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FINITE DISTORTION FUNCTIONS AND DOUGLAS-DIRICHLET FUNCTIONALS

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Abstract. In this paper, we estimate the Douglas-Dirichlet functionals of harmonic mappings, namely Euclidean harmonic mapping and flat harmonic mapping, by using the extremal dilatation of finite distortion functions with given boundary value on the unit circle. In addition, $\bar{\partial}$-Dirichlet functionals of harmonic mappings are also investigated.

Keywords: Douglas-Dirichlet functional; ϱ-harmonic mapping; finite distortion functions; extremal quasiconformal mapping; Dirichlet's principle

MSC 2010: 30C62, 30C70, 31A05

1. Introduction

Let Δ be the unit disk and $\partial\Delta$ be the unit circle. If H is a continuous homeomorphism mapping from $\partial\Delta$ onto $\partial\Delta$, consider the family of quasiconformal mappings

$$Q(\mathrm{H}) = \{f\colon\ f\colon\ \Delta \to \Delta \text{ is a quasiconformal mapping},\ f|_{\partial\Delta} = \mathrm{H}(\mathrm{e}^{\mathrm{i}\theta}),\ \theta \in [0, 2\pi)\}.$$

Hence, the extremal complex dilatation K_{H} of $Q(\mathrm{H})$ can be defined by

$$K_{\mathrm{H}} = \inf_{f \in Q(\mathrm{H})} K[f], \quad K[f] = \operatorname*{ess\,sup}_{z\in\Delta} K(z, f) = \operatorname*{ess\,sup}_{z\in\Delta} \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_z|}.$$

Thus $f_0 \in Q(\mathrm{H})$ is said to be an extremal quasiconformal mapping if f_0 satisfies $K[f_0] = K_{\mathrm{H}}$. It is well known that extremal quasiconformal mappings play an important role in the theories of quasiconformal mapping and Teichmüller space.

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Quasiconformal mapping is a special case of finite distortion function which is defined as follows. Let Ω and Ω' be two bounded domains of $\mathbb{C}$. Then a sense-preserving homeomorphism mapping $f\colon \Omega \to \Omega'$ is said to be a finite distortion function if $f \in \mathbb{W}^{1,1}_{\mathrm{loc}}(\Omega, \Omega')$ and there is a measure function $K(z) \in [1, \infty)$ such that

$$|Df|^2 \leqslant K(z)J(z, f),$$

where

$$|Df| = |f_z| + |f_{\bar z}|, \;\; J(z, f) = |f_z|^2 - |f_{\bar z}|^2.$$

If H is a continuous homeomorphism mapping from $\partial\Delta$ onto $\partial\Delta$ or more generally $\mathrm{H} \in L^2(\partial\Delta)$, it is natural to define the family of finite distortion functions which have the same boundary value on $\partial\Delta$ as

$$\mathcal{FD}(\mathrm{H}) = \{f\colon\; f\colon\; \Delta \to \Delta \text{ is a finite distortion function, } f|_{\partial\Delta} = \mathrm{H}(\mathrm{e}^{\mathrm{i}\theta}),\; \theta \in [0, 2\pi)\}.$$

Note that the boundary function of $f \in \mathcal{FD}(\mathrm{H})$ belongs to $L^2(\partial\Delta)$ in the trace sense here. Denote the outer distortion function $\mathbb{K}(z, f)$ as $\|Df\|^2/J(z, f)$ when $J(z, f) > 0$, and $\mathbb{K}(z, f) = 1$ when $J(z, f) = 0$, where $\|Df\| = \sqrt{|f_z|^2 + |f_{\bar z}|^2}$ is a normalized Hilbert-Schmidt norm. Analogously to the extremal quasiconformal mapping, the extremal finite distortion function is the function $f_0 \in \mathcal{FD}(\mathrm{H})$ whose dilatation satisfies $\mathbb{K}[f_0] = \mathbb{K}_{\mathrm{H}}$, here

$$\mathbb{K}_{\mathrm{H}} = \inf_{f \in \mathcal{FD}(\mathrm{H})} \mathbb{K}[f], \quad \mathbb{K}[f] = \operatorname*{ess\,sup}_{z \in \Delta} \mathbb{K}(z, f) = \operatorname*{ess\,sup}_{z \in \Delta} \frac{|f_z|^2 + |f_{\bar z}|^2}{|f_z|^2 - |f_{\bar z}|^2}.$$

Since

$$\mathbb{K}(z, f) = \frac{1}{2}\Big(K(z, f) + \frac{1}{K(z, f)}\Big), \quad z \in \Delta$$

is a convex function and therefore

$$\mathbb{K}[f] = \frac{1}{2}\Big(K[f] + \frac{1}{K[f]}\Big), \quad \text{when } f \in \mathcal{FD}(\mathrm{H}).$$

The extensive study on the finite distortion function was initiated in [2], [1], [7].

Given a smooth metric $\varrho(z)|\mathrm{d}z|^2$ on Ω' with $\|\varrho\|_1 = \iint_{\Omega'} |\varrho| \,\mathrm{d}x\,\mathrm{d}y < \infty$, suppose that $\omega = f \in C^2\colon\; \Omega \to \Omega'$. Then the Douglas-Dirichlet functional is defined as

$$D_\varrho[f] = \iint_\Omega \varrho(f(z))(|f_z|^2 + |f_{\bar z}|^2)\,\mathrm{d}x\,\mathrm{d}y, \tag{1.1}$$

for $z = x + \mathrm{i}y \in \Omega$. It is well known that the critical point f of (1.1) satisfies the following Euler-Lagrange equation

$$f_{z\bar z}(z) + (\log \varrho(\omega))_\omega f_z(z) f_{\bar z}(z) = 0, \quad \omega = f(z). \tag{1.2}$$

Then f is said to be a harmonic mapping with respect to ϱ (or briefly ϱ-harmonic mapping), if $\omega = f(z) \in C^2$ satisfies the Euler-Lagrange equation on Ω. Especially, when $\varrho \equiv 1$, f becomes a Euclidean harmonic mapping; when $\varrho = 1/|\omega|^2$, f just corresponds to a nonvanishing logharmonic mapping; when $\varrho = |\varphi|$ for a nonvanishing analytic function φ on Ω', then f is said to be a flat harmonic mapping whose Gaussian curvature is zero. For further research on Euclidean harmonic mapping and ϱ-harmonic mapping readers can refer to [3], [4], [8], [12], [15] for more details.

In 1985, Reich [13] first estimated the upper bound of $D_\varrho[f]$ for $f \in Q(\mathrm{H})$ as

$$\sup_{\varrho\in\mathcal{P}} \inf_{f\in Q(\mathrm{H})} \frac{D_\varrho[f]}{\|\varrho\|_1} = \frac{1}{2}\Big(K_{\mathrm{H}} + \frac{1}{K_{\mathrm{H}}}\Big),$$

where $\mathcal{P}$ is the set of all measure functions ϱ which satisfy the conditions that $\varrho(z) \geqslant 0$ and $\|\varrho\|_1 < \infty$. Consider the class of holomorphic functions

$$\mathrm{Hol}_2(\Delta) = \{\varphi\colon\ \Delta \to \mathbb{C} \text{ is a holomorphic function and } \|\varphi'^2\|_1 < \infty\},$$

and let $P(\mathrm{H})$ be the Poisson integral of boundary function H on $\partial\Delta$. Denote the Douglas-Dirichlet functional $D[f]$ for short when $\varrho \equiv 1$. Reich also proved the following statement:

Theorem A ([14]). *For any $\varphi \in \mathrm{Hol}_2(\Delta)$, it holds that*

$$\frac{D[P(\varphi\circ \mathrm{H})]}{D[\varphi]} \leqslant \frac{1}{2}\Big(K_{\mathrm{H}} + \frac{1}{K_{\mathrm{H}}}\Big). \tag{1.3}$$

In addition, when $K_{\mathrm{H}} > 1$, the equality in (1.3) *holds if and only if there exists $\psi_0 \in \mathrm{Hol}_2(\Delta)$ and an extremal quasiconformal mapping $f_0 \in Q(\mathrm{H})$ such that f_0 satisfies*

$$\frac{f_{0\bar z}}{f_{0z}} = k_{\mathrm{H}}\frac{\overline{\psi_0'}}{\psi_0'}, \quad k_{\mathrm{H}} = \frac{K_{\mathrm{H}} - 1}{K_{\mathrm{H}} + 1}. \tag{1.4}$$

Define the family of (Euclidean) harmonic mappings

$$D(\Delta) = \{\varphi\colon\ \Delta \to \mathbb{C} \text{ is a harmonic mapping and } D[\varphi] < \infty\}.$$

In 1999, a more extensive conclusion of Theorem A is obtained in [16] as follows.

Theorem B ([16]). *For any $\varphi \in D(\Delta)$, we have*

$$\frac{1}{K_{\mathrm{H}}} \leqslant \frac{D[P(\varphi \circ \mathrm{H})]}{D[\varphi]} \leqslant K_{\mathrm{H}}. \tag{1.5}$$

In addition, (1.5) *is accurate and the extremal function is the same as in* (1.4).

In 2016, Feng in [5] used the quantity of $\mathbb{K}_{\mathrm{H}}$ instead of K_{H} to estimate the Douglas-Dirichlet functional as follows.

Theorem C ([5]). *For any $\varphi \in \mathrm{Hol}_2(\Delta)$, it holds that*

$$\frac{1}{\mathbb{K}_{\mathrm{H}}} \leqslant \frac{D[P(\varphi \circ \mathrm{H})]}{D[\varphi]} \leqslant \mathbb{K}_{\mathrm{H}} \tag{1.6}$$

and the equality in (1.6) *can be attained the same way as in* (1.4).

Theorem D ([5]). *For any $\varphi \in D(\Delta)$, it holds that*

$$\frac{D[P(\varphi \circ \mathrm{H})]}{D[\varphi]} \leqslant \mathbb{K}_{\mathrm{H}} + \sqrt{\mathbb{K}_{\mathrm{H}}^2 - 1}. \tag{1.7}$$

Meanwhile, (1.7) *is also accurate the same way as* (1.4).

The following Dirichlet's principle (see [10], [11]) is crucial in the estimate of Douglas-Dirichlet functional.

Theorem E (Dirichlet's principle). *Suppose that g is continuous on $\bar{\Delta}$ and has the first partial derivatives which are continuous on Δ. Let f be a Euclidean harmonic mapping on Δ which is continuous on $\partial\Delta$. If $f|_{\partial\Delta} = g|_{\partial\Delta}$ and $D[g] < \infty$, then $D[f] \leqslant D[g]$, where the inequality equals if and only if $f \equiv g$ on Δ.*

In addition, if $f \in Q(\mathrm{H})$, then $D_\varrho[f] \leqslant D_\varrho[g] < \infty$ holds for all $g \in Q(\mathrm{H})$ if and only if f is a ϱ-harmonic mapping. There exists a similar result on $\bar{\partial}$-Dirichlet functional, see [17], [18], [19] and [20] for more details. Analogously to Douglas-Dirichlet functional in (1.1), $\bar{\partial}$-Dirichlet functional is given in [9] which is defined as

$$D'_\varrho[f] = \iint_\Omega \varrho(f(z))|f_{\bar{z}}(z)|^2 \,\mathrm{d}x\,\mathrm{d}y. \tag{1.8}$$

The critical point f of (1.8) also satisfies the Euler-Lagrange equation (1.2), but its Hopf differential $\varrho(f) f_z \overline{f_{\bar{z}}}\,\mathrm{d}z^2$ is not necessarily holomorphic. Using the method shown in [10], we find that Dirichlet's principle also holds for $\bar{\partial}$-Dirichlet functional of a Euclidean harmonic mapping, which refers to Theorem 3.1.

The paper is organized as follows. Firstly, the concrete lower bound of (1.7) is given in Theorem 2.1. Then Douglas-Dirichlet functionals $D_\varrho[f]$ for some special ϱ-harmonic mappings f are estimated by using extremal dilatation $\mathbb{K}_{\mathrm{H}}$ of finite distortion functions $\mathcal{FD}(\mathrm{H})$, refer to Theorems 2.2 and 2.3. In Section 3, $\bar{\partial}$-Dirichlet functionals $D'_\varrho[f]$ of Euclidean harmonic mapping, flat harmonic mapping (and $1/|\omega|^4$-harmonic mapping) are also investigated. It is not difficult to verify that $D_\varrho[f]$ and $D'_\varrho[f]$ satisfy the equation

$$D'_\varrho[f] = \frac{1}{2}D_\varrho[f] - \frac{1}{2}\iint_\Delta \varrho(f(z))(|f_z|^2 - |f_{\bar{z}}|^2)\,\mathrm{d}x\,\mathrm{d}y.$$

Thus some results on the estimation of $D'_\varrho[f]$ can be deduced from the corresponding theorems in Section 2, Theorem B and Theorem C directly.

Notes. It is obvious that quasiconformal mappings belong to the family of finite distortion functions and the existence of extremal quasiconformal mappings has been intensively studied by many scholars over the past decades. But there are few studies on the existence of extremal finite distortion functions among the functions which are not quasiconformal mappings. The purpose of this paper is to estimate the Douglas-Dirichlet functions of φ-harmonic mappings, thus we do not focus on the difference between the quantities $\mathbb{K}_{\mathrm{H}}$ and K_{H}.

2. Estimations of Douglas-Dirichlet functions

In this section, we firstly give the accuracy of lower bound estimation of $D[P(\varphi\circ\mathrm{H})]$ for any $\varphi \in D(\Delta)$ in Theorem D as follows.

Theorem 2.1. *For any $\varphi(z) \in D(\Delta)$, the inequalities*

$$\frac{1}{\mathbb{K}_{\mathrm{H}} + \sqrt{\mathbb{K}_{\mathrm{H}}^2 - 1}} \leqslant \frac{D[P(\varphi\circ\mathrm{H})]}{D[\varphi]} \leqslant \mathbb{K}_{\mathrm{H}} + \sqrt{\mathbb{K}_{\mathrm{H}}^2 - 1}, \tag{2.1}$$

hold true. Especially, when $K_{\mathrm{H}} > 1$, the equalities in (2.1) hold if and only if there exist $\varphi_0 \in \mathrm{Hol}_2(\Delta)$ and an extremal function $f_0 \in Q(\mathrm{H})$ such that the Beltrami coefficient of f_0 satisfies

$$\frac{f_{0\bar{z}}(z)}{f_{0z}(z)} = k_{\mathrm{H}}\frac{\overline{\varphi_0'(z)}}{\varphi_0'(z)}, \quad k_{\mathrm{H}} = \frac{K_{\mathrm{H}} - 1}{K_{\mathrm{H}} + 1}.$$

P r o o f. Suppose that $\omega = f(z) \in \mathcal{FD}(\mathrm{H})$ is an extremal function. For any $\varphi \in D(\Delta)$, consider the composite function $F(z) = \varphi \circ f(z)$. Then we have

$$F_z(z) = \varphi_\omega(\omega) f_z(z) + \varphi_{\overline{\omega}}(\omega)\overline{f_{\bar z}(z)}, \quad F_{\bar z}(z) = \varphi_\omega(\omega) f_{\bar z}(z) + \varphi_{\overline{\omega}}(\omega)\overline{f_z(z)}. \tag{2.2}$$

Therefore,

$$\begin{aligned} |F_z(z)|^2 + |F_{\bar z}(z)|^2 &= |\varphi_\omega f_z + \varphi_{\overline{\omega}}\overline{f_{\bar z}}|^2 + |\varphi_\omega f_{\bar z} + \varphi_{\overline{\omega}}\overline{f_z}|^2 \\ &= (|\varphi_\omega|^2 + |\varphi_{\overline{\omega}}|^2)(|f_z|^2 + |f_{\bar z}|^2) + 4\Re\varphi_\omega\overline{\varphi_{\overline{\omega}}} f_z f_{\bar z} \\ &\leqslant (|\varphi_\omega|^2 + |\varphi_{\overline{\omega}}|^2)[|f_z|^2 + |f_{\bar z}|^2 \\ &\quad + \sqrt{(|f_z|^2 + |f_{\bar z}|^2)^2 - (|f_z|^2 - |f_{\bar z}|^2)^2}] \\ &= J(z,f)(|\varphi_\omega|^2 + |\varphi_{\overline{\omega}}|^2)\big(\mathbb{K}(z,f) + \sqrt{\mathbb{K}^2(z,f) - 1}\big). \end{aligned} \tag{2.3}$$

Hence the upper and lower bounds of the Douglas-Dirichlet functional of $F(z)$ can be deduced from relation (2.3) as follows.

$$\begin{aligned} D[F] &= \iint_\Delta (|F_z(z)|^2 + |F_{\bar z}(z)|^2)\,\mathrm{d}x\,\mathrm{d}y \\ &\leqslant \iint_\Delta (|\varphi_\omega|^2 + |\varphi_{\overline{\omega}}|^2) J(z,f)\big(\mathbb{K}(z,f) + \sqrt{\mathbb{K}^2(z,f) - 1}\big)\,\mathrm{d}x\,\mathrm{d}y \\ &\leqslant \Big(\mathbb{K}_\mathrm{H} + \sqrt{\mathbb{K}_\mathrm{H}^2 - 1}\Big) \iint_\Delta (|\varphi_\omega|^2 + |\varphi_{\overline{\omega}}|^2)\,\mathrm{d}u\,\mathrm{d}v \\ &= \Big(\mathbb{K}_\mathrm{H} + \sqrt{\mathbb{K}_\mathrm{H}^2 - 1}\Big) D[\varphi] \end{aligned} \tag{2.4}$$

with $z = x + \mathrm{i}y \in \Delta$ and $\omega = u + \mathrm{i}v \in \Delta$. Especially, when $\varphi = P(\varphi \circ \mathrm{H})$, we have

$$D[F] \leqslant \big(\mathbb{K}_\mathbb{H} + \sqrt{\mathbb{K}^2 - 1}\big) D[P(\varphi \circ \mathrm{H})]. \tag{2.5}$$

Applying Theorem E for F and $\varphi \in D(\Delta)$, we get that

$$D[P(\varphi \circ \mathrm{H})] \leqslant D[F] \leqslant \Big(\mathbb{K}_\mathrm{H} + \sqrt{\mathbb{K}_\mathrm{H}^2 - 1}\Big) D[\varphi]$$

and

$$D[\varphi] \leqslant D[F] \leqslant \Big(\mathbb{K}_\mathrm{H} + \sqrt{\mathbb{K}_\mathrm{H}^2 - 1}\Big) D[P(\varphi \circ \mathrm{H})],$$

from (2.4) and (2.5), respectively.

Since the equality in (2.3) holds if and only if $|\varphi_\omega| = |\varphi_{\overline{\omega}}|$ and $\mathbb{K}(z,f) = \mathbb{K}_\mathrm{H}$, by the proof of Theorem 1 in [16], the accuracy of the lower bound estimation in (2.1) can be obtained. The proof of the accuracy is omitted here. $\square$

Remark. We should illustrate that Theorem 2.1 is coincident with Theorem B by the equality $\mathbb{K}_{\mathrm{H}} = \frac{1}{2}(K_{\mathrm{H}} + 1/K_{\mathrm{H}})$ when $f \in Q(\mathrm{H})$.

Notice that the critical point f of the minimal Douglas-Dirichlet functional, i.e., $D[f] \leqslant D[F]$ for all $F \in Q(\mathrm{H})$, can be generated by the Poisson extension of a given boundary value H, that is, $f = P(\mathrm{H})$. But that is not suitable for ϱ-harmonic mappings. Furthermore, Dirichlet's principle is not necessarily established for ϱ-harmonic mappings. Thus it is meaningful to estimate the Douglas-Dirichlet functional $D_\varrho[f]$ of ϱ-harmonic mappings f. Based on the connections between Euclidean harmonic mapping and φ-harmonic mapping f, Douglas-Dirichlet functions $D_\varrho[f]$ have a similar estimation with $D[f]$ as follows.

Theorem 2.2. *For a univalent function $\varphi(z) \in \mathrm{Hol}_2(\Delta)$, the inequalities*

$$\frac{1}{\mathbb{K}_{\mathrm{H}} + \sqrt{\mathbb{K}_{\mathrm{H}}^2 - 1}} \leqslant \frac{D_\varrho[\varphi \circ P(\mathrm{H})]}{D_\varrho[\varphi]} \leqslant \mathbb{K}_{\mathrm{H}} \tag{2.6}$$

hold for $\varrho(z) = 1/|\varphi'(z)|^2$ and $D_\varrho[\varphi] < \infty$.

Proof. Suppose that $\omega = f(z) \in \mathcal{FD}(\mathrm{H})$ is an extremal function, that is, $f\colon \Delta \to \Delta$ is a finite distortion function which satisfies $f|_{\partial\Delta} = \mathrm{H}$ and $\mathbb{K}[f] = \mathbb{K}_{\mathrm{H}}$. For any given $\varphi \in \mathrm{Hol}_2(\Delta)$, consider the composite function $F(z) = \varphi \circ f(z)$. Then we have

$$F_z(z) = \varphi'(\omega) \cdot f_z(z), \quad F_{\bar z}(z) = \varphi'(\omega) \cdot f_{\bar z}(z).$$

Therefore,

$$|F_z(z)|^2 + |F_{\bar z}(z)|^2 = |\varphi'(\omega)|^2(|f_z(z)|^2 + |f_{\bar z}(z)|^2).$$

Integrate on both sides of the above equality to obtain that

$$\begin{aligned} D_\varrho[F] &= \iint_\Delta \varrho(\varphi(\omega))(|F_z(z)|^2 + |F_{\bar z}(z)|^2)\,\mathrm{d}x\,\mathrm{d}y \\ &= \iint_\Delta \varrho(\varphi(\omega))|\varphi'(\omega)|^2 \frac{|f_z(z)|^2 + |f_{\bar z}(z)|^2}{|f_z(z)|^2 - |f_{\bar z}(z)|^2} J(z,f)\,\mathrm{d}x\,\mathrm{d}y \\ &\leqslant \mathbb{K}_{\mathrm{H}} \iint_\Delta \varrho(\varphi(\omega))|\varphi'(\omega)|^2\,\mathrm{d}u\,\mathrm{d}v = \mathbb{K}_{\mathrm{H}} D_\varrho[\varphi], \end{aligned}$$

with $z = x + \mathrm{i}y$ and $\omega = u + \mathrm{i}v$. Especially, when φ is a conformal function and $f = P[\mathrm{H}]$, then $F = \varphi \circ f$ is a φ-harmonic mapping (that is, flat harmonic mapping) due to Lemma 3.6 in [8].

On the other hand, since

$$\begin{aligned} D[f] &= \iint_\Delta |(\varphi^{-1})'|^2(|F_z|^2 + |F_{\bar z}|^2)\,\mathrm{d}x\,\mathrm{d}y \\ &= \iint_\Delta \varrho(F)(|F_z|^2 + |F_{\bar z}|^2)\,\mathrm{d}x\,\mathrm{d}y = D_\varrho[F], \end{aligned}$$

applying Dirichlet principle (Theorem E), we obtain that

$$D_\varrho[\varphi\circ P(\mathrm{H})]\leqslant D_\varrho[F]\leqslant \mathbb{K}_{\mathrm{H}}D_\varrho[\varphi]. \tag{2.7}$$

Then we will estimate the lower bound of $D_\varrho[\varphi\circ P(\mathrm{H})]$. For any given $f\in\mathcal{FD}(\mathrm{H})$ and $\varphi\in\mathrm{Hol}_2(\Delta)$, let $F=\varphi\circ f$, which implies that $\varphi=F\circ f^{-1}$. If $\omega=f(z)$, thus we have

$$(F(f^{-1}))_\omega=F_z(f^{-1})_\omega+F_{\bar z}\overline{(f^{-1})_{\overline{\omega}}},\quad (F(f^{-1}))_{\overline{\omega}}=F_z(f^{-1})_{\overline{\omega}}+F_{\bar z}\overline{(f^{-1})_{\omega}}.$$

Thus, by the fact that $\mathbb{K}_{\mathrm{H}^{-1}}=\mathbb{K}_{\mathrm{H}}$, we have

$$\begin{aligned}
D_\varrho[F(f^{-1})]&=\iint_\Delta \varrho(F(f^{-1}))[|(F(f^{-1}))_\omega|^2+|(F(f^{-1}))_{\overline{\omega}}|^2]\,\mathrm{d}u\,\mathrm{d}v\\
&\leqslant \iint_\Delta \varrho(F(f^{-1}))(|F_z|^2+|F_{\bar z}|^2)[|(f^{-1})_\omega|^2+|(f^{-1})_{\overline{\omega}}|^2\\
&\quad+\sqrt{(|(f^{-1})_\omega|^2+|(f^{-1})_{\overline{\omega}}|^2)^2-(|(f^{-1})_\omega|^2-|(f^{-1})_{\overline{\omega}}|^2)^2}]\,\mathrm{d}u\,\mathrm{d}v\\
&\leqslant \Big(\mathbb{K}_{\mathrm{H}}+\sqrt{\mathbb{K}_{\mathrm{H}}^2-1}\Big)\iint_\Delta \varrho(F(z))(|F_z|^2+|F_{\bar z}|^2)\,\mathrm{d}x\,\mathrm{d}y\\
&=\Big(\mathbb{K}_{\mathrm{H}}+\sqrt{\mathbb{K}_{\mathrm{H}}^2-1}\Big)D_\varrho[F]
\end{aligned}$$

for $z=x+\mathrm{i}y\in\Delta$ and $\omega=u+\mathrm{i}v$. Since $D_\varrho[F(f^{-1})]=D_\varrho[\varphi]$, thus we get that

$$D_\varrho[F]\geqslant\frac{1}{\mathbb{K}_{\mathrm{H}}+\sqrt{\mathbb{K}_{\mathrm{H}}^2-1}}D_\varrho[\varphi]$$

for all $f\in\mathcal{FD}(\mathrm{H})$. Especially, when $f=P(\mathrm{H})$, the first inequality of (2.6) is obtained. Therefore, the proof of this theorem is complete. □

Next, we shall verify the above estimates by analyzing the corresponding problem of a particular class of functions of φ-harmonic mappings. Let $A(1,r)=\{z\in\mathbb{C}\colon\ 1\leqslant|z|\leqslant r\}$ be an annulus. It was proved that $R/f\colon\ A(1,r)\to A(1,R)$ is a Euclidean harmonic mapping if $\omega=f(z)$ is a $1/|\omega|^4$-harmonic mapping from annulus $A(1,r)$ onto $A(1,R)$ in [6]. Let

$$\mathrm{Hol}_2(\Delta,\Delta)=\{\varphi\colon\ \Delta\to\Delta \text{ is an analytic function and } \|\varphi'^2\|_1<\infty\}.$$

Analogously to the estimation of Douglas-Dirichlet functionals in Theorem C and Theorem 2.2, we conclude the following result, which is a special case of Theorem 2.2.

Theorem 2.3. *For any $\varphi\in\mathrm{Hol}_2(\Delta,\Delta)$, it holds that*

$$\frac{1}{\mathbb{K}_{\mathrm{H}}+\sqrt{\mathbb{K}_{\mathrm{H}}^2-1}}\leqslant\frac{D_\varrho[1/(2+P(\varphi\circ\mathrm{H}))]}{D[\varphi]}\leqslant\mathbb{K}_{\mathrm{H}} \tag{2.8}$$

for $\varrho(\omega)=1/|\omega|^4$, where $\omega=1/(2+P(\varphi\circ\mathrm{H}))$.

P r o o f. Suppose that $\omega = f(z) \in \mathcal{FD}(\mathrm{H})$ is an extremal function, and for any $\varphi \in \mathrm{Hol}_2(\Delta,\ \Delta)$, consider the composite function $F(z) = 1/(2+\varphi\circ f(z))$. Then we have

$$F_z = -\frac{\varphi'(f)f_z}{(2+\varphi\circ f)^2}, \quad F_{\bar z} = -\frac{\varphi'(f)f_{\bar z}}{(2+\varphi\circ f)^2},$$

and

$$|F_z|^2 + |F_{\bar z}|^2 = \frac{|\varphi'(f)|^2}{|2+\varphi\circ f|^4}(|f_z|^2 + |f_{\bar z}|^2).$$

Integrate on Δ to obtain that

$$\begin{aligned} D_\varrho[F] &= \iint_\Delta \varrho(F)(|F_z|^2 + |F_{\bar z}|^2)\,\mathrm{d}x\,\mathrm{d}y \\ &= \iint_\Delta \varrho(F)\frac{|\varphi'(f)|^2}{|2+\varphi\circ f|^4}(|f_z|^2 + |f_{\bar z}|^2)\,\mathrm{d}x\,\mathrm{d}y \\ &= \iint_\Delta \varrho(F)\frac{|\varphi'(f)|^2}{|2+\varphi\circ f|^4}\frac{|f_z|^2 + |f_{\bar z}|^2}{|f_z|^2 - |f_{\bar z}|^2}J(z,f)\,\mathrm{d}x\,\mathrm{d}y \\ &\leqslant \mathbb{K}_\mathrm{H}\iint_\Delta |\varphi'(\omega)|^2\,\mathrm{d}u\,\mathrm{d}v = \mathbb{K}_\mathrm{H} D[\varphi] \end{aligned} \tag{2.9}$$

for $\varrho(\omega) = 1/|\omega|^4$, $z = x + \mathrm{i}y$ and $\omega = f(z) = u + \mathrm{i}v$. According to the fact that

$$D[\varphi\circ f] = \iint_\Delta \frac{1}{|F|^4}(|F_z|^2 + |F_{\bar z}|^2)\,\mathrm{d}x\,\mathrm{d}y = D_\varrho[F],$$

Dirichlet's principle directly deduce to

$$D_\varrho\Big[\frac{1}{2+P(\varphi\circ\mathrm{H})}\Big] = D[P(\varphi\circ\mathrm{H})] \leqslant D[\varphi\circ f] = D_\varrho[F],$$

that is,

$$D_\varrho\Big[\frac{1}{2+P(\varphi\circ\mathrm{H})}\Big] \leqslant D_\varrho[F] \leqslant \mathbb{K}_\mathrm{H} D[\varphi].$$

Consider the extremal mapping in $\mathcal{FD}(\mathrm{H}^{-1})$, analogously to the proof of the first inequality in Theorem 2.2, we obtain the lower bound estimation in (2.8). The proof of Theorem 2.3 is complete. $\square$

3. Estimations on $\bar{\partial}$-Dirichlet functionals

Recall that Li Zhong in [9] introduced $\bar{\partial}$-Dirichlet functional, which is similar to Douglas-Dirichlet functional, as

$$D'_{\varrho}[f] = \iint_{\Omega} \varrho(f(z))|f_{\bar{z}}(z)|^2 \,\mathrm{d}x\,\mathrm{d}y \tag{3.1}$$

for any C^1-diffeomorphism $f\colon \Omega \to \Omega'$ and conformal metric density ϱ on Ω'. The critical point f of (3.1) also satisfies the Euler-Lagrange equation, that is to say, f is a solution of equation (1.2), but its Hopf differential $\varrho(f)f_z\overline{f_{\bar{z}}}\,\mathrm{d}z^2$ is not always holomorphic on Ω. In addition, the existence of ϱ-harmonic mapping defined by (1.3) is studied in [19] and [20].

Analogously to Douglas-Dirichlet functional, the upper bound of this new kind of Dirichlet functional is also estimated in this section. First, Dirichlet's principle for $\bar{\partial}$-Dirichlet functional of a Euclidean harmonic mapping is proved as follows.

Lemma 3.1 ([10]). *Suppose that f and h are continuous on $\bar{\Delta}$ with $h \equiv 0$ for $z \in \partial\Delta$. If f is a Euclidean harmonic mapping on Δ and h has continuous partial derivatives of the first order on Δ, $D[f] < \infty$ and $D[h] < \infty$. Then*

$$D[f,h] := \iint_{\Delta} (f_x h_x + f_y h_y)\,\mathrm{d}x\,\mathrm{d}y = 0, \quad z = x + \mathrm{i}y.$$

Theorem 3.1. *Suppose that g is a continuous function on $\bar{\Delta}$ which has the first partial derivatives which are continuous on Δ. Let f be a Euclidean harmonic mapping which is continuous on $\partial\Delta$ and satisfies $f|_{\partial\Delta} = g|_{\partial\Delta}$ and $D'[f] < \infty$, $D'[g] < \infty$. Then $D'[f] \leqslant D'[g]$, where the inequality equals if and only if $f = g$ on Δ.*

Proof. Let $h = g - f$. Thus $h \equiv 0$ on $\partial\Delta$. Then

$$D'[g] = \iint_{\Delta} |f_{\bar{z}} + h_{\bar{z}}|^2 \,\mathrm{d}x\,\mathrm{d}y = D'[f] + D'[h] + \frac{1}{2}D[f,h] = D'[f] + D'[h] \geqslant D'[f].$$

The equality holds if and only if $h \equiv 0$ on Δ, that is $f \equiv g$. □

The proof of Theorem 3.1 is due to Mateljević. Applying Theorem 3.1, we want to estimate the upper bounds of $\bar{\partial}$-Dirichlet functionals of some special φ-harmonic mappings.

Theorem 3.2. *For any $\varphi \in \mathrm{Hol}_2(\Delta)$, we have*

$$\frac{D'[P(\varphi\circ H)]}{D[\varphi]} \leqslant \frac{1}{2}(\mathbb{K}_{\mathrm{H}}-1). \tag{3.2}$$

Especially, when $K_{\mathrm{H}}>1$, the equality in (3.2) *holds true if and only if there exist $\varphi_0 \in \mathrm{Hol}_2(\Delta)$ and an extremal function $f_0 \in Q(\mathrm{H})$ such that the Beltrami coefficient of f_0 satisfies*

$$\frac{f_{0\bar z}(z)}{f_{0z}(z)} = k_{\mathrm{H}}\frac{\overline{\varphi_0'(z)}}{\varphi_0'(z)}, \quad k_{\mathrm{H}} = \frac{K_{\mathrm{H}}-1}{K_{\mathrm{H}}+1}. \tag{3.3}$$

Proof. Suppose that $\omega = f(z) \in \mathcal{FD}(\mathrm{H})$ is an extremal function, that is, $f\colon \Delta\to\Delta$ is a finite distortion function which satisfies $f|_{\partial\Delta}=\mathrm{H}$ and $\mathbb{K}[f]=\mathbb{K}_{\mathrm{H}}$. For any given $\varphi\in\mathrm{Hol}_2(\Delta)$, consider the composite function $F(z)=\varphi\circ f(z)$. Then we have

$$F_{\bar z}(z)=\varphi'(\omega)f_{\bar z}(z), \quad z\in\Delta.$$

Therefore, the upper bound estimation of $D'_{\varrho}[F]$ is

$$\begin{aligned} D'[F] &= \iint_{\Delta} |F_{\bar z}(z)|^2\,\mathrm{d}x\,\mathrm{d}y = \iint_{\Delta}|\varphi'(\omega)|^2|f_{\bar z}(z)|^2\,\mathrm{d}x\,\mathrm{d}y \\ &= \iint_{\Delta}|\varphi'(\omega)|^2\frac{|f_{\bar z}(z)|^2}{|f_z(z)|^2-|f_{\bar z}(z)|^2}J(z,f)\,\mathrm{d}x\,\mathrm{d}y \\ &\leqslant \frac{\mathbb{K}_{\mathrm{H}}-1}{2}\iint_{\Delta}|\varphi'(\omega)|^2\,\mathrm{d}u\,\mathrm{d}v = \frac{\mathbb{K}_{\mathrm{H}}-1}{2}D[\varphi] \end{aligned} \tag{3.4}$$

for $z=x+\mathrm{i}y$ and $\omega=u+\mathrm{i}v$. By Theorem 3.1, we get

$$D'[P(\varphi\circ\mathrm{H})]\leqslant D'[F]\leqslant\frac{\mathbb{K}_{\mathrm{H}}-1}{2}D[\varphi].$$

Since

$$D'[F]=\frac{1}{2}D[F]-\frac{1}{2}\iint_{\Delta}(|F_z|^2-|F_{\bar z}|^2)\,\mathrm{d}x\,\mathrm{d}y=\frac{1}{2}(D[F]-D[\varphi]), \tag{3.5}$$

the accuracy of (3.2) can be obtained from Theorem C. □

Theorem 3.3. *For any $\varphi\in D(\Delta)$, we have*

$$\frac{1}{2}\Bigl(\mathbb{K}_{\mathrm{H}}-1-\sqrt{\mathbb{K}_{\mathrm{H}}^2-1}\Bigr)\leqslant\frac{D'[P(\varphi\circ\mathrm{H})]}{D[\varphi]}-\frac{D'[\varphi]}{D[\varphi]}\leqslant\frac{1}{2}\Bigl(\mathbb{K}_{\mathrm{H}}-1+\sqrt{\mathbb{K}_{\mathrm{H}}^2-1}\Bigr). \tag{3.6}$$

P r o o f. Suppose that $\omega = f(z) \in \mathcal{FD}(\mathrm{H})$ is an extremal function. For any given $\varphi \in D(\Delta)$, consider the composite function $F(z) = \varphi \circ f(z)$. Then we have

$$F_z(z) = \varphi_\omega(\omega) f_z(z) + \varphi_{\overline{\omega}}(\omega)\overline{f_{\bar z}(z)},\ F_{\bar z}(z) = \varphi_\omega(\omega) f_{\bar z}(z) + \varphi_{\overline{\omega}}(\omega)\overline{f_z(z)}.$$

Then

$$\begin{aligned}\iint_\Delta (|F_z|^2 - |F_{\bar z}|^2)\,\mathrm{d}x\,\mathrm{d}y &= \iint_\Delta (|\varphi_\omega|^2 - |\varphi_{\overline{\omega}}|^2)(|f_z|^2 - |f_{\bar z}|^2)\,\mathrm{d}x\,\mathrm{d}y \\ &= \iint_\Delta (|\varphi_\omega|^2 - |\varphi_{\overline{\omega}}|^2)\,\mathrm{d}u\,\mathrm{d}v = D[\varphi] - 2D'[\varphi],\end{aligned}$$

for $z = x + \mathrm{i}y$ and $\omega = u + \mathrm{i}v$, which implies that

$$D'[F] = \frac{1}{2}D[F] - \frac{1}{2}\iint_\Delta (|F_z|^2 - |F_{\bar z}|^2)\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{2}D[F] - \frac{1}{2}D[\varphi] + D'[\varphi]. \tag{3.7}$$

Substituting (2.5) into (3.7) we get that

$$D'[F] \leqslant \frac{1}{2}\Big(\mathbb{K}_\mathrm{H} - 1 + \sqrt{\mathbb{K}_\mathrm{H}^2 - 1}\Big) D[\varphi] + D'[\varphi].$$

Applying Theorem 3.1, we obtain the second inequality of (3.6) when $F = P[\varphi \circ \mathrm{H}]$. Substituting the first inequality of (2.1) into (3.7), the other side of (3.6) also can be obtained. □

Notice that $\mathbb{K}_\mathrm{H} - 1 - \sqrt{\mathbb{K}_\mathrm{H}^2 - 1} \leqslant 0$ for $\mathbb{K}_\mathrm{H} \in [1, \infty)$, from relation (3.7), the lower bound in relation (3.6) is reasonable for any $\varphi \in D(\Delta)$ by the fact that

$$\frac{1}{2}\Big(\mathbb{K}_\mathrm{H} - 1 - \sqrt{\mathbb{K}_\mathrm{H}^2 - 1}\Big) D[\varphi] + D'[\varphi] > \frac{1}{2}\Big(\mathbb{K}_\mathrm{H} + 1 - \sqrt{\mathbb{K}_\mathrm{H}^2 - 1}\Big) D'[\varphi] \geqslant 0$$

and this estimation is sharp when $\mathbb{K}_\mathrm{H} > 1$.

From the identical equation

$$D'_\varrho[F] = \frac{1}{2}D_\varrho[F] - \frac{1}{2}\iint_\Delta \varrho(F)(|F_z|^2 - |F_{\bar z}|^2)\,\mathrm{d}x\,\mathrm{d}y,$$

we find that $D'_\varrho[\varphi \circ P(\mathrm{H})]$ and $D'_\varrho[1/(2 + P(\varphi \circ \mathrm{H}))]$ can be investigated similarly as in Theorems 2.2 and 2.3, respectively.

Theorem 3.4. *For a univalent function $\varphi \in \mathrm{Hol}_2(\Delta)$ and $\varrho(z) = 1/|\varphi'(z)|^2$, we have*

$$\frac{D'_\varrho[\varphi \circ P(\mathrm{H})]}{D_\varrho[\varphi]} \leqslant \frac{1}{2}(\mathbb{K}_\mathrm{H} - 1). \tag{3.8}$$

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