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Kybernetika, Vol. 54 (2018), No. 6, 1231–1246

Persistent URL: `http://dml.cz/dmlcz/147607`

KYBERNETIKA — VOLUME *54* (2018), NUMBER 6, PAGES 1231–1246

STOCHASTIC OPTIMIZATION PROBLEMS WITH SECOND ORDER STOCHASTIC DOMINANCE CONSTRAINTS VIA WASSERSTEIN METRIC

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Optimization problems with stochastic dominance constraints are helpful to many real–life applications. We can recall e. g., problems of portfolio selection or problems connected with energy production. The above mentioned constraints are very suitable because they guarantee a solution fulfilling partial order between utility functions in a given subsystem $\mathcal{U}$ of the utility functions. Especially, considering $\mathcal{U} := \mathcal{U}_1$ (where $\mathcal{U}_1$ is a system of non decreasing concave nonnegative utility functions) we obtain second order stochastic dominance constraints. Unfortunately it is also well known that these problems are rather complicated from the theoretical and the numerical point of view. Moreover, these problems goes to semi–infinite optimization problems for which Slater's condition is not necessary fulfilled. Consequently it is suitable to modify the constraints. A question arises how to do it.

The aim of the paper is to suggest one of the possibilities how to modify the original problem with an "estimation" of a gap between the original and a modified problem. To this end the stability results obtained on the base of the Wasserstein metric corresponding to $\mathcal{L}_1$ norm are employed. Moreover, we mention a scenario generation and an investigation of empirical estimates. At the end attention will be paid to heavy tailed distributions.

Keywords: stochastic programming problems, second order stochastic dominance constraints, stability, Wasserstein metric, relaxation, scenario generation, empirical estimates, light– and heavy–tailed distributions, crossing

Classification: 90C15

1. INTRODUCTION

Let $(\Omega, \mathcal{S}, P)$ be a probability space, $\xi := \xi(\omega) = (\xi_1(\omega), \ldots, \xi_s(\omega))$ an s–dimensional random vector defined on $(\Omega, \mathcal{S}, P)$, $F := F_\xi$ the distribution function of ξ, P_F, and Z_F the probability measure and the support corresponding to F, respectively. Let, moreover, $g_0,\ g : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^1$ be real-valued functions, $Y (:= Y(\xi(\omega))$ random variable defined on $(\Omega, \mathcal{S}, P)$, $X \subset \mathbb{R}^n$ a nonempty "deterministic" set; E_F denote the operator of mathematical expectation corresponding to the distribution function F.

DOI: 10.14736/kyb-2018-6-1231

If for $x \in X$ there exist finite $\mathsf{E}_F g(x, \xi)$, $\mathsf{E}_F Y(\xi)$ and if

$$F^2_{g(x,\,\xi)}(u) = \int\limits_{-\infty}^{u} F_{g(x,\,\xi)}(y)\,\mathrm{d}y, \quad F^2_Y(u) = \int\limits_{-\infty}^{u} F_Y(y)\,\mathrm{d}y, \quad u \in \mathbb{R}^1,$$

then we can define the second order stochastic dominance constraints set X_F by

$$X_F \;=\; \{x \in X : F^2_{g(x,\,\xi)}(u) \le F^2_Y(u) \quad \text{for every} \quad u \in \mathbb{R}^1\}. \tag{1}$$

Remark 1.1. Stochastic dominance of second order corresponds to order in the space of non decreasing concave nonnegative utility functions.

To define a stochastic programming problem with the second order stochastic dominance constraints we assume that there exist finite mathematical expectations $\mathsf{E}_F g(x,\,\xi)$, $\mathsf{E}_F g_0(x,\,\xi)$, $\mathsf{E}_F Y(\xi)$ for $x \in X$. The corresponding optimization problem can be then defined in the form:

$$\text{to find} \quad \varphi(F, X_F) = \inf\bigl\{\mathsf{E}_F g_0(x,\xi) | x \in X_F\bigr\}, \tag{2}$$

where

$$X_F = \{x \in X :\, \mathsf{E}_F(u - g(x,\xi))^+ \le \mathsf{E}_F(u - Y(\xi))^+ \quad \text{for every} \quad u \in \mathbb{R}^1\}. \tag{3}$$

The equivalence of the constraints (1) and (3) can be found in [8]; see also Section 3.

Evidently, a type of the problems introduced by (2), (3) is complicated as from the theoretical so from the numerical point of view. On one side the probability measure appears there in the form of the mathematical expectation but on the other side these problems belong to semi–infinite optimization problems for which Slater's condition is not fulfilled, generally. Consequently to this fact Dentcheva and Ruszczyński in [1] have suggested to relax the problem and to replace the interval $(-\infty,\,+\infty)$ in (3) by a compact interval $\langle a,\, b\rangle$, $a,\, b \in \mathbb{R}^1$. They defined new relaxed problem:

$$\text{to find} \quad \varphi^{a,\,b}(F, X_F^{a,\,b}) = \inf\bigl\{\mathsf{E}_F g_0(x,\xi) | x \in X_F^{a,\,b}\bigr\}, \tag{4}$$

where

$$X_F^{a,\,b} = \{x \in X :\, \mathsf{E}_F(u - g(x,\xi))^+ \le \mathsf{E}_F(u - Y(\xi))^+ \quad \text{for every} \quad u \in \langle a,\, b\rangle\}. \tag{5}$$

However they did not specified how to choose a, b, generally. Surely, it is desirable to determine a, b to hold a "difference" between X_F and $X_F^{a,\,b}$ small. More precisely, it is desirable to hold small the difference between

$$\inf\bigl\{\mathsf{E}_F g_0(x,\xi) |\, x \in X_F^{a,\,b}\bigr\} \quad \text{and} \quad \inf\bigl\{\mathsf{E}_F g_0(x,\xi) |\, x \in X_F\bigr\}. \tag{6}$$

It seems that it is not a problem to select a, b in the case of the known distributions with thin tails, however likely the problem can arise with heavy tailed distributions. To deal with this problem generally we employ the stability results obtained on the base of the Wasserstein metric based on $\mathcal{L}_1$ norm. Furthermore we suggest an approximation obtained by replacing the "underlying" distribution by a discrete one with finite number of atoms and the following scenario generation. We mention also the case when the problem has to be solved on the data base or on stable distributions.

The paper is organized as follows: Section 2 is devoted to an analysis and a justification of the problem relaxation in a dependence on the distribution tails. A brief survey of results from the literature can be found in Section 3 (in details, Subsection 3.1 recalls the suitable results on the stability based on the Wassertein metric, in Subsection 3.2 can be found simple auxiliary assertion, Subsection 3.3 recalls results for problems with discrete probability measure). In Section 4 first a new possible relaxation is suggested and further also a discrete approximation is introduced there. The results of this paper are summarized in Section 5. The cases when the "underlying" distribution is replaced by empirical one or when it belongs to the stable distributions are mentioned in Section 6. The text is finished by the Conclusion and a list of References.

2. PROBLEM ANALYSIS

The problem introduced by (2), (3) belongs to semi–infinite programming problems for which Slater's condition is not necessary fulfilled. Dentcheva and Ruszczyński in [1] suggested to replace constraints set (3) by constraints set (5). In this section we try to analyze the above mentioned situation in the dependence on the distribution tails.

Comparing problems (2) and (4) with constraints (3) and (5) we can see that

$$X_F \subset X_F^{a,\,b}, \quad \varphi^{a,\,b}(F,\, X_F^{a,\,b}) \leq \varphi(F,\, X_F); \quad a,\, b \in \mathbb{R}^1,\, a < b.$$

Namely, events $Y(\xi) > b,\ g(x,\, \xi) > b$ for some $x \in X$ have not to be in the relation (5), generally, included. Consequently, there are not included, maybe, events with the probability

$$P\{\omega : Y(\xi) > b \cup g(x,\, \xi) > b \quad \text{for some} \quad x \in X\}. \tag{7}$$

Analyzing the situation in more details we can see that the following equation holds

$$\begin{aligned} X_F^{a,\,b} \;=\; & \{x \in X : \mathsf{E}_F(u - g(x,\, \xi))^+ \leq \mathsf{E}_F(u - Y(\xi))^+ \quad \text{for every} \quad u \in \mathbb{R}^1\} \\ & \cup \{x \in X : \mathsf{E}_F(u - g(x,\, \xi))^+ \leq \mathsf{E}_F(u - Y(\xi))^+ \quad \text{for every} \quad u \in \langle a,\, b\rangle \text{ and,} \\ & \qquad \text{simultaneously, there exists} \quad u \in (-\infty,\, a) \cup (b,\, +\infty) \\ & \qquad \text{such that } \mathsf{E}_F(u - g(x,\, \xi))^+ > \mathsf{E}_F(u - Y(\xi))^+\}. \end{aligned} \tag{8}$$

We can also write $X_F^{\mathbb{R}^1} = X_F^{a,\,b} \bigcap X_F^{\mathbb{R}^1 - \langle a,\, b\rangle}$.

Consequently, the distribution tails of the random variables $Y(\xi),\, g(x,\, \xi)$ for $x \in X$ determine a relationship between X_F and $X_F^{a,\,b}$. Since the above mentioned tails depend generally also on the "underlying" distribution of ξ it is suitable also to recall the tails for different F_ξ. To this end we summarize the corresponding quantiles in two tables.

Table 1 presents quantiles of the stable distributions $S_\alpha(\sigma,\, \beta,\, \mu)$ (for the definition of $S_\alpha(\sigma,\, \beta,\, \mu)$ see, e. g., [5, 9, 11]).

α	$F_{S_\alpha(1,0,0)}(q_{95})$	$F_{S_\alpha(1,0,0)}(q_{99})$	$F_{S_\alpha(1,0,0)}(q_{99.5})$	$F_{S_\alpha(1,0,0)}(q_{99.99})$
1.05	0.876452	0.912226	0.920843	0.953008
1.10	0.881909	0.918085	0.926646	0.957888
1.20	0.892181	0.928994	0.937373	0.966491
1.30	0.901660	0.938941	0.947056	0.973743
1.40	0.910409	0.948050	0.955829	0.979842
1.50	0.918477	0.956428	0.963812	0.984956
1.60	0.925903	0.964165	0.971108	0.989225
1.70	0.932727	0.971336	0.9707805	0.992769
1.80	0.938988	0.978001	0.983979	0.995689
1.90	0.944732	0.984210	0.989694	0.998072
2.00	0.950000	0.990000	0.995000	0.999990

Tab. 1. $S_\alpha(\sigma, \beta, \mu)$.

A comparison of quantiles for some heavy tailed distributions (not stable) and the normal distribution are introduced in Table 2. To this end we employ normal, Weibul, Pareto and lognormal distributions (for their definitions see, e. g., [6]). We denote:

1. N – Normal distribution,

2. W – Weibull distribution with probability density

$$f(z) = \begin{array}{ll} \frac{c}{\nu}(\frac{z-z_0}{\nu})^{c-1}\exp\{-(z-z_{0)}/\nu)^c & \text{for} \quad z > z_0, \\ 0 & \text{for} \quad z \le z_0; \quad c > 0,\, \nu > 0,\, z_0, \end{array}$$

3. P – Pareto distribution with probability density

$$f(z) = \begin{array}{ll} \alpha C^\alpha z^{-\alpha-1} & \text{for} \quad z \ge C, \\ 0 & \text{for} \quad z < C; \quad C > 0,\, \alpha > 0, \end{array}$$

4. L – Lognormal distribution.

Distr.	Quantiles											
	0.999999		0.99999		0.9999		0.999		0.99		0.9	
P	0.0	179.07	0.0	68.99	0.0	26.58	0.0	10.24	0.0	3.95	—	
W	0.0	27.63	0.0	23.03	0.0	1842	0.0	13.82	0.0	9.21	0.0	4.61
L	0.0	1455.05	0.0	739.18	0.0	346.82	0.0	145.06	0.0	50.31	0.0	11.82
N	-17.01	21.01	-15.06	9.06	-12.88	16.88	-10.36	14.36	-7.31	11.31	-3.13	7.13

Tab. 2. The value in the table are calculated for a mean value 2 and a variance 4; the numerical results have been obtained by K. Odintsov [7] and V. Omelčenko [3].

Evidently, it follows from the relations (7), (8), Table 1 and Table 2 that it is suitable to choose a, b with respect to the "underlying" distribution function F and the functions g_0, g, Y. To this end, first, we recall the stability results based on the Wasserstein metric and $\mathcal{L}_1$ norm.

3. A BRIEF SURVEY OF DEFINITIONS, FORMER RESULTS AND AUXILIARY ASSERTIONS

3.1. Wasserstein Metric

The problem introduced by (2), (3) depends on the distribution function F. Replacing F by another s–dimensional distribution function G (for which the problem is well defined) we obtain a modified problem. Employing triangular inequality we have

$$|\varphi(F, X_F) - \varphi(G, X_G)| \le |\varphi(F, X_F) - \varphi(G, X_F)| + |\varphi(G, X_F) - \varphi(G, X_G)|. \qquad (9)$$

To recall the first auxiliary assertion based on the Wasserstein metric, let $\mathcal{P}(\mathbb{R}^s)$ denote the set of all (Borel) probability measures on $\mathbb{R}^s$ and let the system $\mathcal{M}_1^1(\mathbb{R}^s)$ be defined by the relation:

$$\mathcal{M}_1^1(\mathbb{R}^s) := \left\{ \nu \in \mathcal{P}(\mathbb{R}^s) : \int_{\mathbb{R}^s} \|z\|_1 \, \mathrm{d}\nu(z) < \infty \right\}, \quad \|\cdot\|_1^s := \|\cdot\|_1 \text{ denotes } \mathcal{L}_1 \text{ norm in } \mathbb{R}^s. \qquad (10)$$

If the assumptions **A.0**, **A.1** are defined by

A.0 $g_0(x, z)$ is for $x \in X$ a Lipschitz function of $z \in \mathbb{R}^s$ with the Lipschitz constant L (corresponding to the $\mathcal{L}_1$ norm) not depending on x,

A.1 $g_0(x, z)$ is either a uniformly continuous function on $X \times \mathbb{R}^s$ or there exists $\varepsilon > 0$ such that $g_o(x, z)$ is a convex function on $X(\varepsilon)$ and bounded on $X(\varepsilon) \times \mathbb{R}^s$ ($X(\varepsilon)$ denotes the ε–neighborhood of the set X),

and if F_i, $G_i, i = 1, \ldots, s$ denote one-dimensional marginal distribution functions corresponding to F and G, then the following assertion has been proven.

Proposition 3.1. (Kaňková and Houda [2]) Let $P_F, P_G \in \mathcal{M}_1^1(\mathbb{R}^s)$. If Assumption **A.0** is fulfilled, then

$$|\mathsf{E}_F g_0(x, \xi) - \mathsf{E}_G g_0(x, \xi)| \le L \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)| \, \mathrm{d}z_i \quad \text{for} \quad x \in X. \qquad (11)$$

If, moreover, X is a compact set and Assumption **A.1** is fulfilled, then also

$$|\inf_{x \in X} \mathsf{E}_F g_0(x, \xi) - \inf_{x \in X} \mathsf{E}_G g_0(x, \xi)| \le L \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)| \, \mathrm{d}z_i. \qquad (12)$$

Lemma 3.2. (Kaňková [4]) Let $g(x, z)$, $Y(z)$ be for every $x \in X$ Lipschitz functions of $z \in \mathbb{R}^s$ with the Lipschitz constant L_g not depending on $x \in X$. Let, moreover, $P_F \in \mathcal{M}_1^1(\mathbb{R}^s)$. If X_F is defined by the relation (1), then

1.
$$X_F = \{x \in X : \mathsf{E}_F(u - g(x, \xi))^+ \leq \mathsf{E}_F(u - Y(\xi))^+ \quad \text{for every} \quad u \in \mathbb{R}^1\},$$

2.
$$(u - g(x, z))^+, \ (u - Y(z))^+, \quad u \in \mathbb{R}^1, \, x \in \mathbb{R}^n$$
are Lipschitz functions of $z \in \mathbb{R}^s$ with the Lipschitz constant L_g not depending on $u \in \mathbb{R}^1$, $x \in \mathbb{R}^n$.

(The results of [8] has been employed to verify Lemma 3.2.)

Investigating the problem defined by (2), (3) it can be reasonable to define for $\varepsilon \in \mathbb{R}^1$ the sets X_F^{ε}

$$X_F^{\varepsilon} = \{x \in X : \mathsf{E}_F(u - g(x, \xi))^+ - \mathsf{E}_F(u - Y(\xi))^+ \leq \varepsilon \quad \text{for every} \quad u \in \mathbb{R}^1\}, \quad \varepsilon \in \mathbb{R}^1 \tag{13}$$

(for more details see, e. g., [2] or [4]). Evidently $X_F^0 = X_F$.

If the assumptions of Lemma 3.2 are fulfilled, P_F, $P_G \in \mathcal{M}_1^1(\mathbb{R}^s)$, $u \in \mathbb{R}^1$, $x \in X$, then it follows from Proposition 3.1 that

$$\begin{aligned} |\mathsf{E}_F(u - g(x, \xi))^+ - \mathsf{E}_G(u - g(x, \xi))^+| &\leq L_g \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)| \, \mathrm{d}z_i, \\ |\mathsf{E}_F(u - Y(\xi))^+ - \mathsf{E}_G(u - Y(\xi))^+| &\leq L_g \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)| \, \mathrm{d}z_i. \end{aligned} \tag{14}$$

Consequently

$$x \in X_F \Longrightarrow x \in X_G^{\varepsilon}, \quad x \in X_G \Longrightarrow x \in X_F^{\varepsilon} \quad \text{with} \quad \varepsilon = 2L_g \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)| \, \mathrm{d}z_i.$$

Generally

$$X_G^{\delta - \varepsilon} \subset X_F^{\delta} \subset X_G^{\delta + \varepsilon} \quad \text{for} \quad \delta \in \mathbb{R}^1. \tag{15}$$

Lemma 3.3. (Kaňková [4]) Let X be a nonempty compact set, P_F, $P_G \in \mathcal{M}_1^1(\mathbb{R}^s)$, Assumption A.1 be fulfilled. Let, moreover, $g(x, z)$, $Y(z)$ be for every $x \in X$ Lipschitz functions of $z \in Z_F \cup Z_G$ with the Lipschitz constant L_g not depending on $x \in X$. If

1. $g_0(x, z)$ is a Lipschitz function on X with the Lipschitz constant $\bar{L}$ not depending on $z \in Z_F \cup Z_G$,

2. X_F, X_G defined by (2) or (4) are nonempty compact sets,

3. there exists a constant $D > 0$ such that

$$\Delta[X_F^{\varepsilon'}, X_F^{\varepsilon''}] \le D\varepsilon \quad \text{for every} \quad \varepsilon', \varepsilon'' \in \langle -3\varepsilon, 3\varepsilon\rangle,$$

   with $\varepsilon = 2L_g \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)| \, \mathrm{d}z_i$,

then

$$|\inf_{x\in X_F} \mathsf{E}_F g_0(x, \xi) - \inf_{x\in X_G} \mathsf{E}_F g_0(x, \xi)| \le 2D\bar{L}L_g \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)| \, \mathrm{d}z_i. \qquad (16)$$

($\Delta[\cdot, \cdot] := \Delta_n[\cdot, \cdot]$ denotes the Hausdorff distance in the subsets of n-dimensional Euclidean space $\mathbb{R}^n$; for more details see, e. g., [10].)

Proposition 3.4. (Kaňková [4]) Let X be a nonempty compact set, $P_F, P_G \in \mathcal{M}_1^1(\mathbb{R}^s)$, Assumptions **A.0**, **A.1** be fulfilled. Let, moreover, $g(x, z)$, $Y(z)$ be for every $x \in X$ Lipschitz functions of $z \in Z_F \cup Z_G$ with the Lipschitz constant L_g not depending on $x \in X$. If

1. $g_0(x, z)$ is a Lipschitz function on X with the Lipschitz constant $\bar{L}$ not depending on $z \in Z_F \cup Z_G$,

2. X_F, X_G, defined by (3) are nonempty compact sets,

3. there exists a constant $D > 0$ such that

$$\Delta[X_F^{\varepsilon'}, X_F^{\varepsilon''}] \le D\varepsilon \quad \text{for every} \quad \varepsilon', \varepsilon'' \in \langle -3\varepsilon, 3\varepsilon\rangle$$

   with $\varepsilon = 2L_g \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)| \, \mathrm{d}z_i$,

then

$$|\inf_{x\in X_F} \mathsf{E}_F g_0(x, \xi) - \inf_{x\in X_G} \mathsf{E}_G g_0(x, \xi)| \le (2D\bar{L}L_g + L) \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)| \, \mathrm{d}z_i. \quad (17)$$

(Proposition 3.1, Lemma 3.3 and the relation (9) have been employed to prove Proposition 3.4.)

3.2. Simple auxiliary assertion

We prove one simple assertion. To this end let $\zeta = \zeta(\omega)$ be a random variable defined on $(\Omega, \mathcal{S}, P)$ and let its corresponding support Z_ζ fulfils the condition $Z_\zeta \subset \langle a,\, b\rangle,\ a,\, b \in R^1$, $a < b$, then

$$\begin{array}{llll} u < a & \Longrightarrow & (u-\zeta)^+ = 0 & \text{almost surely,} \\ u \in \langle a,\, b\rangle & \Longrightarrow & (u-\zeta)^+ = u - \zeta & \text{for} \quad \zeta < u, \\ & & 0 & \text{for} \quad \zeta \geq u, \\ u \geq b & \Longrightarrow & (u-\zeta)^+ = u - \zeta & \text{almost surely,} \\ u \geq b,\, u' > 0 & \Longrightarrow & (u + u' - \zeta)^+ = u - \zeta + u' & \text{almost surely.} \end{array}$$

Consequently

$$\begin{array}{lll} u < a & \Longrightarrow & \mathsf{E}_{F_\zeta}(u-\zeta)^+ = 0, \\ u \in \langle a,\, b\rangle & \Longrightarrow & \mathsf{E}_{F_\zeta}(u-\zeta)^+ = \displaystyle\int\limits_a^u (u-z)\,\mathrm{d}F_\zeta(z), \\ u \geq b,\, u' \geq 0 & \Longrightarrow & \mathsf{E}_{F_\zeta}(u+u'-\zeta)^+ = \mathsf{E}_{F_\zeta}(u-\zeta) + u', \\ . & & \mathsf{E}_{F_\zeta}(u+u'-\zeta)^+ = \mathsf{E}_{F_\zeta}(b-\zeta)^+ + u + u^{'} - b. \end{array}$$

Evidently, setting successively $\zeta := Y(\xi)$; $\zeta := g(x,\, \xi))$, $x \in X$ and supposing that almost surely $Y(\xi) \in \langle a,\, b\rangle$, $g(x,\, \xi) \in \langle a,\, b\rangle$, $x \in X$ we can obtain

$$\begin{aligned} &\{x \in X : \mathsf{E}_{F_\xi}(u - g(x,\, \xi))^+ \leq \mathsf{E}_{F_\xi}(u - Y(\xi))^+ \quad \text{for} \quad u \in \mathbb{R}^1\} \\ &= \{x \in X : \mathsf{E}_{F_\xi}(u - g(x,\, \xi))^+ \leq \mathsf{E}_{F_\xi}(u - Y(\xi))^+ \quad \text{for} \quad u \leq a\} \\ &\cap \{x \in X : \mathsf{E}_{F_\xi}(u - g(x,\, \xi))^+ \leq \mathsf{E}_{F_\xi}(u - Y(\xi))^+ \quad \text{for} \quad u \in \langle a,\, b\rangle\} \\ &\cap \{x \in X : \mathsf{E}_{F_\xi}(b - g(x,\, \xi))^+ \leq \mathsf{E}_{F_\xi}(b - Y(\xi))^+ \quad \text{for} \quad u > b\} \\ &= \{x \in X : \mathsf{E}_{F_\xi}(u - g(x,\, \xi))^+ \leq \mathsf{E}_{F_\xi}(u - Y(\xi))^+ \quad \text{for} \quad u \in \langle a,\, b\rangle\}. \end{aligned}$$

We have proven the next assertion.

Lemma 3.5. Let $a,\, b \in \mathbb{R}^1$, $a < b$. If $Y(\xi)$, $g(x,\, \xi)$, $x \in X$ are random variables such that $Y(\xi) \in \langle a,\, b\rangle$, $g(x,\, \xi) \in \langle a,\, b\rangle$, $x \in X$ almost surely, then we can set

$$X_F^{a,\, b} = \{x \in X : \mathsf{E}_{F_\xi}(u - g(x,\, \xi))^+ \leq \mathsf{E}_{F_\xi}(u - Y(\xi))^+ \quad \text{for} \quad u \in \langle a,\, b\rangle\}.$$

Remark 3.6. Evidently, to define modified constraints set $X_F^{a,\,b}$ (introduced in Lemma 3.5) we have generally to modify, first, the corresponding distribution functions.

3.3. Second order stochastic dominance constraints via discrete distribution

In this subsection we recall one very suitable assertion proven by Dentcheva and Ruszczyński.

Proposition 3.7. (Dentcheva and Ruszczyński [1]) Let $\bar{Y} := \bar{Y}(\xi)$ be a random variable defined on $(\Omega, \mathcal{S}, P)$. Let, moreover, $Y(\xi)$ has a discrete distribution with realizations y_i, $i = 1, \ldots, m$, where $a \le y_i \le b$, $a, b \in \mathbb{R}^1$ for all i. Then the inequality

$$\mathsf{E}_{F_{\bar{Y}}}(u - \bar{Y}(\xi))^+ \le \mathsf{E}_{F_Y}(u - Y(\xi))^+ \quad \text{for all} \quad u \in \langle a,\, b\rangle$$

is equivalent to

$$\mathsf{E}_{F_{\bar{Y}}}(y_i - \bar{Y})^+ \le \mathsf{E}_{F_Y}(y_i - Y)^+ \quad \text{for} \quad i = 1, \ldots, m.$$

Furthermore, evidently, if $Y(\xi)$ is a general random variable defined on $(\Omega, \mathcal{S}, P)$ $(\xi := \xi(\omega))$ and if $a < b,\ a, b \in \mathbb{R}^1$, then we can define a random variable $Y^{a,\,b} := Y^{a,\,b}(\xi)$ by

$$\begin{aligned} Y^{a,\,b}(\xi) = \quad & Y(a) \quad \text{if} \quad Y(\xi) \le a, \\ & Y(\xi) \quad \text{if} \quad Y(\xi) \in (a,\, b), \\ & Y(b) \quad \text{if} \quad Y(\xi) \ge b \end{aligned} \tag{18}$$

and note the corresponding distribution function by $F_Y^{a,\,b}$.

4. APPROACH TO DEFINITION OF RELAX PROBLEMS AND DISCRETE APPROXIMATION

$Y(\xi)$, $g(x,\, \xi)$, $x \in X$ are functions of the random vector $\xi = (\xi_1, \ldots, \xi_s)$ and, simultaneously, they are functions of components $\xi_1, \ldots, \xi_s$. Consequently, it is often fulfilled the following assumption:

C.1 for $a_1,\, b_1 \in \mathbb{R}^1$, $a_1 < b_1$ there exist bounds $a,\, b \in \mathbb{R}^1$, $a < b$ such that, employing (18) $a_1,\, b_1$ defines random variables $\xi_i^{a_1,\, b_1} := \xi_i^{a_1,\, b_1}(\xi)$, $i = 1, \ldots, s$ with the supports being generally subset of $\langle a_1,\, b_1\rangle$ and one–dimensional distributions $F_i^{a_1,\, b_1}$; simultaneously there exists a distribution function $F^{a_1,\, b_1} := F_\xi^{a,\, b} = F^{a,\, b}$ with the support being (generally) subset of $\prod_{i=1}^{s}\langle a_1,\, b_1\rangle$; moreover it holds

$$\xi \in \prod_{i=1}^{s}\langle a_1,\, b_1\rangle \Longrightarrow a < Y(\xi) < b,\, a < g(x,\, \xi) < b \quad \text{for} \quad x \in X.$$

Consequently (if the assumption C.1 is fulfilled) we can define the following optimization problems:

$$\text{to find} \quad \varphi^{a_1, b_1}(F, X^{a_1, b_1}_{F^{a_1, b_1}}) = \inf\{\mathsf{E}_F g_0(x, \xi) | x \in X^{a_1, b_1}_F\}, \tag{19}$$

$$\text{to find} \quad \varphi^{a_1, b_1}(F^{a_1, b_1}, X^{a_1, b_1}_F) = \inf\{\mathsf{E}_{F^{a_1, b_1}} g_0(x, \xi) | x \in X^{a_1, b_1}_F\}, \tag{20}$$

where

$$X^{a, b}_F := X^{a_1, b_1}_F = \{x \in X : \mathsf{E}_{F^{a_1, b_1}}(u - g(x, \xi))^+ \le \mathsf{E}_{F^{a_1, b_1}}(u - Y(\xi))^+ \text{ for every } u \in \langle a, b\rangle\}. \tag{21}$$

According to (15), if $g(x, z)$, $Y(z)$ (for every $x \in X$) are Lipschitz functions of $z \in \mathbb{R}^s$ with the Lipschitz constant L_g not depending on $x \in X$, then we can obtain

$$X^{\delta - \varepsilon}_{F^{a_1, b_1}} \subset X^{\delta}_F \subset X^{\delta + \varepsilon}_{F^{a_1, b_1}} \text{ with } \varepsilon = 2L_g \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - F^{a_1, b_1}_i(z_i)| \, \mathrm{d}z_i, \ \delta \in \mathbb{R}^1; X^0_F = X^{a_1, b_1}_F. \tag{22}$$

F^{a_1, b_1}_i, $i = 1, \ldots, s$ can be considered as one- dimensional "marginal distribution functions" corresponding to F^{a_1, b_1}.

To deal with a discrete approximation we define the following assumption:

C.2 To a natural number m, points $y_1, \ldots, y_m \in \prod_{i=1}^{s} \langle a_1, b_1\rangle$ and $F^{a,b}_Y$ we can set a discrete distribution function $\bar{F}^{a,b; m}_Y$ with atoms $y_1, \ldots, y_m$; $\bar{F}^{a,b;m}_Y$ approximates $F^{a,b}_Y$.

According to the condition **C.2** it is possible to define discrete distribution function $\bar{F}^{a, b; m}_Y$ corresponding to discrete modification of the random variable $Y^{a, b}$ and, moreover, according to the condition **C.1** it is easy to see that there exists the "underlying discrete" distribution function $\bar{F}^{a_1, b_1; m}$ having a support $\bar{Z}^{a_1, b_1; m} \subset \prod_{i=1}^{s} \langle a_1, b_1\rangle$. So we can define the optimization problems:

$$\text{to find} \quad \bar{\varphi}^{a_1, b_1; m}(F_\xi, \bar{X}^{a_1, b_1; m}_{\bar{F}}) = \inf\{\mathsf{E}_{F_\xi} g_0(x, \xi) | x \in \bar{X}^{a_1, b_1; m}_{\bar{F}}\}, \tag{23}$$

$$\text{to find} \quad \bar{\varphi}^{a_1, b_1; m}(F^{a_1, b_1}, \bar{X}^{a_1, b_1; m}_{\bar{F}}) = \inf\{\mathsf{E}_{F^{a_1, b_1}} g_0(x, \xi) | x \in \bar{X}_{\bar{F}^{a_1, b_1; m}}\}, \tag{24}$$

$$\text{to find} \quad \bar{\varphi}^{a_1, b_1; m}(\bar{F}^{a_1, b_1; m}, \bar{X}_{\bar{F}^{a_1, b_1; m}} = \inf\{\mathsf{E}_{\bar{F}^{a_1, b_1; m}} g_0(x, \xi) | x \in \bar{X}^{a_1, b_1; m}_{\bar{F}})\}, \tag{25}$$

where

$$\bar{X}^{a_1, b_1; m}_{\bar{F}} = \{x \in X : E_{F^{a_1, b_1}}(y_i - g(x, \xi))^+ \le \mathsf{E}_{\bar{F}^{a_1, b_1; m}}(y_i - Y(\xi))^+ \text{ for every } i = i, \ldots, m\}. \tag{26}$$

Remark 4.1.

- To obtain the relation (26) we have employed the assertion of Proposition 3.7.
- Employing Assumption **C.2** we can see that the constraints set given by (26) is simpler then the constraints sets X_F, X^{a_1, b_1}_F. Evidently, (25) can be considered as a "discrete approximation" of the original problem.

5. MAIN RESULTS

Now already we can summarize the former analysis, auxiliary assertions and to introduce the properties of the relax problem with an estimation of the relax gap. To this end we suppose: the relax problem is defined by the relation (19) with the constraints set given by (21).

Theorem 5.1. Let Assumptions **A.1**, **C.1** be fulfilled. Let moreover X_F, $X_F^{a,\,b}$ be nonempty compact sets, $P_F \in \mathcal{M}_1^1(\mathbb{R}^s)$. If

1. $g_0(x,\, z)$ is for every $z \in Z_F$ a Lipschitz function of $x \in X$ with the Lipschitz constant $\bar{L}$ not depending on $z \in Z_F$,
2. $g(x, z),\ Y(z)$ are for every $x \in X$ Lipschitz functions of $z \in \mathbb{R}^s$ with the Lipschitz constant L_g not depending on $x \in X$,
3. there exists a constant $D > 0$ such that

$$\Delta[X_F^{\varepsilon'}, X_F^{\varepsilon''}] \le D\varepsilon \text{ for every } \varepsilon', \varepsilon'' \in \langle -3\varepsilon, 3\varepsilon \rangle,$$

with $\varepsilon = 2L_g \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - F_i^{a_1,\,b_1}(z_i)|\, \mathrm{d}z_i$,

then

1. $$\varphi(F, X_F) - \varphi^{a_1,b_1}(F, X_F^{a_1,b_1})| \le 2D\bar{L}L_g \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - F_i^{a_1,b_1}(z_i)|\, \mathrm{d}z_i. \quad (27)$$

2. If, moreover, Assumption **A.0** is fulfilled, then

$$|\varphi(F, X_F) - \varphi^{a_1,b_1}(F^{a_1,b_1}, X_F^{a_1,b_1})| \le (2L + 2D\bar{L}L_g) \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - F_i^{a_1,b_1}(z_i)|\, \mathrm{d}z_i. \quad (28)$$

P r o o f. To prove the assertions of Theorem 5.1 we employ the assertion of Lemma 3.3 and we set $G := F^{a_1,\,b_1}$ there. Now already we can see that the assertion 1 holds. Replacing Lemma 3.3 by Proposition 3.4 we can see that the assertion 2 of Theorem 5.1 holds also. □

Theorem 5.1 suggests (under rather general conditions) a possibility to define the relax problem to (2), (3). Employing the relation (27) we can see that the following Corollary is valid.

Corollary 5.2 Let Assumptions **A.1**, **C.1** be fulfilled. Let moreover X_F, $X_F^{a,\,b}$ be nonempty compact sets, $P_F \in \mathcal{M}_1^1(\mathbb{R}^s)$, $\kappa > 0$. If

1. $g_0(x,\, z)$ is for every $z \in Z_F$ a Lipschitz function of $x \in X$ with the Lipschitz constant $\bar{L}$ not depending on $z \in Z_F$,

2. $g(x, z)$, $Y(z)$ are for every $x \in X$ a Lipschitz functions of $z \in \mathbb{R}^s$ with the Lipschitz constant L_g not depending on $x \in X$,

3. there exists a constant $D > 0$ such that

$$\Delta[X_F^{\varepsilon'}, X_F^{\varepsilon''}] \le D\varepsilon \quad \text{for every} \quad \varepsilon', \varepsilon'' \in \langle -3\varepsilon, 3\varepsilon \rangle,$$

with $\varepsilon = 2L_g \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - F_i^{a_1, b_1}(z_i)| \, \mathrm{d}z_i$.

Then for a_1, b_1 such that

$$2D\bar{L}L_g \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - F_i^{a_1, b_1}(z_i)| \, \mathrm{d}z_i \le \kappa,$$

it holds for a, b that

$$|\inf\{\mathsf{E}_F g_0(x, \xi) | \, x \in X_F^{a, b}\} - \inf\{\mathsf{E}_F g_0(x, \xi) | \, x \in X_F\}| \le \kappa. \tag{29}$$

P r o o f. The assertion of Corollary 1 follows from Theorem 5.1 (especially from the relation (27)). □

Remark 5.3. Evidently, it follows from the Corollary 5.2 that employing the above mentioned approach we can obtain the relaxed problem with given relaxed gap. However to obtain this result we have to modify first the "underlying" distribution function (especially its one dimensional marginal distribution functions); for details see Lemma 3.5.

To introduce the next assertion we employ the relation (15) successively with $G := F^{a_1, b_1}$, $G := \bar{F}^{a_1, b_1; m}$ and $X_G := X_{F^{a_1, b_1}}$, $X_G := \bar{X}_F^{a_1, b_1; m}$. According to (15) if $a_1, b_1 \in \mathbb{R}^1$; $m > 0$ is a natural number, the assumptions **C.1**, **C.2** are fulfilled and if $g(x, z)$, $Y(z)$ are Lipschitz functions of $z \in \mathbb{R}^s$ (with the Lipschitz constant L_g not depending on $x \in X$), then for $\delta \in \mathbb{R}^1$ holds

$$\bar{X}^{\delta-\varepsilon}_{\bar{F}^{a_1, b_1; m}} \subset X^{\delta}_{F^{a_1, b_1}} \subset \bar{X}^{\delta+\varepsilon}_{F^{a_1, b_1}; m} \quad \text{with} \quad \varepsilon = 2L_g \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i^{a_1, b_1}(z_i) - \bar{F}_i^{a_1, b_1; m}(z_i)| \, \mathrm{d}z_i,$$

$$\bar{X}^{\delta-\varepsilon}_{\bar{F}^{a_1, b_1; m,}} \subset X^{\delta}_{F} \subset \bar{X}^{\delta+\varepsilon}_{F^{a_1, b_1; m}} \quad \text{with} \quad \varepsilon = 2L_g \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - \bar{F}_i^{a_1, b_1; m}(z_i)| \, \mathrm{d}z_i.$$

($\bar{F}_i^{a_1, b_1; m}$ are one–dimensional distribution functions corresponding to $\bar{F}^{a_1, b_1 m}$.)

Theorem 5.4. Let Assumptions **A.1**, **C.1**, **C.2** be fulfilled. Let, moreover, X_F, $X_F^{a_1, b_1}$, $\bar{X}_{\bar{F}}^{a_1, b_1; m}$ be nonempty compact sets, $P_F \in \mathcal{M}_1^1(\mathbb{R}^s)$. If

1. $g_0(x, z)$ is for every $z \in Z_F$ a Lipschitz function of $x \in X$ with the Lipschitz constant $\bar{L}$ not depending on $z \in Z_F$,

2. there exists a constant $D > 0$ such that
$$\Delta[X_F^{\varepsilon'}, X_F^{\varepsilon''}] \le D\varepsilon \quad \text{for every} \quad \varepsilon', \varepsilon'' \in \langle -3\varepsilon, 3\varepsilon\rangle,$$
with $\varepsilon = 2L_g \max[\sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i^{a_1, b_1}(z_i) - \bar{F}_i^{a_1, b_1; m}(z_i)| \, \mathrm{d}z_i, \ \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - F_i^{a_1, b_1}(z_i)| \, \mathrm{d}z_i]$,

then

1.
$$\begin{aligned} |\varphi(F, X_F) &- \bar{\varphi}^{a_1, b_1, m}(F, \bar{X}_{\bar{F}}^{a_1, b_1; m})| \\ &\le 2D\bar{L}L_g \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - \bar{F}_i^{a_1, b_1; m}(z_i)| \, \mathrm{d}z_i, \\ |\varphi(F, X_F^{a_1, b_1}) &- \bar{\varphi}^{a_1, b_1, m}(F, \bar{X}_{\bar{F}}^{a_1, b_1; m})| \\ &\le 2D\bar{L}L_g \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i^{a_1, b_1}(z_i) - \bar{F}_i^{a_1, b_1; m}(z_i)|, \\ |\varphi(F^{a_1, b_1}, X_{\bar{F}}^{a_1, b_1}) &- \bar{\varphi}^{a_1, b_1, m}(F^{a_1, b_1}, \bar{X}_{F}^{a_1, b_1; m})| \\ &\le 2D\bar{L}L_g \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i^{a_1, b_1}(z_i) - \bar{F}_i^{a_1, b_1; m}(z_i)| \, \mathrm{d}z_i. \end{aligned}$$

2. If, moreover, Assumptions **A.0** is fulfilled, then
$$\begin{aligned} &|\varphi(F, X_F) - \bar{\varphi}^{a_1, b_1; m}(\bar{F}^{a_1, b_1; m}, \bar{X}_{\bar{F}}^{a_1, b_1; m})| \\ &\qquad \le (2L + 2D\bar{L}L_g)[\sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i^{a_1, b_1}(z_i) - \bar{F}_i^{a_1, b_1; m}(z_i)| \, \mathrm{d}z_i \\ &\qquad\qquad + \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - F_i^{a_1, b_1}(z_i)| \, \mathrm{d}z_i], \end{aligned} \tag{30}$$
$$\begin{aligned} &|\varphi(F, X_F) - \bar{\varphi}^{a_1, b_1; m}(\bar{F}^{a_1, b_1. m}, \bar{X}_{\bar{F}}^{a_1, b_1;m})| \\ &\qquad \le (2L + 2D\bar{L}L_g)[\sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - \bar{F}_i^{a_1, b_1; m}(z_i)| \, \mathrm{d}z_i, \end{aligned}$$

$$|\varphi(F^{a_1,\,b_1}\, X_F^{a_1,\,b_1}) - \bar{\varphi}^{a_1,\,b_1;\,m}(\bar{F}^{a_1,\,b_1;\,m},\, \bar{X}_{\bar{F}}^{a_1,\,b_1;\,m})| \\ \leq (2L + 2D\bar{L}L_g)[\sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i^{a_1,\,b_1}(z_i) - \bar{F}_i^{a_1,\,b_1;\,m}(z_i)|\, \mathrm{d}z_i.$$

Proof. The assertion of Theorem 5.4 follows from Lemma 3.3, Proposition 3.4 and the triangular inequality. □

Remark 5.5. Evidently to construct the function $\bar{F}^{a_1,\,b_1;\,m}$ it is reasonable to start with the corresponding one-dimensional marginal distributions to $F^{a_1,\,b_1}$.

Evidently the results 2 of Theorem 5.4 can be employed for a scenario generation of the original problem defined by (2) and (3).

6. A NOTE ON EMPIRICAL ESTIMATES AND STABLE DISTRIBUTIONS

Very often it is necessary to solve the problem on the data base. Mathematically said, it means mostly that it is necessary to replace "theoretical underlying" distribution function by empirical one to obtain empirical estimates. It is known that a great attention has been paid (in the stochastic literature) to such situation, generally. To recall at least two results we introduce the following assumptions:

A.2
- $\{\xi^i\}_{i=1}^{\infty}$ is an independent random sequence corresponding to F,
- F^N is an empirical distribution function determined by $\{\xi^i\}_{i=1}^{N}, N = 1, 2, \ldots,$

A.3 $P_{F_i}, i = 1, \ldots, s$ are absolutely continuous w. r. t. the Lebesgue measure on R^1. (F_i^N, $i = 1, \ldots, s$ denote one–dimensional marginal distributions corresponding to F^N.)

Proposition 6.1. (Kaňková and Houda [2]) Let $P_F \in \mathcal{M}_1^1(R^s)$, $g(x,\, z), Y(z)$ be Lipschitz functions of $z \in Z_F$ with the Lipschitz constant L_g not depending on $x \in X$, $\delta > 0$. Let moreover X_F be a nonempty set, then

$$X_{F^N}^{\delta-\varepsilon(N)} \subset X_F^{\delta} \subset X_{F^N}^{\delta+\varepsilon(N)} \quad \text{with} \quad \varepsilon(N) = 2L_g \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - F_i^N(z_i)|\, \mathrm{d}z_i.$$

If, moreover the assumptions **A.0**, **A.1**, **A.2** and **A.3** are fulfilled, then also

$$\begin{array}{ccccc} \varphi(F,\, X_{F^N}^{\varepsilon(N)}) & \leq & \varphi(F,\, X_F^0) & \leq & \varphi(F,\, X_{F^N}^{-\varepsilon(N)}), \\ \varphi(F^N,\, X_{F^N}^{\varepsilon(N)}) & \leq & \varphi(F^N,\, X_F^0) & \leq & \varphi(F^N,\, X_{F^N}^{-\varepsilon(N)}). \end{array}$$

Proposition 6.2. (Kaňková and Houda [2]) Let $P_F \in \mathcal{M}_1^1(R^s)$, $t > 0$. Let moreover X_F be a nonempty compact set. If

1. assumptions **A.0**, **A.1**, **A.2**, **A.3** are fulfilled, X_F be nonempty set,
2. - $g(x,\, z)$, $Y(z)$ is a Lipschitz function of $z \in Z_F$ with the Lipschitz constant not depending on $x \in X$,
   - $g_0(x,\, z)$ is Lipschitz function on X with the Lipschitz constant $L^{'}$ not depending on $z \in Z_F$,
3. there exists $\varepsilon_0 > 0$ such that X_F^{ε} are nonempty compact sets for every $\varepsilon \in \langle -\varepsilon_0,\, \varepsilon_0 \rangle$ and, moreover, there exists a constant $\hat{C} > 0$ such that
$$\Delta_n[X_F^{\varepsilon},\; X_F^{\varepsilon^{'}}] \;\leq \hat{C}|\varepsilon - \varepsilon^{'}| \quad \text{for} \quad \varepsilon, \varepsilon^{'} \in \langle -\varepsilon_0,\, \varepsilon_0 \rangle,$$
4. for some $r > 2$ it holds that $\mathsf{E}_{F_i}|\xi_i|^r < +\infty$, $i = 1, \ldots, s$ and a constant γ fulfils the inequalities
$$0 \,<\, \gamma < 1/2 - 1/r$$

then
$$P\{\omega :\, N^{\gamma}|\varphi(F,\, X_F^0) - \varphi(F^N,\, X_{F^N}^0)| > t\} \underset{N \to \infty}{\longrightarrow} 0. \tag{31}$$

Evidently replacing the "underlying" distribution function by empirical one we obtain "good" estimate of the problem (2), (3). However, it is over the possibility of this work to define and to analyze exactly empirical estimates of the bounds a_1, b_1, $i = 1, \ldots, s$ in the approximate empirical problems. But, evidently, it is possible to await that approximate problems will have very good properties also.

The situation is rather complicated in the case of the distributions with heavy tails, especially with the stable distributions. There very often appears crossing; see, e. g., [3]. Consequently to this fact the set X_F can be empty. Evidently, this case also needs a special investigation.

7. CONCLUSION

In the paper we have considered stochastic programming problems with second order stochastic dominance constraints. It is known that these problems goes to semi–infinite optimization problems for which Slater's condition is not necessary fulfilled. Consequently we have tried to introduce the modified problem for which this condition is already fulfilled. To this end we have recalled the stability assertion based on the Wasserstein metric corresponding to the $\mathcal{L}_1$ norm. The gap between the original and the relax problem has been estimated. Further, employing the stability results and the results of [1] we have suggested a "discrete approximation". We obtain by this approach the optimization problem with relative simple constraints set. The approximation error can be estimated.

At the end we have discussed the case when the theoretical distribution function has to be replaced by empirical one and the case of the stable distributions. However both these cases are rather complicated and there are beyond the scope of this paper.

ACKNOWLEDGEMENT.

This research was supported by the the Czech Science Foundation under Grant No. 18-02739S.

(Received December 21, 2017)

REFERENCES

[1] D. Dentcheva and A. Ruszczyński: Optimization with stochastic dominance constraints. SIAM J. Optim. *14* (2003), 548–566. DOI:10.1137/s1052623402420528

[2] V. Kaňková and M. Houda: Thin and heavy tails in stochastic programming. Kybernetika *51* (2015), 3, 433–456. DOI:10.14736/kyb-2015-3-0433

[3] V. Kaňková and V. Omelchenko: Empirical estimates in stochastic programs with probability and second order stochastic dominance constraints. Acta Math. Universitatis Comenianae *84* (2015), 2, 267–281.

[4] V. Kaňková: Stability, empirical estimates and scenario generation in stochastic optimization–applications in finance. Kybernetika *53* (2017), 6, 1026–1046. DOI:10.14736/kyb-2017-6-1026

[5] L. B. Klebanov: Heavy Tailed Distributions. MATFYZPRESS, Prague 2003. DOI:10.1007/springerreference_205369

[6] S. Kotz, N. Balakrishnan, and N. J. Johnson: Continuous Multivariate Distributions (Volume 1: Models and Applications.) John Wiley and Sons, New York 2000.

[7] K. Odintsov: Decision Problems and Empirical Data; Applicationns to New Types of Problems (in Czech). Diploma Thesis, Faculty of Mathematics and Physics, Charles University Prague, Prague 2012.

[8] W. Ogryczak and A. Ruszczyński: From stochastic dominance to mean–risk models: Semideviations as risk measures. Europ. J. Oper. Res. *116* (1999), 33–50. DOI:10.1016/s0377-2217(98)00167-2

[9] V. Omelčenko: Problems of Stochastic Optimization under Uncertainty, Quantitative Methods, Simulations, Applications in Gas Storage Valuation. Doctoral Thesis, Faculty of Mathematics and Physics, Charles University Prague, Prague 2016.

[10] R. Rockafellar and R. J. B. Wets: Variational Analysis. Springer, Berlin 1983. DOI:10.1007/978-3-642-02431-3

[11] G. Samarodnitsky and M. Taqqu: Stable Non-Gaussian Random Processes. Chapman and Hall, New York 1994. DOI:10.1017/s0266466600005673

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