.\end{aligned}$$

We list below the values of the functions $u_j := 256\partial^j a_6/\partial w^j$, $j = 0, \ldots, 5$ for $w = 6.71$. They are all positive for $t \in (0, \frac{1}{2}]$ (this can be checked numerically). From the Taylor series of a_6 for $w = 6.71$ one concludes that $a_6 > 0$ for $(t,w) \in (0, \frac{1}{2}] \times [6.71, \infty)$. Here is the list:

$$\begin{aligned}u_0 := {} & -16t^6 + 173.44t^5 + 3764.7464t^4 - 2037.93476t^3 - 52440.84297t^2 \\ & - 44774.66948t + 35543.86077, \\ u_1 := {} & 64t^5 + 1139.68t^4 + 1127.0520t^3 - 28669.71244t^2 - 41261.71907t \\ & + 35244.43996, \\ u_2 := {} & 208t^4 + 906.40t^3 - 10051.0092t^2 - 27388.66364t + 25772.93608, \\ u_3 := {} & 240t^3 - 1793.04t^2 - 12021.1320t + 12880.8240, \\ u_4 := {} & -24t^2 - 2954.40t + 3844.80, \\ u_5 := {} & 240(2-t).\end{aligned}$$

In the same way we consider the values for $w = 5$ of these same functions, see the list below. One can check that they are all positive for $t \in (0, 0.3]$ and by analogy we conclude that $a_6 > 0$ for $(t,w) \in (0, 0.3] \times [5, \infty)$.

$$\begin{aligned}u_0 := {} & -16t^6 + 64t^5 + 2120t^4 - 2840t^3 - 16625t^2 - 5266t + 3534, \\ u_1 := {} & 64t^5 + 784t^4 - 72t^3 - 14084t^2 - 9626t + 6972, \\ u_2 := {} & 208t^4 + 496t^3 - 7020t^2 - 10952t + 8968, \\ u_3 := {} & 240t^3 - 1752t^2 - 7320t + 7008, \\ u_4 := {} & -24t^2 - 2544t + 3024, \\ u_5 := {} & 240(2-t).\end{aligned}$$

$\square$

Lemma 25. *For* $(t, w) \in (0, \frac{1}{2}] \times [4, 6.71] \supset \Omega_3^-$ *the coefficient* a_6 *is a decreasing function in* t*. For* $t = 0$*,* $w \in [4, 6.71]$ *one has* $a_6 \geqslant 0$ *with equality only for* $w = 4$*.*

Proof. The second claim of the lemma follows from

$$256 a_6|_{t=0} = 4w^5 + 26w^4 - 352w^3 + 864w^2 - 768w + 1024,$$

whose real roots are $-13.978\ldots$, $3.110\ldots$ and 4. To prove the first claim, we list the derivatives $\eta_j := 256\partial^j a_6/\partial t^j|_{t=0}$, $j = 1, \ldots, 6$ and their real roots (η_4 has no real roots):

$$\begin{aligned}
\eta_1 :=& -2w^5 - 56w^4 + 400w^3 - 576w^2 + 384w - 1536\\
& -34.115\ldots,\ 2.782\ldots,\ 4,\\
\eta_2 :=& -2w^4 - 544w^3 + 1440w^2 - 768w + 3840\\
& -274.626\ldots,\ 2.948\ldots,\\
\eta_3 :=& \, 240w^3 - 2112w^2 + 2688w - 7680,\ 7.894\ldots,\\
\eta_4 :=& \, 2496w^2 - 6144w + 19200,\\
\eta_5 :=& \, 7680w - 30720,\ 4,\\
\eta_6 :=& -11520.
\end{aligned}$$

As we see, for $w \in [4, 6.71]$ one has $\eta_1 \leqslant 0$, $\eta_2 < 0$, $\eta_3 < 0$, $\eta_4 > 0$, $\eta_5 \geqslant 0$ and $\eta_6 < 0$. One can majorize the Taylor series for $t = 0$ of

$$256\frac{\partial a_6}{\partial t} = \eta_1 + t\big(\eta_2 + \tfrac{1}{2}t\eta_3 + \tfrac{1}{6}t^2\eta_4 + \tfrac{1}{24}t^3\eta_5 + \tfrac{1}{120}t^4\eta_6\big)$$

by omitting the nonpositive terms η_1, $\frac{1}{2}t^2\eta_3$ and $\frac{1}{120}t^5\eta_6$ and by giving to t inside the brackets its maximal value $\frac{1}{2}$. This gives the polynomial

$$t\big(\eta_2 + \tfrac{1}{24}\eta_4 + \tfrac{1}{192}\eta_5\big) = t(-2w^4 - 544w^3 + 1544w^2 - 984w + 4480)$$

with real roots $-274.815\ldots$ and $3.083\ldots$, hence negative for $w \in [4, 6.71]$. □

Lemma 26. *Consider the quantity* H^* *(see (6.2)) as a polynomial in* t*. For* $w \in [4, 6.71]$ *it has a single root* $\tau(w) \in [0, \frac{1}{2}]$*:*

$$\tau = \frac{-4w^2 + 5w - 4 + \sqrt{(4w^2 + 19w + 4)(4w^2 - 13w + 4)}}{4(4w - 1)}.$$

One has $H^* < 0$ *(hence* $a_{10} < 0$*) for* $t < \tau$ *and* $H^* > 0$*,* $a_{10} > 0$ *for* $t > \tau$*. The equality* $\tau = 0$ *takes place only for* $w = 4$*.*

P r o o f. The statements about τ are to be checked directly. The signs of H^* follow easily from $H^*|_{t=0} = 2w(4-w) \leqslant 0$ with equality only for $w = 4$. □

Lemma 27. *Consider a_6 as a function in (t, w). Then with τ as defined in Lemma 26 one has $a_6(\tau, w) \geqslant 0$ for $w \in [4, 5]$ with equality only for $w = 4$.*

R e m a r k 5. The lemma implies that at least one of the inequalities $a_6 < 0$ and $a_{10} < 0$ fails in Ω_3^-. Indeed, for $t \geqslant \tau$ this is $a_{10} < 0$ (see Lemma 26), for $t < \tau$ this is $a_6 < 0$ (see Lemmas 25 and 27).

P r o o f of Lemma 27. Set $Y := \sqrt{(4w^2 + 19w + 4)(4w^2 - 13w + 4)}$. One checks numerically that

$$256a_6(\tau, w) = \frac{wC_0 + (4w^2 + 19w + 4)C_1 Y}{(4w-1)^6},$$

where

$$\begin{aligned} C_0 :=\ & 6144w^{10} - 6144w^9 - 224512w^8 + 2284416w^7 - 6369192w^6 \\ & + 6270368w^5 - 3922014w^4 + 1993629w^3 - 860272w^2 \\ & + 234384w - 25728, \\ C_1 :=\ & 384w^7 - 2496w^6 + 632w^5 - 4064w^4 + 4730w^3 - 1355w^2 - 136w + 64. \end{aligned}$$

(With $t = \tau(w)$, a_6 becomes a degree 6 polynomial in Y with coefficients in $\mathbb{R}(t)$. Using the fact that Y^2 is a polynomial in t, one obtains the above form of $256a_6$.) All real roots of C_0 are smaller than 4, so $C_0 > 0$ for $w \in [4, 5]$. The real roots of C_1 equal $-0.192\ldots$, $0.269\ldots$ and $6.455\ldots$, so C_1 is negative for $w \in [4, 5]$. Hence, $wC_0 - (4w^2 + 19w + 4)C_1 Y > 0$ and the inequality $wC_0 + (4w^2 + 19w + 4)C_1 Y > 0$ is equivalent to $w^2 C_0^2 - (4w^2 + 19w + 4)^2 C_1^2 Y^2 > 0$. The left-hand side of the last inequality equals $128(w-4)C_2(4w-1)^6$ with

$$\begin{aligned} C_2 :=\ & 55296w^{12} + 82944w^{11} - 1638912w^{10} + 6310368w^9 - 13847224w^8 \\ & + 10530920w^7 - 8336710w^6 + 5520431w^5 - 2256796w^4 \\ & + 758480w^3 - 378304w^2 + 63488w + 2048. \end{aligned}$$

The largest real root of C_2 equals $3.045\ldots < 4$, so $C_2 > 0$ for $w \in [4, 5]$ and the lemma is proved. This finishes the proof of Lemma 13. □

P r o o f of Lemma 14.

N o t a t i o n 3. If ζ_1, $\zeta_2, \dots, \zeta_k$ are distinct roots of the polynomial P (not necessarily simple), then by $P_{\zeta_1}, P_{\zeta_1,\zeta_2}, \dots, P_{\zeta_1,\zeta_2,\dots,\zeta_k}$ we denote the polynomials $P/(x-\zeta_1)$, $P/(x-\zeta_1)(x-\zeta_2), \dots, P/(x-\zeta_1)(x-\zeta_2)\dots(x-\zeta_k)$.

Denote by u, v, w and t the four distinct roots of P (all nonzero). Hence $P = (x-u)^m(x-v)^n(x-w)^p(x-t)^q$, $m+n+p+q = 11$. For $j = 1, 5$ or 6 we show that the Jacobian (3×4)-matrix $J := (\partial(a_{10}, a_9, a_j)/\partial(u, v, w, t))^t$ (where a_{10}, a_9, a_j are the corresponding coefficients of P expressed as functions of (u, v, w, t)) is of rank 3. (The entry in position $(2, 3)$ of J is $\partial a_9/\partial w$.) Hence, one can vary the values of (u, v, w, t) in such a way that a_{10} and a_j remain fixed (the value of a_{10} being -1) and a_9 takes all possible nearby values. Hence, the polynomial is not a_9-maximal.

The entries of the four columns of J are the coefficients of x^{10}, x^9 and x^j of the polynomials $-mP_u = \partial P/\partial u$, $-nP_v$, $-pP_w$ and $-qP_t$. By abuse of language we say that the linear space $\mathcal{F}$ spanned by the columns of J is generated by the polynomials P_u, P_v, P_w and P_t. As $P_{u,v} = (P_u - P_v)/(v-u)$, $P_{u,w} = (P_u - P_w)/(w-u)$ and $P_{u,t} = (P_u - P_t)/(t-u)$, one can choose as generators of $\mathcal{F}$ the quadruple $(P_u,\ P_{u,v},\ P_{u,w},\ P_{u,t})$; in the same way one can choose $(P_u,\ P_{u,v},\ P_{u,v,w},\ P_{u,v,t})$ or $(P_u,\ P_{u,v},\ P_{u,v,w},\ P_{u,v,w,t})$ (the latter polynomials are of respective degrees 10, 9, 8 and 7). As $(x-t)P_{u,v,w,t} = P_{u,v,w}$, $(x-w)P_{u,v} = P_{u,v,w}$ etc., one can choose as generators the quadruple $\psi :=(x^3P_{u,v,w,t},\ x^2P_{u,v,w,t},\ xP_{u,v,w,t},\ P_{u,v,w,t})$. Set $P_{u,v,w,t} := x^7 + Ax^6 + \dots + G$. The coefficients of x^{10}, x^9 and x^6 of the quadruple ψ define the matrix $J^* := \begin{pmatrix} 1 & 0 & 0 & 0 \\ A & 1 & 0 & 0 \\ D & C & B & A \end{pmatrix}$. Its columns span the space $\mathcal{F}$, hence $\operatorname{rank} J^* = \operatorname{rank} J$. As at least one of the coefficients B and A is nonzero (see Lemma 7), one has $\operatorname{rank} J^* = 3$ and the lemma follows (for the case $j = 6$). In the cases $j = 5$ and $j = 1$, the last row of J^* equals $(EDCB)$ and $(00GF)$, respectively, and in the same way $\operatorname{rank} J^* = 3$. □

P r o o f of Lemma 15. We are using Notation 3 and the method of proof of Lemma 14. Denote by u, v, w, t, h the five distinct real roots of P (not necessarily simple). Thus, using Lemma 10 one can assume that

$$(7.1)\quad P = (x+u)^l(x+v)^m(x+w)^n(x-t)^2(x-h), \quad u, v, w, t, h > 0,\ l+m+n = 8.$$

Set $J := (\partial(a_{10}, a_9, a_j, a_1)/\partial(u, v, w, t, h))^t$, $j = 5$ or $j = 6$. The columns of J span a linear space $\mathcal{L}$ defined by analogy with the space $\mathcal{F}$ from the proof of Lemma 14, but spanned by 4-vector-columns.

Set $P_{u,v,w,t,h} := x^6 + ax^5 + bx^4 + cx^3 + dx^2 + fx + g$. Consider the vector-column

$$(0, 0, 0, 0, 1, a, b, c, d, f, g)^t.$$

The similar vector-columns defined when using the polynomials $x^s P_{u,v,w,t,h}$, $1 \leqslant s \leqslant 4$, instead of $P_{u,v,w,t,h}$ are obtained from this one by successive shifts by one position upward. To obtain generators of $\mathcal{L}$ one has to restrict these vector-columns to the rows corresponding to x^{10} (first), x^9 (second), x^j ($(11-j)$th) and x (tenth row).

Further we assume that $a_1 = 0$. If this is not the case, then at most one of the conditions $a_5 = 0$ and $a_6 = 0$ is fulfilled and the proof of the lemma can be finished by analogy with the proof of Lemma 14.

First consider the case $j = 6$. Hence, the rank of J is the same as the rank of the matrix

$$M := \begin{array}{c} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ a & 1 & 0 & 0 & 0 \\ d & c & b & a & 1 \\ 0 & 0 & 0 & g & f \end{pmatrix} \end{array} \begin{array}{c} x^{10} \\ x^9 \\ x^6 \\ x \end{array}.$$

One has $\operatorname{rank} M = 2 + \operatorname{rank} N$, where $N = \left(\begin{smallmatrix} b & a & 1 \\ 0 & g & f \end{smallmatrix}\right)$. Given that $g \neq 0$, one can have $\operatorname{rank} N < 2$ only if $b = 0$ and $af = g$. We show that the condition $b = 0$ leads to the contradiction that one must have $a_{10} > 0$. We set $u = 1$ to reduce the number of parameters, so we require only the inequality $a_{10} < 0$ to hold but not the equality $a_{10} = -1$. We have to consider the following cases for the values of the triple (l, m, n) (see (7.1)):

$(6, 1, 1)$, $(5, 2, 1)$, $(4, 3, 1)$, $(4, 2, 2)$, $(3, 3, 2)$.

Notice that

$$P_{1,v,w,t,h} = (x+1)^{l-1}(x+v)^{m-1}(x+w)^{n-1}(x-t).$$

Case 1. Triple $(6, 1, 1)$. One has $b = 10 - 5t$, so $t = 2$. For $t = 2$ one has $a_1 = 4vw - 20vwh - 4hv - 4hw$ and the condition $a_1 = 0$ yields $h = h_1 := vw/(5vw + v + w) < \frac{1}{5}$. Notice that $a_{10} = 2 + v + w - h$, which for $h = h_1$ is positive—a contradiction.

Case 2. Triple $(5, 2, 1)$. We obtain $b = 6u^2 + 4uv - 4ut - tv$, hence $t = t_2 := 2(3 + 2v)/(4 + v)$. One has $a_1 = -tv(-vwt - 2vwh + thv + 5thvw + 2thw)$ and for $t = t_2$ the condition $a_1 = 0$ gives

$$h = h_2 := \frac{vw(3 + 2v)}{9v^2w + 3v + 2v^2 + 15vw + 6w} < w.$$

Observe that $a_{10} = 5 + 2v - 2t + (w - h) > 5 + 2v - 2t$. However, for $t = t_2$ one has $5 + 2v - 2t_2 = (8 + 5v + 2v^2)/(4 + v) > 0$.

Case 3. Triple $(4,3,1)$. One gets $b = 3+6v+v^2-3t-2tv = 0$, so $t = t_3 := (3+6v+v^2)/(3+2v)$. As $a_1 = -tv^2(-vwt-2vwh+thv+4thwv+3thw) = 0$, for $t = t_3$ one obtains

$$h = h_3 := \frac{vw(3+6v+v^2)}{24vw+23v^2w+3v+6v^2+v^3+4wv^3+9w} < w.$$

One has $a_{10} = 4+3v-2t+(w-h) > 4+3v-2t$. For $t = t_3$ one checks directly that

$$4+3v-2t_3 = \frac{6+5v+4v^2}{3+2v} > 0, \quad \text{i.e. } a_1 > 0.$$

Case 4. Triple $(4,2,2)$. One has $b = 3+3v+3w+vw-3t-tv-tw$, therefore $t = t_4 := (3+3v+3w+vw)/(3+v+w)$. As $a_1 = -tvw(-vwt-2vwh+4thwv+2thv+2thw)$, for $t = t_4$ it follows from $a_1 = 0$ that

$$h = h_4 := \frac{vw(3+3v+3w+vw)}{2(9vw+6v^2w+6vw^2+2v^2w^2+3v+3v^2+3w+3w^2)},$$

which is less than $\frac{1}{2}w$. One has $a_{10} = 4+2v+2w-2t-h$, which for $h = h_4$ and $t = t_4$ is

$$> 4+2v+\frac{3}{2}w-2t_4 = \frac{1}{2}\frac{12+8v+5w+4v^2+3vw+3w^2}{3+v+w} > 0.$$

Case 5. Triple $(3,3,2)$. One has $b = 1+4v+v^2+2w+2vw-2t-2tv-tw$, therefore

$$t = t_5 := \frac{1+4v+v^2+2w+2vw}{2+2v+w}.$$

As $a_1 = -tv^2w(-vwt-2vwh+3thwv+2thv+3thw)$, the condition $a_1 = 0$ yields

$$h = h_5 := \frac{vw(1+4v+v^2+2w+2vw)}{15vw+15v^2w+10vw^2+3wv^3+6v^2w^2+2v+8v^2+2v^3+3w+6w^2},$$

which is less than $\frac{1}{2}w$. One has $a_{10} = 3+3v+2w-2t-h$, which for $t = t_5$, $h = h_5$ is

$$> 3+3v+\frac{3}{2}w-2t_5 = \frac{1}{2}\frac{8+8v+4w+8v^2+4vw+3w^2}{2+2v+w} > 0.$$

Now consider the case $j = 5$. The matrices M and N equal

$$M := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ a & 1 & 0 & 0 & 0 \\ f & d & c & b & a \\ 0 & 0 & 0 & g & f \end{pmatrix}, \quad N = \begin{pmatrix} c & b & a \\ 0 & g & f \end{pmatrix},$$

respectively. One has rank $N < 2$ only for $c = 0$ and $bf = ag$. Similarly to the case $j = 6$ we show that the equality $c = 0$ leads to the contradiction $a_{10} > 0$. We define the cases 1–5 in the same way as above.

Case 1. One has $c = 10 - 10t$, so $t = 1$. As $a_1 = vw - 4vwh - hv - hw$, the equality $a_1 = 0$ implies $h = h^1 := vw/(4vw + v + w) < \frac{1}{4}$. One has $a_{10} = 4 + v + w - h$, which for $h = h^1$ is positive—a contradiction.

Case 2. One gets $c = -2u(-2u^2 - 3uv + 3ut + 2tv)$, so $c = 0$ implies $t = t^2 := (2 + 3v)/(3 + 2v)$. From $a_1 = -kv(-vwk - 2vwh + thv + 5thwv + 2thw) = 0$ one gets for $t = t^2$

$$h = h^2 := \frac{vw(2 + 3v)}{11v^2w + 2v + 3v^2 + 10vw + 4w} < w.$$

From $a_{10} = 5 + 2v + w - 2t - h$ one sees that for $h = h^2$, $t = t^2$ it is true that

$$a_{10} > 5 + 2v - 2t^2 = \frac{11 + 10v + 4v^2}{3 + 2v} > 0.$$

Case 3. One obtains $c = 1 + 6v + 3v^2 - 3t - 6tv - v^2t$, so $t = t^3 := (1 + 6v + 3v^2)/(3 + 6v + v^2)$. The condition $a_1 = -tv^2(-vwt - 2vwh + thv + 4thwv + 3thw) = 0$ with $t = t^3$ implies

$$h = h^3 := \frac{vw(1 + 6v + 3v^2)}{16vw + 21v^2w + 10wv^3 + v + 6v^2 + 3v^3 + 3w} < w.$$

But then from $a_{10} = 4 + 3v + w - 2t - h$ with $t = t^3$, $h = h^3$ it follows

$$a_{10} > 4 + 3v - 2t^3 = \frac{10 + 21v + 16v^2 + 3v^3}{3 + 6v + v^2} > 0.$$

Case 4. One has $c = 1 + 3v + 3w + 3vw - 3t - 3tv - 3tw - vwt$, so $c = 0$ implies $t = t^4 := (1 + 3v + 3w + 3vw)/(3 + 3v + 3w + vw)$. For $t = t^4$ the condition $a_1 = -tvw(-vwt - 2vwh + 4thwv + 2thv + 2thw) = 0$ implies

$$h = h^4 := \frac{1}{2}\frac{vw(1 + 3v + 3w + 3vw)}{5vw + 6v^2w + 6vw^2 + 5v^2w^2 + v + 3v^2 + w + 3w^2},$$

which is less than $\frac{1}{2}w$. Thus, $a_{10} = 4 + 2v + 2w - 2t - h$ with $t = t^4$, $h = h^4$ implies

$$\begin{aligned} a_{10} &> 4 + 2v + \frac{3}{2}w - 2t^4 \\ &= \frac{20 + 24v + 21w + 17vw + 12v^2 + 4v^2w + 9w^2 + 3vw^2}{2(3 + 3v + 3w + vw)} > 0. \end{aligned}$$

Case 5. We get $c = 2v + 2v^2 + w + 4vw + v^2w - t - 4tv - v^2t - 2tw - 2vwt$ and $c = 0$ implies

$$t = t^5 := \frac{2v + 2v^2 + w + 4vw + v^2w}{1 + 4v + v^2 + 2w + 2vw}.$$

For $t = t^5$ the equalities $a_1 = -tv^2w(-vwt - 2vwh + 3thwv + 2thv + 3thw) = 0$ yield

$$h = h^5 := \frac{vw(2v + 2v^2 + w + 4vw + v^2w)}{6vw + 12v^2w + 6wv^3 + 11vw^2 + 11v^2w^2 + 3w^2v^3 + 4v^2 + 4v^3 + 3w^2}$$

which is less than $\frac{1}{2}w$. Hence, $a_{10} = 3 + 3v + 2w - 2t - h$ with $t = t^5$, $h = h^5$ implies

$$\begin{aligned} a_{10} &> 3 + 3v + \frac{3}{2}w - 2t^5 \\ &= \frac{6 + 22v + 22v^2 + 11w + 20vw + 6v^3 + 11v^2w + 6w^2 + 6vw^2}{2(1 + 4v + v^2 + 2w + 2vw)} > 0. \end{aligned}$$

$\square$

P r o o f of Lemma 16. We use the same ideas and notation as in the proof of Lemma 15. Six of the six or more real roots of P are denoted by (u, v, w, t, h, q). The space $\mathcal{L}$ is defined by analogy with the one of the proof of Lemma 15. The Jacobian matrix J is of the form

$$J := \left(\frac{\partial(a_{10}, a_9, a_j, a_1)}{\partial(u, v, w, t, h, q)}\right)^t.$$

Set $P_{u,v,w,t,h,q} := x^5 + ax^4 + bx^3 + cx^2 + dx + f$ and consider the vector-column

$$(0, 0, 0, 0, 0, 1, a, b, c, d, f)^t.$$

Its successive shifts by one position upward correspond to the polynomials $x^s P_{u,v,w,t,h,q}$, $s \leqslant 5$. In the case $j = 6$, the matrices M and N look like this:

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ a & 1 & 0 & 0 & 0 & 0 \\ d & c & b & a & 1 & 0 \\ 0 & 0 & 0 & 0 & f & d \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} b & a & 1 & 0 \\ 0 & 0 & f & d \end{pmatrix}.$$

One has $\operatorname{rank} M = 2 + \operatorname{rank} N$ and $\operatorname{rank} N = 2$ because $f \neq 0$ and at least one of the two coefficients b and a is nonzero (Lemma 7). Hence, $\operatorname{rank} M = 4$ and the lemma is proved by analogy with Lemmas 14 and 15. In the case $j = 5$ the third row of M equals $(fdcba1)$, the first row of N equals $(cba1)$, at least one of the two coefficients c and b is nonzero and again $\operatorname{rank} M = 4$. $\square$

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Author's address: *Vladimir Petrov Kostov*, Université Côte d'Azur, Le Centre national de la recherche scientifique (CNRS), Laboratoire Jean-Alexandre Dieudonné (LJAD), 28 Avenue de Valrose, 06108 Nice CEDEX 2, France, e-mail: `vladimir.kostov@unice.fr`.