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ON σ-PERMUTABLY EMBEDDED SUBGROUPS OF FINITE GROUPS

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Abstract. Let $\sigma = \{\sigma_i \colon i \in I\}$ be some partition of the set of all primes $\mathbb{P}$, G be a finite group and $\sigma(G) = \{\sigma_i \colon \sigma_i \cap \pi(G) \neq \emptyset\}$. A set $\mathcal{H}$ of subgroups of G is said to be a complete Hall σ-set of G if every non-identity member of $\mathcal{H}$ is a Hall σ_i-subgroup of G and $\mathcal{H}$ contains exactly one Hall σ_i-subgroup of G for every $\sigma_i \in \sigma(G)$. G is said to be σ-full if G possesses a complete Hall σ-set. A subgroup H of G is σ-permutable in G if G possesses a complete Hall σ-set $\mathcal{H}$ such that $HA^x = A^xH$ for all $A \in \mathcal{H}$ and all $x \in G$. A subgroup H of G is σ-permutably embedded in G if H is σ-full and for every $\sigma_i \in \sigma(H)$, every Hall σ_i-subgroup of H is also a Hall σ_i-subgroup of some σ-permutable subgroup of G.

By using the σ-permutably embedded subgroups, we establish some new criteria for a group G to be soluble and supersoluble, and also give the conditions under which a normal subgroup of G is hypercyclically embedded. Some known results are generalized.

Keywords: finite group; σ-subnormal subgroup; σ-permutably embedded subgroup; σ-soluble group; supersoluble group

MSC 2010: 20D10, 20D20, 20D35

1. Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. If n is an integer, then the symbol $\pi(n)$ denotes the set of all primes dividing n; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of G.

In what follows, $\sigma = \{\sigma_i \colon i \in I\}$ is some partition of all primes $\mathbb{P}$, that is, $\mathbb{P} = \bigcup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$. We write $\sigma(G) = \{\sigma_i \colon \sigma_i \cap \pi(G) \neq \emptyset\}$.

Following [20], [29], [30], G is said to be *σ-primary* if $|\sigma(G)| \leqslant 1$; *σ-soluble* if every chief factor of G is σ-primary. A set $\mathcal{H}$ of subgroups of G is said to be a *complete*

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Hall σ-set of G if every non-identity member of $\mathcal{H}$ is a Hall σ_i-subgroup of G for some σ_i and $\mathcal{H}$ contains exactly one Hall σ_i-subgroup for every $\sigma_i \in \sigma(G)$. G is said to be *σ-full* if G possesses a complete Hall σ-set; *σ-nilpotent* if G has a complete Hall σ-set $\mathcal{H} = \{H_1, H_2, \ldots, H_t\}$ such that $G = H_1 \times H_2 \times \ldots \times H_t$. Clearly, a σ-nilpotent group is σ-soluble. G is said to be a *σ-full group of Sylow type* if every subgroup of G is a D_{σ_i}-group for all $\sigma_i \in \sigma(G)$. A subgroup H of G is said to be *σ-subnormal* in G if there exists a subgroup chain $H = H_0 \leqslant H_1 \leqslant \ldots \leqslant H_n = G$ such that either H_{i-1} is normal in H_i or $H_i/(H_{i-1})_{H_i}$ is σ-primary for all $i = 1, \ldots, n$.

Let $\mathcal{L}$ be some nonempty set of subgroups of G and $K \leqslant G$. Following [29], a subgroup H of G is called *$\mathcal{L}$-permutable* if $HA = AH$ for all $A \in \mathcal{L}$; *$\mathcal{L}^K$-permutable* if $HA^x = A^xH$ for all $A \in \mathcal{L}$ and all $x \in K$. In particular, a subgroup H of G is *σ-permutable* in G if G has a complete Hall σ-set $\mathcal{H}$ such that H is $\mathcal{H}^G$-permutable.

It is well known that embedded subgroups play an important role in the theory of finite groups. For example, a subgroup H of G is said to be *normally embedded* in G (see [12], page 250) if for each prime p dividing the order of H, a Sylow p-subgroup of H is also a Sylow p-subgroup of some normal subgroup of G. A subgroup H of G is said to be *permutably embedded* in G (see [5]) if for each prime p dividing the order of H, a Sylow p-subgroup of H is also a Sylow p-subgroup of some permutable subgroup of G. (Note that a subgroup H of G is said to be *permutable* in G if $HS = SH$ for any subgroup S of G.) A subgroup H of G is said to be *s-permutably embedded* in G (see [8]) if for each prime p dividing the order of H, a Sylow p-subgroup of H is also a Sylow p-subgroup of some s-permutable subgroup of G. (Note that a subgroup H of G is said to be *s-permutable* in G if $HP = PH$ for any Sylow subgroup P of G.) A subgroup H of G is *σ-permutably embedded* in G (see [19]) if H is σ-full and for every $\sigma_i \in \sigma(H)$, every Hall σ_i-subgroup of H is also a Hall σ_i-subgroup of some σ-permutable subgroup of G. By using the above embedded subgroups, the researchers have obtained a series of interesting results (see, for example, [3], [5], [8], [19], [24], [31]).

Some properties of σ-permutably embedded subgroups were analysed in [19]. In this paper, we continue the research of σ-permutably embedded subgroups.

We first obtain the following result.

Theorem 1.1. *Let G be a σ-full group of Sylow type and $\mathcal{H} = \{H_1, H_2, \ldots, H_t\}$ be a complete Hall σ-set of G such that H_i is a supersoluble σ_i-subgroup for all $i = 1, \ldots, t$. If every maximal subgroup of non-cyclic H_i is σ-permutably embedded in G, then G is supersoluble.*

Recall that a normal subgroup E of G is called *hypercyclically embedded* in G ([25], page 217) if every chief factor of G below E is cyclic. Hypercyclically embedded subgroups play an important role in the theory of soluble groups (see [6], [9], [15], [25]) and the conditions under which a normal subgroup is hypercyclically embedded in G were found by many authors (see books [6], [9], [15], [25] and, for example, the recent papers [16], [18], [23], [27], [28], [32]).

In this paper, we also get the following results in this line researches.

Theorem 1.2. *Let G be a σ-full group of Sylow type and $\mathcal{H} = \{H_1, H_2, \dots, H_t\}$ be a complete Hall σ-set of G such that H_i is a nilpotent σ_i-subgroup for all $i = 1, \dots, t$. Let E be a normal subgroup of G. If every maximal subgroup of any non-cyclic $H_i \cap E$ is σ-permutably embedded in G, then E is hypercyclically embedded in G.*

Theorem 1.3. *Let G be a σ-full group of Sylow type and $\mathcal{H} = \{H_1, H_2, \dots, H_t\}$ be a complete Hall σ-set of G such that H_i is a supersoluble σ_i-subgroup for all $i = 1, \dots, t$. Let E be a normal subgroup of G. If every cyclic subgroup H of any non-cyclic $H_i \cap E$ of prime order and order 4 (if the Sylow 2-subgroup of E is non-abelian and $H \nleq Z_\infty(G)$) is σ-permutably embedded in G, then E is hypercyclically embedded in G.*

In Section 3 and Section 4 we give the proofs of Theorems 1.1, 1.2 and 1.3. In Section 5 we will give some applications of our results.

All unexplained terminologies and notations are standard. The reader is referred to [12], [15] if necessary.

2. Preliminaries

We use $\mathfrak{S}_\sigma$ to denote the class of all σ-soluble groups and $F_\sigma(G)$ to denote the product of all normal σ-nilpotent subgroups of G.

Lemma 2.1 ([29], Lemma 2.1)**.** *The class $\mathfrak{S}_\sigma$ is closed under taking direct products, homomorphic images and subgroups. Moreover, any extension of a σ-soluble group by a σ-soluble group is a σ-soluble group as well.*

Lemma 2.2 ([29], Lemma 2.6 (11))**.** *Let G be a σ-full group and A be a σ-subnormal subgroup of G. If A is σ-nilpotent, then A is contained in $F_\sigma(G)$.*

Lemma 2.3 ([19], Lemma 2.6 (i))**.** *$F_\sigma(G)$ is σ-nilpotent.*

Lemma 2.4 ([29], Lemma 2.8)**.** *Let H, K and N be subgroups of G. Let $\mathcal{H} = \{H_1, H_2, \dots, H_t\}$ be a complete Hall σ-set of G and $\mathcal{L} = \mathcal{H}^K$. Suppose that H is $\mathcal{L}$-permutable and N is normal in G.*

(1) *The subgroup HN/N is $\mathcal{L}^*$-permutable, where*

$$\mathcal{L}^* = \{H_1N/N, \dots, H_tN/N\}^{KN/N}.$$

In particular, if H is σ-permutable in G, then HN/N is σ-permutable in G/N.

(2) *If G is a σ-full group of Sylow type and E/N is a σ-permutable subgroup of G/N, then E is σ-permutable in G.*

Lemma 2.5 ([29], Theorem B)**.** *Let H be a subgroup of a σ-full group G. If H is σ-permutable in G, then H is σ-subnormal in G and H^G/H_G is σ-nilpotent.*

Lemma 2.6. *Let $H \leqslant K$ and N be subgroups of G. Suppose that N is normal in G.*

(1) *If G is a σ-full group of Sylow type and H is σ-permutably embedded in G, then H is σ-permutably embedded in K.*

(2) *If H is σ-permutably embedded in G, then HN/N is σ-permutably embedded in G/N.*

(3) *If G is a σ-full group of Sylow type and H/N is σ-permutably embedded in G/N, then H is σ-permutably embedded in G.*

P r o o f. (1)–(2) can be found in [19], Lemma 2.2.

(3) Let H_i be a Hall σ_i-subgroup of H, where $\sigma_i \in \sigma(H)$. Then H_iN/N is a Hall σ_i-subgroup of H/N. By the hypothesis, there exists a σ-permutable subgroup T/N of G/N such that H_iN/N is a Hall σ_i-subgroup of T/N. Then T is σ-permutable in G by Lemma 2.4 (2). Since $N \leqslant H$, $H_iN \leqslant H$ and so H_i is a Hall σ_i-subgroup of H_iN. It follows that $|T : H_i| = |T : H_iN||H_iN : H_i|$ is a σ_i'-number. Hence, H_i is a Hall σ_i-subgroup of T. This shows that H is σ-permutably embedded in G. □

Following [29], [20], we use $O^{\sigma_i}(G)$ to denote the subgroup of G generated by all σ_i'-subgroups of G, and $O_{\sigma_i}(G)$ and $O_{\sigma_i'}(G)$ to denote the subgroup of G generated by all normal σ_i-subgroups and normal σ_i'-subgroups of G, respectively.

Lemma 2.7 ([29], Lemma 3.1)**.** *Let H be a σ_1-subgroup of a σ-full group G. Then H is σ-permutable in G if and only if $O^{\sigma_1}(G) \leqslant N_G(H)$.*

Let P be a p-group. If P is not a non-abelian 2-group, then we use $\Omega(P)$ to denote $\Omega_1(P)$. Otherwise, $\Omega(P) = \Omega_2(P)$.

Lemma 2.8 ([17], Lemma 4.3)**.** *Let C be a Thompson critical subgroup of a p-group P (see* [13]*, page 185).*

(1) *If p is odd, then the exponent of $\Omega(C)$ is p.*

(2) *If P is a non-abelian 2-group, then the exponent of $\Omega(C)$ is 4.*

The following lemma is a corollary of [17], Lemma 4.4 and [11], Lemma 2.12.

Lemma 2.9. *Let P be a normal p-subgroup of G and C be a Thompson critical subgroup of P. If either $P/\Phi(P)$ is hypercyclically embedded in $G/\Phi(P)$ or $\Omega(C)$ is hypercyclically embedded in G, then P is hypercyclically embedded in G.*

Lemma 2.10 ([28], Theorem C)**.** *Let E be a normal subgroup of G. If $F^*(E)$ is hypercyclically embedded in G, then E is hypercyclically embedded in G.*

In this lemma, $F^*(E)$ is the generalized Fitting subgroup of E, that is, the largest normal quasinilpotent subgroup of E (see [22], Chapter X).

Recall that a class of groups $\mathfrak{F}$ is said to be a *formation* provided that (i) if $G \in \mathfrak{F}$ and $N \trianglelefteq G$, then $G/N \in \mathfrak{F}$, and (ii) if $G/M \in \mathfrak{F}$ and $G/N \in \mathfrak{F}$, then $G/(M \cap N) \in \mathfrak{F}$ for any normal subgroup M, N of G. A formation $\mathfrak{F}$ is said to be *saturated* if $G/\Phi(G) \in \mathfrak{F}$ implies that $G \in \mathfrak{F}$.

Lemma 2.11 ([26], Lemma 2.16 or [15], Theorem 1.2.7 (b))**.** *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups and E be a normal subgroup of G such that $G/E \in \mathfrak{F}$. If E is cyclic, then $G \in \mathfrak{F}$.*

3. Proofs of Theorems 1.1 and 1.2

The following fact is one of the main steps in the proofs of Theorems 1.1 and 1.2.

Proposition 3.1. *Let G be a σ-full group of Sylow type and $\mathcal{H} = \{H_1, H_2, \dots, H_t\}$ be a complete Hall σ-set of G such that H_i is a soluble σ_i-subgroup for all $i = 1, \dots, t$, and let the smallest prime p of $\pi(G)$ belong to σ_1. If every maximal subgroup of H_1 is σ-permutably embedded in G, then G is soluble.*

P r o o f. Assume that this is false and let (G, H_1) be a counterexample with minimal $|G| + |H_1|$. Then $p = 2 \in \pi(H_1)$ by the Feit-Thompson theorem.

(1) *G is not σ-soluble, and so $|\sigma(G)| > 1$.* Assume that G is σ-soluble. Then for every chief factor, H/K of G is σ-primary, that is, H/K is a σ_i-group for some i. But since H_i is soluble, H/K is an elementary abelian group. It follows that G is soluble. This contradiction shows that (1) holds.

(2) $O_{\sigma_1}(G) = 1$. Assume that $O_{\sigma_1}(G) \neq 1$. Let $N = O_{\sigma_1}(G)$. If $N = H_1$, then G/N is soluble by Feit-Thompson theorem, and so G is σ-soluble, contrary to claim (1). Hence $N \neq H_1$, and so H_1/N is a non-identity Hall σ_1-subgroup of G/N. Let M/N be a maximal subgroup of H_1/N. Then M is a maximal subgroup of H_1. By the hypothesis and Lemma 2.6 (2), M/N is σ-permutably embedded in G/N.

Then, clearly, the hypothesis holds for $(G/N, H_1/N)$. Hence G/N is soluble by the choice of (G, H_1). Consequently, G is σ-soluble, which contradicts claim (1). Hence we have (2).

(3) $O_{\sigma_1'}(G) = 1$. Assume that $K = O_{\sigma_1'}(G) \neq 1$. Then H_1K/K is a Hall σ_1-subgroup of G/K. Let W/K be a maximal subgroup of H_1K/K. Then $W = (H_1 \cap W)K$ is a maximal subgroup of H_1K. If $H_1 \cap W$ is not a maximal subgroup of H_1, then there exists a subgroup E of H_1 such that $H_1 \cap W < E < H_1$. Since $(|H_1|, |K|) = 1$, $W < EK < H_1K$. This contradiction shows that $H_1 \cap W$ is a maximal subgroup of H_1. By the hypothesis and Lemma 2.6 (2), W/K is σ-permutably embedded in G/K. This shows that $(G/K, H_1K/K)$ satisfies the hypothesis, so G/K is soluble by the choice of (G, H_1). But since K is soluble by Feit-Thompson theorem, it follows that G is soluble. This contradiction shows that (3) holds.

(4) *Let R be a minimal normal subgroup of G. Then R is not σ-soluble and G/R is soluble.* Assume that R is σ-soluble. Then R is a σ_i-group for some i. Hence, $R \leqslant O_{\sigma_1}(G)$ or $R \leqslant O_{\sigma_1'}(G)$, which contradicts claim (2) or (3). Hence, R is not σ-soluble. By the hypothesis and Lemma 2.6 (1), it is easy to see that (RH_1, H_1) satisfies the hypothesis. If $RH_1 < G$, then RH_1 is soluble by the choice of G. It follows that R is soluble, a contradiction. Hence, $G = RH_1$, and so $G/R = H_1R/R \cong H_1/(H_1 \cap R)$ is soluble since H_1 is soluble.

(5) *R is the unique minimal normal subgroup of G and $F_\sigma(G) = 1$.* This directly follows from claim (4).

(6) *Final contradiction.* Let L be any maximal subgroup of H_1. By the hypothesis, there exists a σ-permutable subgroup T of G such that L is a Hall σ_1-subgroup of T. If $T_G = 1$, then T is σ-nilpotent and σ-subnormal in G by Lemma 2.5. Then by Lemma 2.2 and claim (5), we get that $T \leqslant F_\sigma(G) = 1$. This implies that $L = 1$, and so $|G| = 2n$, where n is an odd number. It follows that G is soluble, a contradiction. Hence, $T_G \neq 1$, and so $R \leqslant T$ by claim (5). Then $T \geqslant RL$. Hence, $L = T \cap H_1 \geqslant RL \cap H_1 = (R \cap H_1)L$ for any maximal subgroup L of H_1. This implies that $R \cap H_1 \leqslant \Phi(H_1)$. Then by [21], Lemma IV.4.6, there exists a normal subgroup M of R such that R/M is a σ_1-group and $|R \cap H_1| \mid |R/M|$. It follows that $O^{\sigma_1}(R) \leqslant M$. Since $O^{\sigma_1}(R)$ char $R \trianglelefteq G$, we have $O^{\sigma_1}(R) \trianglelefteq G$, so $O^{\sigma_1}(R) = 1$ or R by claim (5). If $O^{\sigma_1}(R) = 1$, then $R \leqslant H_1$, witch contradicts claim (2). Hence $O^{\sigma_1}(R) = R$, and therefore $M = R$. Moreover, since $|R \cap H_1| \mid |R/M|$, we obtain that $R \cap H_1 = 1$. But clearly, $R \cap H_1$ is a Hall σ_1-subgroup of R. Thus, R is a σ_1'-subgroup, so $R \leqslant O_{\sigma_1'}(G) = 1$, contrary to claim (4). This completes the proof. □

P r o o f of Theorem 1.1. Assume that this is false and let G be a counterexample of minimal order. Then: (1) *G is soluble.* By Feit-Thompson theorem, we may assume that $2 \mid |G|$. Without loss of generality, we may assume that $2 \in \pi(H_1)$.

If H_1 is cyclic, then G has a cyclic Sylow 2-subgroup. Hence G is 2-nilpotent by [21], Theorem IV.2.8, and so G is soluble. If H_1 is non-cyclic, then G is soluble by Proposition 3.1.

(2) *Let R be a minimal normal subgroup of G. Then G/R is supersoluble.* It is clear that $\overline{\mathcal{H}} = \{H_1R/R, H_2R/R, \ldots, H_tR/R\}$ is a complete Hall σ-set of G/R and $H_iR/R \cong H_i/H_i \cap R$ is supersoluble. By claim (1), R is an elementary abelian p-group for some prime p. Without loss of generality, we may assume that $R \leqslant H_1$. Assume that H_1/R is non-cyclic. Then H_1 is non-cyclic. Let M/R be a maximal subgroup of H_1/R. Then M is a maximal subgroup of H_1. By the hypothesis and Lemma 2.6 (2), M/R is σ-permutably embedded in G/R. Now let M_i/R be a maximal subgroup of H_iR/R, where $i \neq 1$, and suppose that H_iR/R is non-cyclic. Then $M_i = (H_i \cap M_i)R$ is a maximal subgroup of H_iR. With the same discussion as for claim (3) in the proof of Proposition 3.1, we have that $H_i \cap M_i$ is a maximal subgroup of H_i. Then by the hypothesis and Lemma 2.6 (2), M_i/R is σ-permutably embedded in G/R. This shows that the hypothesis holds for G/R. The choice of G implies that G/R is supersoluble.

(3) *R is the unique minimal normal subgroup of G, $\Phi(G) = 1$, $C_G(R) = R$, R is an elementary abelian p-group for some prime p and $|R| > p$.* This directly follows from claims (1), (2) and [12], Theorem A.15.2.

Without loss of generality, we may assume that $p \in \pi(H_1)$. Then $R \leqslant H_1$.

(4) *Final contradiction.* Since $\Phi(G) = 1$, $R \nleqslant \Phi(H_1)$ by [21], Lemma III.3.3. Hence, there exists a maximal subgroup K of H_1 such that $H_1 = RK$. Let $E = R \cap K$. By claim (3), we have that $E \trianglelefteq H_1$. Since H_1 is supersoluble, $|R : E| = |RK : K| = |H_1 : K|$ is a prime. Hence, E is a maximal subgroup of R, and so $E \neq 1$ by claim (3). Since R is not cyclic by claim (3) and $R \leqslant H_1$, H_1 is non-cyclic. Then by the hypothesis, there exists a σ-permutable subgroup T of G such that K is a Hall σ_1-subgroup of T. If $T_G = 1$, then T is σ-nilpotent and σ-subnormal in G by Lemma 2.5. It follows from Lemma 2.2 that $T \leqslant F_\sigma(G)$. But since $R \leqslant F_\sigma(G)$ and $C_G(R) = R$, we have that T is a σ_1-group by Lemma 2.3, so $T = K$. It follows from Lemma 2.7 that $O^{\sigma_1}(G) \leqslant N_G(K)$. Hence $O^{\sigma_1}(G) \leqslant N_G(K \cap R) = N_G(E)$. This implies that $E \trianglelefteq G$, which contradicts the minimality of R. Hence $T_G \neq 1$. Then by claim (3), $R \leqslant T_G \leqslant T$. Consequently, $K < H_1 \leqslant T$. But K is a Hall σ_1-subgroup of T, a contradiction. This completes the proof. □

P r o o f of Theorem 1.2. Assume that this is false and let (G, E) be a counterexample with minimal $|G| + |E|$. Then: (1) *E is supersoluble.* It is clear that $\mathcal{H}^* = \{H_1 \cap E, H_2 \cap E, \ldots, H_t \cap E\}$ is a complete Hall σ-set of E, $H_i \cap E$ is nilpotent and E is a σ-full group of Sylow type. By Lemma 2.6 (1) and Theorem 1.1, we get that E is supersoluble.

(2) *Let R be a minimal normal subgroup of G contained in E. Then R is an elementary abelian p-group for some prime p, E/R is hypercyclically embedded in G/R and R is non-cyclic.* By claim (1), R is an elementary abelian p-group for some prime p. Without loss of generality, we may assume that $R \leqslant H_1$. Clearly, $\overline{\mathcal{H}} = \{H_1/R, H_2R/R, \ldots, H_tR/R\}$ is a complete Hall σ-set of G/R and $H_iR/R \cong H_i/(H_i \cap R)$ is nilpotent. Assume that $(H_1/R) \cap (E/R)$ is non-cyclic. Then $H_1 \cap E$ is non-cyclic. Let M/R be a maximal subgroup of $(H_1/R) \cap (E/R)$. Then M is a maximal subgroup of $H_1 \cap E$. Hence, M/R is σ-permutably embedded in G/R by the hypothesis and Lemma 2.6 (2). Now assume that M_i/R is a maximal subgroup of some non-cyclic $(H_iR/R) \cap (E/R)$, where $i \neq 1$. Then $H_iR \cap E$ is non-cyclic and $M_i = (H_i \cap M_i)R$ is a maximal subgroup of $H_iR \cap E$. With the same discussion as for claim (3) in the proof of Proposition 3.1, we have that $H_i \cap M_i$ is a maximal subgroup of $H_i \cap E$. Then by the hypothesis and Lemma 2.6 (2), M_i/R is σ-permutably embedded in G/R. This shows that $(G/R, E/R)$ satisfies the hypothesis. Hence E/R is hypercyclically embedded in G/R by the choice of (G, E).

(3) *R is the unique minimal normal subgroup of G contained in E.* Let L be a minimal normal subgroup of G contained in E such that $L \neq R$. Then E/L is also hypercyclically embedded in G/L by claim (2). It follows that RL/L is hypercyclically embedded in G/L. Then $|R| = p$ for $RL/L \cong R$, contrary to claim (2). Hence we have (3).

Without loss of generality, we may assume that $p \in \pi(H_1)$.

(4) *E is a p-group, and so $E \leqslant H_1$.* Let Q be a Sylow q-subgroup of E, where q is the largest prime belong to $\pi(E)$. Since E is supersoluble by claim (1), we obtain that Q char $E \trianglelefteq G$ and so $Q \trianglelefteq G$. Hence, $R \leqslant Q$, $p = q$ and $F(E) = Q$ is a Sylow p-subgroup of E by claim (3). It follows from [14], Theorem 1.8.18, that $C_E(Q) \leqslant Q$. Moreover, since $Q \leqslant H_1 \cap E$ and H_1 is nilpotent, we obtain that $Q = H_1 \cap E$. Hence, $H_1 \cap Q = Q = H_1 \cap E$ and $H_i \cap Q = 1$ for all $i = 2, \ldots, t$. This implies the hypothesis holds for (G, Q). Assume that $Q < E$. Then Q is hypercyclically embedded in G by the choice of (G, E). It follows that R is hypercyclically embedded in G, so R is cyclic, contrary to claim (2). Hence $E = Q$ is a p-group, and so $E \leqslant H_1$.

(5) *$\Phi(E) = 1$, so E is an elementary abelian p-group.* Assume that $\Phi(E) \neq 1$. Then $R \leqslant \Phi(E)$ by claim (3). Hence $E/\Phi(E)$ is hypercyclically embedded in $G/\Phi(E)$ by claim (2) and [15], Theorem 1.2.6 (d). It follows from claim (4) and Lemma 2.9 that E is hypercyclically embedded in G. This contradiction shows that (5) holds.

(6) *Final contradiction.* Let R_1 be a maximal subgroup of R such that $R_1 \trianglelefteq H_1$. Then $|R_1| > 1$ by claim (3). By claim (5), there exists a complement S of R in E (maybe $S = 1$). Let $V = R_1S$. Then clearly $R_1 = R \cap V$ and V is a maximal subgroup of E. By the hypothesis and claims (2)–(5), there exists a σ-permutable subgroup T of G such that V is a Hall σ_1-subgroup of T. We show that V is also

σ-permutable in G. Let L be a Hall σ_i-subgroup of G. If $i = 1$, then $V \leqslant L$ by claim (4). This implies that $VL = LV$. If $i \neq 1$, then V is a Hall σ_1-subgroup of $TL = LT$. (Note that since T is σ-permutable in G, $TL = LT$.) Since E is normal in G, V is subnormal in G by claim (4), so V is also subnormal in TL. But as V is a Hall σ_1-subgroup of T, V is normal in TL. Hence $VL = LV$. This implies that V is σ-permutable in G. Then by Lemma 2.7, $O^{\sigma_1}(G) \leqslant N_G(V)$, and so $O^{\sigma_1}(G) \leqslant N_G(V \cap R) = N_G(R_1)$. Moreover, since $R_1 \trianglelefteq H_1$, we obtain that $R_1 \trianglelefteq G$. This contradiction completes the proof. □

4. Proof of Theorem 1.3

In order to prove Theorem 1.3, we first prove the following:

Lemma 4.1. *Let G be a σ-full group of Sylow type and $\mathcal{H} = \{H_1, H_2, \ldots, H_t\}$ be a complete Hall σ-set of G such that H_i is a supersoluble σ_i-subgroup for all $i = 1, \ldots, t$. Let P be a normal p-group of G and $P \leqslant H_j$ for some j. If every cyclic subgroup H of P of prime order and order 4 (if P is a non-abelian 2-group and $H \nleqslant Z_\infty(G)$) is σ-permutably embedded in G, then P is hypercyclically embedded in G.*

Proof. Assume that this is false and let (G, P) be a counterexample with minimal $|G| + |P|$. Without loss of generality, we may assume that $j = 1$.

(1) *Let P/N be a chief factor of G. Then N is hypercyclically embedded in G. Hence, N is a unique normal subgroup of G such that P/N is a chief factor of G and $|P/N| > p$.*

It is clear that (G, N) satisfies the hypothesis. Hence, N is hypercyclically embedded in G by the choice of (G, P). Assume that G has another normal subgroup $R \neq N$ such that P/R is a chief factor of G. Then R is also hypercyclically embedded in G. It follows that $P/N = RN/N$ is hypercyclically embedded in G/N. Hence, P is hypercyclically embedded in G. This contradiction shows that N is a unique normal subgroup such that P/N is a chief factor of G. It is also clear that $|P/N| > p$.

(2) *The exponent of P is p or 4 (if P is a non-abelian 2-group).* Let C be a Thompson critical subgroup of P (see [13], page 185). If $\Omega(C) < P$, then $\Omega(C) \leqslant N$ is hypercyclically embedded in G by claim (1). Hence, by Lemma 2.9, P is hypercyclically embedded in G, a contradiction. Hence, $\Omega(C) = P$, so by Lemma 2.8, the exponent of P is p or 4 (if P is a non-abelian 2-group).

(3) *Final contradiction.* Since H_1/N is supersoluble and $|P/N| > p$, H_1/N has a minimal normal subgroup L/N such that $1 \neq L/N < P/N$ and L/N is cyclic. Let $x \in L \setminus N$ and $H = \langle x \rangle$. Then $L = HN$ and $|H| = p$ or 4 (if P is a non-abelian 2-group)

by claim (2). If $H \leqslant Z_\infty(G)$, then $L/N = HN/N \leqslant Z_\infty(G)N/N \leqslant Z_\infty(G/N)$ by [15], Theorem 1.2.6 (d). So $Z_\infty(G/N) \cap P/N \neq 1$. Hence, $P/N \leqslant Z_\infty(G/N)$ since P/N is a chief factor of G. It follows from claim (1) that P is hypercyclically embedded in G. This contradiction shows that $H \nleqslant Z_\infty(G)$. Then by the hypothesis, there exists a σ-permutable subgroup T of G such that H is a Hall σ_1-subgroup of T. With a similar argument as for claim (6) in the proof of Theorem 1.2, we have that H is σ-permutable in G. Then HN/N is σ-permutable in G/N by Lemma 2.4 (1). Hence, $O^{\sigma_1}(G/N) \leqslant N_{G/N}(HN/N)$ by Lemma 2.7. Moreover, since $L/N \trianglelefteq H_1/N$, we obtain that $HN/N = L/N \trianglelefteq G/N$, and so $L \trianglelefteq G$. This contradiction completes the proof. $\square$

P r o o f of Theorem 1.3. Assume that this is false and let (G, E) be a counterexample with minimal $|G| + |E|$. Let P be a Sylow p-subgroup of E, where p is the smallest prime contained in $\pi(E)$. Without loss of generality, we may assume that $P \leqslant H_1 \cap E$.

(1) *$H_1 \cap E$ is non-cyclic.* Assume that $H_1 \cap E$ is cyclic. Then P is cyclic. By [21], Theorem IV.2.8, E is p-nilpotent. Let $E_{p'}$ be a normal Hall p'-subgroup of E. Then $E_{p'} \trianglelefteq G$. If $E_{p'} = 1$, then E is cyclic, so E is hypercyclically embedded in G, a contradiction. Hence $E_{p'} \neq 1$. Clearly, $H_i \cap E_{p'} = H_i \cap E$ for $i = 2, \ldots, t$. This shows the hypothesis holds for $(G, E_{p'})$, so $E_{p'}$ is hypercyclically embedded in G by the choice of (G, E). But as $E/E_{p'} \cong P$ is cyclic, it follows that E is hypercyclically embedded in G. This contradiction shows that (1) holds.

(2) *If $E = P$, then E is hypercyclically embedded in G.* This directly follows from Lemma 4.1 and claim (1).

(3) *E is not p-nilpotent.* Assume that E is p-nilpotent. Let $E_{p'}$ be a normal Hall p'-subgroup of E. Then $E_{p'} \trianglelefteq G$. By claim (2), $E_{p'} \neq 1$. Clearly, $\overline{\mathcal{H}} = \{H_1E_{p'}/E_{p'}, H_2E_{p'}/E_{p'}, \ldots, H_tE_{p'}/E_{p'}\}$ is a complete Hall σ-set of $G/E_{p'}$ and $H_iE_{p'}/E_{p'} \cong H_i/H_i \cap E_{p'}$ is supersoluble.

We claim that the hypothesis holds for $(G/E_{p'}, E/E_{p'})$. In fact, $H_iE_{p'}/E_{p'} \cap E/E_{p'} = 1$ for $i = 2, \ldots, t$ and $H_1E_{p'}/E_{p'} \cap E/E_{p'} = E/E_{p'}$. It is trivial when $E/E_{p'}$ is cyclic. We may therefore assume that $E/E_{p'}$ is non-cyclic. Let $H/E_{p'}$ be a cyclic subgroup of $E/E_{p'}$ of order p or 4 (if the Sylow 2-subgroup of $E/E_{p'}$ is non-cyclic and $H/E_{p'} \nleqslant Z_\infty(G/E_{p'})$). Then by Schur-Zassenhaus theorem, $H = E_{p'} \rtimes L$ and without loss of generality, we may assume that $L \leqslant E \cap H_1$. Note that if $L \leqslant Z_\infty(G)$, then $H/E_{p'} = LE_{p'}/E_{p'} \leqslant Z_\infty(G)E_{p'}/E_{p'} \leqslant Z_\infty(G/E_{p'})$ by [15], Theorem 1.2.6 (d). Hence, L is of order p or 4 (if the Sylow 2-subgroup of E is non-cyclic and $L \nleqslant Z_\infty(G)$). Then by Lemma 2.6 (2), we see that the hypothesis holds for $(G/E_{p'}, E/E_{p'})$. Hence, $E/E_{p'}$ is hypercyclically embedded in $G/E_{p'}$ by the choice of (G, E). On the other hand, it is clear that the hypothesis holds for

$(G, E_{p'})$, so $E_{p'}$ is hypercyclically embedded in G by the choice of (G, E). Therefore E is hypercyclically embedded in G, a contradiction. Hence we have (3).

(4) *Final contradiction.* By claim (3), [21], Theorem IV.5.4, and [14], Theorem 3.4.11, E has a p-closed Schmit subgroup $S = P_1 \rtimes Q$, where P_1 is a Sylow p-subgroup of S of exponent p or 4 (if P_1 is non-abelian 2-group), Q is a Sylow q-subgroup of S for some prime $q \neq p$, $P_1/\Phi(P_1)$ is an S-chief factor, $Z_\infty(S) = \Phi(S)$ and $\Phi(S) \cap P_1 = \Phi(P_1)$.

We claim that $|P_1 : \Phi(P_1)| = p$. If $q \in \pi(H_1)$, then S is a σ_1-group, and so $S \leqslant H_1^g$ for some $g \in G$ since G is a σ-full group of Sylow type. Since H_1 is supersoluble and $P_1/\Phi(P_1)$ is an S-chief factor, $|P_1 : \Phi(P_1)| = p$. Now we consider that $q \notin \pi(H_1)$. Assume that there exists a minimal subgroup $D/\Phi(P_1)$ of $P_1/\Phi(P_1)$ such that $D/\Phi(P_1)$ is not σ-permutable in $S/\Phi(P_1)$. Let $x \in D \setminus \Phi(P_1)$ and $U = \langle x \rangle$. Then $D = U\Phi(P_1)$ and $|U| = p$ or 4 (if P_1 is non-abelian 2-group). If $U \leqslant Z_\infty(G)$, then $U \leqslant Z_\infty(S) \cap P_1 = \Phi(S) \cap P_1 = \Phi(P_1)$, a contradiction. Hence $U \nleqslant Z_\infty(G)$. Then by the hypothesis and Lemma 2.6 (1), there exists a σ-permutable subgroup T of S such that U is a Hall σ_1-subgroup of T. Let K be a Hall σ_i-subgroup of S, where $\sigma_i \cap \pi(S) \neq \emptyset$. If $i = 1$, then $K = P_1$, and so $UK = KU = P_1$. If $i \neq 1$, then U is a Hall σ_1-subgroup of $TK = KT$. But as $D < P_1 \leqslant S$ and $p \in \sigma_1$, we have that $TK < S$. Hence, TK is nilpotent, and so $U \trianglelefteq TK$. Thus $UK = KU$. This implies that U is σ-permutable in S. It follows from Lemma 2.4 (1) that $D/\Phi(P_1) = U\Phi(P_1)/\Phi(P_1)$ is σ-permutable in $S/\Phi(P_1)$. This contradiction shows that every minimal subgroup of $P_1/\Phi(P_1)$ is σ-permutable in $S/\Phi(P_1)$. Consequently, every minimal subgroup of $P_1/\Phi(P_1)$ is s-permutable in $S/\Phi(P_1)$ since $\pi(S) = \{p, q\}$ and $q \notin \pi(H_1)$. Then by [26], Lemma 2.12, we also have that $|P_1 : \Phi(P_1)| = p$. Hence, P_1 is cyclic of exponent p. This implies that P_1 is a group of order p. Since $N_S(P_1)/C_S(P_1) \lesssim \mathrm{Aut}(P_1)$ is a group of order $p - 1$ and p is the smallest prime contained in $\pi(E)$, it follows that $C_S(P_1) = N_S(P_1) = S$. Thus $Q \trianglelefteq S$. This contradiction completes the proof. □

5. Some applications of the results

It is clear that every σ-permutable subgroup of G is σ-permutably embedded in G. In the case when $\sigma = \{\{2\}, \{3\}, \ldots\}$, every normal subgroup, every normally embedded subgroup, every permutable subgroup, every permutably embedded subgroup, every s-permutable subgroup and every s-permutably embedded subgroup of G are all σ-permutably embedded in G. However, the converse is not true in general (see [19], Example 1.2). Hence, the following results directly follow from Theorem 1.1.

Corollary 5.1 ([31], Theorem 1). *If all maximal subgroups of every Sylow subgroup of G are normal in G, then G is supersoluble.*

Corollary 5.2 ([5], Theorem 4). *If all maximal subgroups of every Sylow subgroup of G are permutably embedded in G, then G is supersoluble.*

Corollary 5.3 ([31], Theorem 2). *If all maximal subgroups of every Sylow subgroup of G are s-permutable in G, then G is supersoluble.*

Corollary 5.4 ([8], Theorem 1). *If all maximal subgroups of every Sylow subgroup of G are s-permutably embedded in G, then G is supersoluble.*

By Theorems 1.2 and 1.3, we may obtain the following results.

Corollary 5.5. *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups and let E be a normal subgroup of G with $G/E \in \mathfrak{F}$. Suppose that G is a σ-full group of Sylow type and $\mathcal{H} = \{H_1, H_2, \dots, H_t\}$ is a complete Hall σ-set of G such that H_i is a nilpotent σ_i-subgroup for all $i = 1, \dots, t$. If every maximal subgroup of any non-cyclic $H_i \cap E$ is σ-permutably embedded in G, then $G \in \mathfrak{F}$.*

Corollary 5.6. *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups and let E be a normal subgroup of G with $G/E \in \mathfrak{F}$. Suppose that G is a σ-full group of Sylow type and $\mathcal{H} = \{H_1, H_2, \dots, H_t\}$ is a complete Hall σ-set of G such that H_i is a supersoluble σ_i-subgroup for all $i = 1, \dots, t$. If every cyclic subgroup H of any non-cyclic $H_i \cap E$ of prime order and order 4 (if the Sylow 2-subgroup of E is non-abelian and $H \nleq Z_\infty(G)$) is σ-permutably embedded in G, then $G \in \mathfrak{F}$.*

Theorems 1.2–1.3 and Corollaries 5.5–5.6 cover a lot of known results, in particular, [4], Theorem 4.1, [8], Corollary, [2], Theorem 1.3, [3], Theorem 3.3, [10], Theorem 3, [1], Theorem 3.1, [24], Theorem 3.3 and [7], Theorem 2 and Theorem 5.

Corollary 5.7. *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups and let E be a normal subgroup of G with $G/E \in \mathfrak{F}$. Suppose that G is a σ-full group of Sylow type and $\mathcal{H} = \{H_1, H_2, \dots, H_t\}$ is a complete Hall σ-set of G such that H_i is a nilpotent σ_i-subgroup for all $i = 1, \dots, t$. If every maximal subgroup of any non-cyclic $H_i \cap F^*(E)$ is σ-permutable embedded in G, then $G \in \mathfrak{F}$.*

P r o o f. By the hypothesis and Theorem 1.2, we have that $F^*(E)$ is hypercyclically embedded in G. Then E is hypercyclically embedded in G by Lemma 2.10. Therefore $G \in \mathfrak{F}$ by Lemma 2.11. □

By using a similar argument as in the proof of Corollary 5.7, we deduce the following corollary from Theorem 1.3.

Corollary 5.8. *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups and let E be a normal subgroup of G with $G/E \in \mathfrak{F}$. Suppose that G is a σ-full group of Sylow type and $\mathcal{H} = \{H_1, H_2, \ldots, H_t\}$ is a complete Hall σ-set of G such that H_i is a supersoluble σ_i-subgroup for all $i = 1, \ldots, t$. If every cyclic subgroup H of any non-cyclic $H_i \cap F^*(E)$ of prime order and order 4 (if the Sylow 2-subgroup of $F^*(E)$ is non-abelian and $H \nleq Z_\infty(G)$) is σ-permutably embedded in G, then $G \in \mathfrak{F}$.*

Corollaries 5.7 and 5.8 also cover many known results, in particular, [24], Theorem 3.1, Theorem 3.4 and Corollary 3.5, [2], Theorem 1.4, [3], Corollary 3.4, [4], Theorem 3.2 and [8], Theorem 2.

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