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COMINIMAXNESS OF LOCAL COHOMOLOGY MODULES

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Abstract. Let R be a commutative Noetherian ring, I an ideal of R. Let $t \in \mathbb{N}_0$ be an integer and M an R-module such that $\mathrm{Ext}^i_R(R/I, M)$ is minimax for all $i \leqslant t+1$. We prove that if $H^i_I(M)$ is $\mathrm{FD}_{\leqslant 1}$ (or weakly Laskerian) for all $i < t$, then the R-modules $H^i_I(M)$ are I-cominimax for all $i < t$ and $\mathrm{Ext}^i_R(R/I, H^t_I(M))$ is minimax for $i = 0, 1$. Let N be a finitely generated R-module. We prove that $\mathrm{Ext}^j_R(N, H^i_I(M))$ and $\mathrm{Tor}^R_j(N, H^i_I(M))$ are I-cominimax for all i and j whenever M is minimax and $H^i_I(M)$ is $\mathrm{FD}_{\leqslant 1}$ (or weakly Laskerian) for all i.

Keywords: local cohomology; $\mathrm{FD}_{\leqslant n}$ modules; cofinite modules; cominimax modules

MSC 2010: 13D45, 13E10, 13C05

1. Introduction

Throughout this paper R is a commutative Noetherian ring with nonzero identity and I an ideal of R. For an R-module M, the ith local cohomology module M with respect to the ideal I is defined as

$$H^i_I(M) \cong \varinjlim_n \mathrm{Ext}^i_R(R/I^n, M).$$

Grothendieck in [18] proposed the following conjecture:

Conjecture 1.1. *Let M be a finitely generated R-module and I an ideal of R. Then $\mathrm{Hom}_R(R/I, H^i_I(M))$ is finite for all $i \geqslant 0$.*

Although the conjecture is not true in general as Hartshorne showed in [19], some authors proved that for some numbers t, the module $\mathrm{Hom}_R(R/I, H^t_I(M))$ is finite under some conditions. See [2], Theorem 3.4, [3], Theorem 3.3, [7], Theorem 2.3,

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[8], Theorem 2.6, [14], Theorem 2.1, and [15], Theorem 6.3.9. Hartshorne also defined a module M to be *I-cofinite* if $\operatorname{Supp}_R(M) \subseteq V(I)$ and $\operatorname{Ext}^i_R(R/I, M)$ is finitely generated for all $i \geqslant 0$, and posed the following question:

Question 1.2. Let M be a finite R-module and I an ideal of R. When are $H^i_I(M)$ I-cofinite for all $i \geqslant 0$?

This question was studied by several authors in [2], [6], [8], [13], [19], [21], [34], [27], [29], and [30].

As a special case of [35], Definition 2.1, and a generalization of FSF modules (see [32], Definition 2.1), in [2], Definition 2.1, the author of the present paper and Bahmanpour introduced the class of $\mathrm{FD}_{\leqslant n}$ modules. A module M is said to be an $\mathrm{FD}_{\leqslant n}$ module, if there exists a finitely generated submodule N of M such that $\dim M/N \leqslant n$. For more details about properties of this class see [2], Lemma 2.3. Note that the class of $\mathrm{FD}_{\leqslant -1}$ is the same as that of finitely generated R-modules. Recall that a module M is a *minimax* module if there is a finitely generated submodule N of M such that the quotient module M/N is Artinian. Minimax modules were studied by Zöschinger in [37]. Note that for a complete Noetherian local ring, the class of minimax modules is the same as the class of Matlis reflexive modules (see [17] and [36]). Since the class of minimax modules is a generalization of Matlis reflexive modules, the study of minimax modules is as important as the study of Matlis reflexive modules. As a generalization of I-cofinite modules, in [4], the authors, introduced the concept of I-cominimax or cominimax modules with respect to I. An R-module M is an I-cominimax module if $\operatorname{Supp}_R(M) \subseteq V(I)$ and $\operatorname{Ext}^i_R(R/I, M)$ is a minimax module for all $i \geqslant 0$. Recall too that an R-module M is called *weakly Laskerian* if $\operatorname{Ass}_R(M/N)$ is a finite set for each submodule N of M. The class of weakly Laskerian modules was introduced in [16]. Bahmanpour in [5], Theorem 3.3, proved that over Noetherian rings, an R-module M is weakly Laskerian if and only if M is an FSF module. Thus the class of weakly Laskerian modules is contained in the class of $\mathrm{FD}_{\leqslant 1}$ modules.

Recently many authors have studied the minimaxness and cominimaxness of local cohomology modules and answered Conjecture 1.1 and Question 1.2 in the class of minimax modules in some cases (see [1], [7], [20], [22] [24], [26]). The purpose of this note is to make a suitable generalization of Conjecture 1.1 and Question 1.2 in terms of minimax modules instead of finitely generated modules. In this direction in Section 2, we generalize [2], Theorem 3.4 and Corollaries 3.5 and 3.6. More precisely, we will show:

Theorem 1.3 (See Theorem 2.7 and Corollary 2.10)**.** *Let R be a Noetherian ring and I an ideal of R. Let $t \in \mathbb{N}_0$ be an integer and M an R-module such that*

$\operatorname{Ext}^i_R(R/I, M)$ *are minimax for all* $i \leqslant t+1$. *Let the* R*-modules* $H^i_I(M)$ *be* $\mathrm{FD}_{\leqslant 1}$ *(or weakly Laskerian)* R*-modules for all* $i < t$. *Then the following conditions hold:*

(i) *The* R*-modules* $H^i_I(M)$ *are* I*-cominimax for all* $i < t$.

(ii) *For all* $\mathrm{FD}_{\leqslant 0}$ *(or minimax) submodules* N *of* $H^t_I(M)$, *the* R*-modules*

$$\operatorname{Hom}_R(R/I, H^t_I(M)/N) \quad \text{and} \quad \operatorname{Ext}^1_R(R/I, H^t_I(M)/N)$$

are minimax. In particular, the set $\operatorname{Ass}_R(H^t_I(M)/N)$ *is finite.*

Corrolary 1.4 (See Corollary 2.8). *Let* R *be a Noetherian ring and* I *an ideal of* R. *Let* M *be an* R*-module such that* $\operatorname{Ext}^i_R(R/I, M)$ *are minimax for all* i *and the* R*-modules* $H^i_I(M)$ *are* $\mathrm{FD}_{\leqslant 1}$ *(or weakly Laskerian)* R*-modules for all* i. *Then:*

(i) *The* R*-modules* $H^i_I(M)$ *are* I*-cominimax for all* i.

(ii) *For any* $i \geqslant 0$ *and for any* $\mathrm{FD}_{\leqslant 0}$ *(or minimax) submodule* N *of* $H^i_I(M)$, *the* R*-module* $H^i_I(M)/N$ *is* I*-cominimax.*

Hartshorne in [19] also asked the following question:

Question 1.5. Does the category $\mathcal{M}(R, I)_{\mathrm{cof}}$ of I-cofinite modules form an Abelian subcategory of the category of all R-modules? That is, if $f\colon M \to N$ is an R-module homomorphism of I-cofinite modules, are $\operatorname{Ker} f$ and $\operatorname{Coker} f$ I-cofinite?

With respect to this question, Hartshorne proved that if I is a prime ideal of dimension one in a complete regular local ring R, then the answer to his question is affirmative. On the other hand, in [13], Delfino and Marley extended this result to arbitrary complete local rings. Recently, Kawasaki in [23] generalized the Delfino and Marley's result to an arbitrary ideal I of dimension one in a local ring R. Finally, Melkersson in [31] completely removed the local assumption on R. More recently, in [2] and [10] it is shown that Hartshorne's question is true for the category of all I-cofinite R-modules M with $\dim M \leqslant 1$ and the class of I-cofinite $\mathrm{FD}_{\leqslant 1}$ modules, respectively, for all ideals I in a commutative Noetherian ring R. Irani in [22], Theorem 2.5, proved that the category of all I-cominimax R-modules M with $\dim M \leqslant 1$ is Abelian. One of the main results of this section is to prove that the class of I-cominimax weakly Laskerian ($\mathcal{WL}(R, I)_{\mathrm{comin}}$) and I-cominimax $\mathrm{FD}_{\leqslant 1}(\mathcal{FD}^1(R, I)_{\mathrm{comin}})$ modules are Abelian categories. (See Theorem 2.11.) Using this fact we generalize [20], Corollary 3.5, as follows:

Corrolary 1.6 (See Corollary 2.13). *Let* I *be an ideal of a Noetherian ring* R, M *a nonzero minimax* R*-module such that* $H^i_I(M)$ *is* $\mathrm{FD}_{\leqslant 1}$ *for all* $i \geqslant 0$. *Then for each finite* R*-module* N, *the* R*-modules* $\operatorname{Ext}^j_R(N, H^i_I(M))$ *and* $\operatorname{Tor}^R_j(N, H^i_I(M))$ *are* I*-cominimax and* $\mathrm{FD}_{\leqslant 1}$ *modules for all* $i \geqslant 0$ *and* $j \geqslant 0$.

Throughout this paper, R will always be a commutative Noetherian ring with nonzero identity and I will be an ideal of R. We denote $\{\mathfrak{p} \in \operatorname{Spec} R\colon \mathfrak{p} \supseteq I\}$ by $V(I)$.

For any unexplained notation and terminology we refer the reader to [11], [12] and [28].

2. Cominimaxness of local cohomology

We begin with an example showing us that the class of cofinite modules with respect to an ideal is strictly contained in the class of cominimax modules with respect to the same ideal.

Example 2.1. Let $(R, \mathfrak{m})$ be a local ring and $\mathfrak{p}$ a prime ideal of R such that $\dim R/\mathfrak{p} = 1$. Then it is easy to see that the R-module $E(R/\mathfrak{p})$ is $\mathfrak{p}$-cominimax but not $\mathfrak{p}$-cofinite.

The following useful lemma will be needed in the proof of Proposition 2.4.

Lemma 2.2. *Let I be an ideal of a Noetherian ring R and M an $\mathrm{FD}_{\leqslant 0}$ R-module such that $\operatorname{Supp}_R(M) \subseteq V(I)$. Then the following statements are equivalent:*

(i) *M is I-cominimax.*

(ii) *The R-module $\operatorname{Hom}_R(R/I, M)$ is minimax.*

Proof. We know by definitions that (ii) follows from (i). Let N be a finite submodule of M such that $\dim M/N \leqslant 0$ and suppose the R-module $\operatorname{Hom}_R(R/I, M)$ is minimax.

The exactness of

$$0 \to \operatorname{Hom}_R(R/I, N) \to \operatorname{Hom}_R(R/I, M) \to \operatorname{Hom}_R(R/I, M/N) \to \operatorname{Ext}^1_R(R/I, N)$$

implies that $\operatorname{Hom}_R(R/I, M/N)$ is minimax. Since $\dim M/N \leqslant 0$, it is easy to see that $\operatorname{Hom}_R(R/I, M/N)$ is an Artinian R-module. As M/N is I-torsion, it follows by Melkersson's theorem that M/N is Artinian. Thus M is minimax. The I-torsionness of M imples that it is I-cominimax. $\square$

The following lemma improves and generalizes the condition of [2], Lemma 2.5.

Lemma 2.3. *Let I be an ideal of a Noetherian ring R and let M be an R-module such that $\dim M = 1$ and $\operatorname{Supp}_R(M) \subseteq V(I)$. If $\operatorname{Hom}_R(R/I, M)$ is a minimax R-module, then there is a finite submodule N of M and an element $x \in I$ such that $\operatorname{Supp}_R(M/(xM + N)) \subseteq \operatorname{Max}(R)$.*

P r o o f. Since $\operatorname{Hom}_R(R/I, M)$ is a minimax R-module we conclude that $\operatorname{Ass}_R(M)$ is finite and therefore $\operatorname{Assh}_R(M) = \{\mathfrak{p} \in \operatorname{Supp} M \colon \dim R/\mathfrak{p} = 1\}$ is finite. Consider $S = R \setminus \bigcup_{\mathfrak{p} \in \operatorname{Assh}_R(M)} \mathfrak{p}$. It is easy to see that $\operatorname{Supp}_{S^{-1}R}(S^{-1}M) \subseteq V(S^{-1}I) \cap \operatorname{Max}(S^{-1}R)$. By the definition of minimax modules it follows that $\operatorname{Hom}_{S^{-1}R}(S^{-1}R/S^{-1}I, S^{-1}M)$ is a finite $S^{-1}R$-module. From [31], Lemma 2.1, we conclude that $S^{-1}M$ is an Artinian $S^{-1}R$-module and $S^{-1}I$-cofinite. By [29], Corollary 1.2, the $S^{-1}R$-module $S^{-1}M/IS^{-1}M$ is finite. Hence there is a finite submodule N of M such that $S^{-1}(M/(IM+N)) = 0$. Put $\overline{M} = M/N$. Then $S^{-1}\overline{M}$ (as a homomorphic image of $S^{-1}M$) is an Artinian $S^{-1}R$-module. Furthermore, $S^{-1}\overline{M} = IS^{-1}\overline{M}$. Then by [25], 2.8, there is $x \in I$ such that $S^{-1}\overline{M} = xS^{-1}\overline{M}$. Therefore $S^{-1}(\overline{M}/x\overline{M}) = 0$ and hence $\operatorname{Supp}_R(\overline{M}/x\overline{M}) \subseteq \operatorname{Supp}_R(M) \setminus \operatorname{Assh}_R(M) \subseteq \operatorname{Max}(R)$. Together with the isomorphism $\overline{M}/x\overline{M} \cong M/(xM+N)$, this proves our assertion. $\square$

The following proposition is the same as [22], Proposition 2.4, but the method of proof is completely different.

Proposition 2.4 (Compare [22], Proposition 2.4)**.** *Let I be an ideal of a Noetherian ring R and M an R-module such that* $\dim M \leqslant 1$ *and* $\operatorname{Supp} M \subseteq V(I)$. *Then the following statements are equivalent:*

(i) *M is I-cominimax.*

(ii) *The R-modules* $\operatorname{Hom}_R(R/I, M)$ *and* $\operatorname{Ext}^1_R(R/I, M)$ *are minimax.*

P r o o f. The conclusion (i) $\Rightarrow$ (ii) is obvious. In order to prove (ii) $\Rightarrow$ (i) using Lemma 2.2, we may assume $\dim M = 1$. Now use Lemma 2.3 instead of [31], Lemma 2.1, and the I-cominimaxness instead of I-cofiniteness in the proof of [31], Theorem 2.3. $\square$

In what follows the next theorem plays an important role.

Theorem 2.5. *Let I be an ideal of a Noetherian ring R and M an* $\mathrm{FD}_{\leqslant 1}$ *R-module such that* $\operatorname{Supp} M \subseteq V(I)$. *Then the following statements are equivalent:*

(i) *M is I-cominimax.*

(ii) *The R-modules* $\operatorname{Hom}_R(R/I, M)$ *and* $\operatorname{Ext}^1_R(R/I, M)$ *are minimax.*

P r o o f. (i) $\Rightarrow$ (ii) is clear. In order to prove (ii) $\Rightarrow$ (i), by definition there is a finitely generated submodule N of M such that $\dim(M/N) \leqslant 1$ and $\operatorname{Supp} M/N \subseteq V(I)$. Also, the exact sequence

$$(*) \qquad 0 \to N \to M \to M/N \to 0$$

induces the exact sequence

$$
\begin{aligned}
&0 \to \operatorname{Hom}_R(R/I, N) \to \operatorname{Hom}_R(R/I, M) \to \operatorname{Hom}_R(R/I, M/N)\\
&\quad \to \operatorname{Ext}^1_R(R/I, N) \to \operatorname{Ext}^1_R(R/I, M) \to \operatorname{Ext}^1_R(R/I, M/N) \to \operatorname{Ext}^2_R(R/I, N).
\end{aligned}
$$

Hence, it follows that the R-modules $\operatorname{Hom}_R(R/I, M/N)$ and $\operatorname{Ext}^1_R(R/I, M/N)$ are finitely generated. Therefore, in view of Proposition 2.4, the R-module M/N is I-cominimax. Now it follows from the exact sequence $(*)$ that M is I-cominimax. □

The following lemma is needed in the proof of the next theorem.

Lemma 2.6. *Let I be an ideal of a Noetherian ring R, M a nonzero R-module and $t \in \mathbb{N}_0$. Suppose that the R-module $H^i_I(M)$ is I-cominimax for all $i < t$, and the R-modules $\operatorname{Ext}^t_R(R/I, M)$ and $\operatorname{Ext}^{t+1}_R(R/I, M)$ are minimax. Then the R-modules $\operatorname{Hom}_R(R/I, H^t_I(M))$ and $\operatorname{Ext}^1_R(R/I, H^t_I(M))$ are minimax.*

Proof. We use induction on t. The exact sequence

$$(*) \qquad 0 \to \Gamma_I(M) \to M \to M/\Gamma_I(M) \to 0$$

induces the exact sequence:

$$
\begin{aligned}
&0 \to \operatorname{Hom}_R(R/I, \Gamma_I(M)) \to \operatorname{Hom}_R(R/I, M) \to \operatorname{Hom}_R(R/I, M/\Gamma_I(M))\\
&\quad \to \operatorname{Ext}^1_R(R/I, \Gamma_I(M)) \to \operatorname{Ext}^1_R(R/I, M).
\end{aligned}
$$

Since $\operatorname{Hom}_R(R/I, M/\Gamma_I(M)) = 0$ so $\operatorname{Hom}_R(R/I, \Gamma_I(M))$ and $\operatorname{Ext}^1_R(R/I, \Gamma_I(M))$ are minimax. Assume inductively that $t > 0$ and that we have established the result for nonnegative integers smaller than t. By applying the functor $\operatorname{Hom}_R(R/I, -)$ to the exact sequence $(*)$, we can deduce that $\operatorname{Ext}^j_R(R/I, M/\Gamma_I(M))$ is minimax for $j = t, t+1$. On the other hand, $H^0_I(M/\Gamma_I(M)) = 0$ and $H^j_I(M/\Gamma_I(M)) \cong H^j_I(M)$ for all $j > 0$. Therefore we may assume that $\Gamma_I(M) = 0$. Let E be an injective hull of M and put $N = E/M$. Then $\operatorname{Hom}_R(R/I, E) = 0 = \Gamma_I(E)$. Hence $\operatorname{Ext}^j_R(R/I, N) \cong \operatorname{Ext}^{j+1}_R(R/I, M)$ and $H^j_I(N) \cong H^{j+1}_I(M)$ for all $j \geqslant 0$. Now, the induction hypothesis yields that $\operatorname{Hom}_R(R/I, H^{t-1}_I(N))$ and $\operatorname{Ext}^1_R(R/I, H^{t-1}_I(N))$ are minimax and so $\operatorname{Hom}_R(R/I, H^t_I(M))$ and $\operatorname{Ext}^1_R(R/I, H^t_I(M))$ are minimax, as required. □

We are now ready to state and prove the main results (Theorem 2.7 and the Corollaries 2.8 and 2.10) which are extensions of Bahmanpour-Naghipour's results in [7] and [8] in terms of minimax modules, [1], Corollary 2.3, [2], Theorem 3.4 and Corollaries 3.5 and 3.6, [24], Corollary 2.3, and Hong Quy's result in [32].

Theorem 2.7. *Let R be a Noetherian ring and I an ideal of R. Let $t \in \mathbb{N}_0$ be an integer and M an R-module such that $\mathrm{Ext}^i_R(R/I, M)$ are minimax for all $i \leqslant t+1$. Let the R-modules $H^i_I(M)$ be $\mathrm{FD}_{\leqslant 1}$ R-modules for all $i < t$. Then the following assertions hold:*

(i) *The R-modules $H^i_I(M)$ are I-cominimax for all $i < t$.*

(ii) *For all $\mathrm{FD}_{\leqslant 0}$ (or minimax) submodules N of $H^t_I(M)$, the R-modules*

$$\mathrm{Hom}_R(R/I, H^t_I(M)/N) \quad \textit{and} \quad \mathrm{Ext}^1_R(R/I, H^t_I(M)/N)$$

are minimax. In particular, the set $\mathrm{Ass}_R(H^t_I(M)/N)$ is finite.

Proof. (i) We proceed by induction on t. In the case $t = 0$ there is nothing to prove. So, let $t > 0$ and suppose the result has been proved for smaller values of t. By the inductive assumption, $H^i_I(M)$ is I-cominimax for $i = 0, 1, \ldots, t-2$. Hence by Lemma 2.6 and the assumption, $\mathrm{Hom}_R(R/I, H^{t-1}_I(M))$ and $\mathrm{Ext}^1_R(R/I, H^{t-1}_I(M))$ are minimax. Therefore by Theorem 2.5, $H^i_I(M)$ is I-cominimax for all $i < t$. This completes the inductive step.

(ii) In view of (i) and Lemma 2.6, $\mathrm{Hom}_R(R/I, H^t_I(M))$ and $\mathrm{Ext}^1_R(R/I, H^t_I(M))$ are minimax. On the other hand, according to Lemma 2.2, N is I-cominimax. Now, the exact sequence

$$0 \to N \to H^t_I(M) \to H^t_I(M)/N \to 0$$

induces the exact sequence

$$\begin{aligned}\mathrm{Hom}_R(R/I, H^t_I(M)) &\to \mathrm{Hom}_R(R/I, H^t_I(M)/N) \to \mathrm{Ext}^1_R(R/I, N)\\ &\to \mathrm{Ext}^1_R(R/I, H^t_I(M)) \to \mathrm{Ext}^1_R(R/I, H^t_I(M)/N) \to \mathrm{Ext}^2_R(R/I, N).\end{aligned}$$

Consequently,

$$\mathrm{Hom}_R(R/I, H^t_I(M)/N) \quad \text{and} \quad \mathrm{Ext}^1_R(R/I, H^t_I(M)/N)$$

are minimax, as required. □

Corollary 2.8. *Let R be a Noetherian ring and I an ideal of R. Let M be an R-module such that $\mathrm{Ext}^i_R(R/I, M)$ are minimax for all i and the R-modules $H^i_I(M)$ are $\mathrm{FD}_{\leqslant 1}$ (or weakly Laskerian) R-modules for all i. Then:*

(i) *The R-modules $H^i_I(M)$ are I-cominimax for all i.*

(ii) *For any $i \geqslant 0$ and for any $\mathrm{FD}_{\leqslant 0}$ (or minimax) submodule N of $H^i_I(M)$, the R-module $H^i_I(M)/N$ is I-cominimax.*

P r o o f. (i) Clear.

(ii) In view of (i) the R-module $H_I^i(M)$ is I-cominimax for all i. Hence the R-module $\operatorname{Hom}_R(R/I, N)$ is minimax, and so it follows from Lemma 2.2 that N is I-cominimax. Now, the exact sequence

$$0 \to N \to H_I^i(M) \to H_I^i(M)/N \to 0$$

and [4], Proposition 3.3, imply that the R-module $H_I^i(M)/N$ is I-cominimax. □

Corollary 2.9. *Let R be a Noetherian ring and I an ideal of R. Let M be an R-module such that the R-modules $H_I^i(M)$ are* $\mathrm{FD}_{\leqslant 1}$ *(or weakly Laskerian) R-modules for all i. Then the following conditions are equivalent:*

(i) *The R-modules $\operatorname{Ext}_R^i(R/I, M)$ are minimax for all i.*

(ii) *The R-modules $H_I^i(M)$ are I-cominimax for all i.*

P r o o f. (i) ⇒ (ii) follows by Corollary 2.8.

(ii) ⇒ (i) follows by [30], Proposition 3.9. □

Corollary 2.10. *Let R be a Noetherian ring and I an ideal of R. Let $t \in \mathbb{N}_0$ be an integer and M an R-module such that $\operatorname{Ext}_R^i(R/I, M)$ are minimax for all $i \leqslant t+1$. Let the R-modules $H_I^i(M)$ be weakly Laskerian for all $i < t$. Then the following conditions hold:*

(i) *The R-modules $H_I^i(M)$ are I-cominimax for all $i < t$.*

(ii) *For all* $\mathrm{FD}_{\leqslant 0}$ *(or minimax) submodule N of $H_I^t(M)$, the R-modules*

$$\operatorname{Hom}_R(R/I, H_I^t(M)/N) \quad \text{and} \quad \operatorname{Ext}_R^1(R/I, H_I^t(M)/N)$$

are minimax. In particular, the set $\operatorname{Ass}_R(H_I^t(M)/N)$ is finite.

P r o o f. Use Theorem 2.7 and note that the category of weakly Laskerian modules is contained in the category of $\mathrm{FD}_{\leqslant 1}$ modules. □

One of the main results of this section is to prove that for an arbitrary ideal I of a Noetherian ring R, the category of I-cominimax $\mathrm{FD}_{\leqslant 1}$ modules is an Abelian category.

Theorem 2.11. *Let I be an ideal of a Noetherian ring R. Let $\mathcal{FD}^1(R, I)_{\mathrm{com}}$ denote the category of I-cominimax* $\mathrm{FD}_{\leqslant 1}$ *R-modules. Then $\mathcal{FD}^1(R, I)_{\mathrm{com}}$ is an Abelian category.*

P r o o f. Let $M, N \in \mathcal{FD}^1(R, I)_{\rm com}$ and let $f\colon M \to N$ be an R-homomorphism. By [2], Lemma 2.3 (v), $\operatorname{Ker} f$ and $\operatorname{Coker} f$ are $\mathrm{FD}_{\leqslant 1}$, so it is enough to show that the R-modules $\operatorname{Ker} f$ and $\operatorname{Coker} f$ are I-cominimax. To this end, the exact sequence

$$0 \to \operatorname{Ker} f \to M \to \operatorname{Im} f \to 0$$

induces an exact sequence

$$\begin{aligned} 0 &\to \operatorname{Hom}_R(R/I, \operatorname{Ker} f) \to \operatorname{Hom}_R(R/I, M) \to \operatorname{Hom}_R(R/I, \operatorname{Im} f) \\ &\to \operatorname{Ext}^1_R(R/I, \operatorname{Ker} f) \to \operatorname{Ext}^1_R(R/I, M) \end{aligned}$$

which implies the R-modules $\operatorname{Hom}_R(R/I, \operatorname{Ker} f)$ and $\operatorname{Ext}^1_R(R/I, \operatorname{Ker} f)$ are minimax. Therefore it follows from Theorem 2.5 that $\operatorname{Ker} f$ is I-cominimax. Now, using [4], Proposition 3.3, the assertion follows from the exact sequences

$$0 \to \operatorname{Ker} f \to M \to \operatorname{Im} f \to 0,$$

and

$$0 \to \operatorname{Im} f \to N \to \operatorname{Coker} f \to 0.$$

□

The following corollary is a generalization of [20], Theorem 3.4.

Corollary 2.12. *Let I be an ideal of a Noetherian ring R. Let M be an $\mathrm{FD}_{\leqslant 1}$ I-cominimax R-module. Then the R-modules $\operatorname{Ext}^i_R(N, M)$ and $\operatorname{Tor}^R_i(N, M)$ are I-cominimax and $\mathrm{FD}_{\leqslant 1}$ modules, for all finitely generated R-modules N and all integers $i \geqslant 0$.*

P r o o f. Since N is finitely generated it follows that N has a free resolution of finitely generated free modules. Now the assertion follows using Theorem 2.11 and computing the modules $\operatorname{Tor}^R_i(N, M)$ and $\operatorname{Ext}^i_R(N, M)$, by this free resolution. □

The following two corollaries are generalizations of [20], Corollary 3.5.

Corollary 2.13. *Let I be an ideal of a Noetherian ring R, M a nonzero minimax R-module such that $H^i_I(M)$ is $\mathrm{FD}_{\leqslant 1}$ for all $i \geqslant 0$. Then for each finite R-module N, the R-modules $\operatorname{Ext}^j_R(N, H^i_I(M))$ and $\operatorname{Tor}^R_j(N, H^i_I(M))$ are I-cominimax and $\mathrm{FD}_{\leqslant 1}$ modules for all $i \geqslant 0$ and $j \geqslant 0$.*

P r o o f. Note that by Corollary 2.8, $H^i_I(M)$ is I-cominimax for all $i \geqslant 0$. Now the assertion followes from Corollary 2.12. □

Corollary 2.14. *Let I be an ideal of a Noetherian ring R, M a nonzero minimax R-module such that* $\dim M/IM \leqslant 1$ (e.g., $\dim R/I \leqslant 1$). *Then for each finite R-module N, the R-modules* $\operatorname{Ext}^j_R(N, H^i_I(M))$ *and* $\operatorname{Tor}^R_j(N, H^i_I(M))$ *are I-cominimax and* $\mathrm{FD}_{\leqslant 1}$ *modules for all $i \geqslant 0$ and $j \geqslant 0$.*

Proof. Note that $\dim \operatorname{Supp} H^i_I(M) \leqslant \dim M/IM \leqslant 1$, thus $H^i_I(M)$ is an $\mathrm{FD}_{\leqslant 1}$ R-module and by Corollary 2.8 it is I-cominimax. □

In the end we give an example to show that in [2], Theorem 3.4, one cannot choose the submodule N in the class of $\mathrm{FD}_{\leqslant 1}$ modules and N must be a minimax or $\mathrm{FD}_{\leqslant 0}$ module.

Example 2.15. Let $(R, \mathfrak{m})$ be a Noetherian Gorenstein local ring of dimension $d = 4$ and x_1, x_2, x_3, x_4 a system of parameters of R. Let $I = Rx_1 + Rx_2$ and $J = Rx_3 + Rx_4$. Then by the proof of [9], Example 2.7, we have $H^3_{I\cap J}(R) \cong E(R/\mathfrak{m})$. Now since $\dim R/I \cap J = 2$, there exists $x \in R$ such that $\dim R/(I \cap J) + Rx = 1$. Let $(I \cap J) + Rx = K$. By [33], Corollary 3.5, we have the exact sequence

$$0 \to H^1_{Rx}(H^2_{I\cap J}(R)) \to H^3_K(R) \to H^0_{Rx}(H^3_{I\cap J}(R)) \to 0.$$

Since $\dim R/K = 1$, hence $H^i_K(R)$ are K-cofinite for all $i \geqslant 0$ by [8], Corollary 2.7. On the other hand, since R is local, they are also weakly Laskerian and so $\mathrm{FD}_{\leqslant 1}$. Consider $H^1_{Rx}(H^2_{I\cap J}(R)) = N$, so by the above exact sequence $N \leqslant H^3_K(R)$ and therefore N is $\mathrm{FD}_{\leqslant 1}$. Applying the functor $\operatorname{Hom}_R(R/K, -)$ to the above exact sequence we obtain the exact sequence

$$\begin{aligned} 0 &\to \operatorname{Hom}_R(R/K, N) \to \operatorname{Hom}_R(R/K, H^3_K(R)) \\ &\to \operatorname{Hom}_R(R/K, E(R/\mathfrak{m})) \to \operatorname{Ext}^1_R(R/K, N). \end{aligned}$$

The R-module $\operatorname{Ext}^1_R(R/K, N)$ cannot be finitely generated, otherwise

$$\operatorname{Hom}_R(R/K, E(R/\mathfrak{m}))$$

is finitely generated as an R-module and R/K-module. This implies R/K is an Artinian ring which is a contradiction. Thus N cannot be K-cofinite.

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