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To the memory of Bohuslav Balcar (1943–2017)

Abstract. The Golomb space $\mathbb{N}_\tau$ is the set $\mathbb{N}$ of positive integers endowed with the topology τ generated by the base consisting of arithmetic progressions $\{a+bn\colon n\ge 0\}$ with coprime a,b. We prove that the Golomb space $\mathbb{N}_\tau$ has continuum many continuous self-maps, contains a countable disjoint family of infinite closed connected subsets, the set Π of prime numbers is a dense metrizable subspace of $\mathbb{N}_\tau$, and each homeomorphism h of $\mathbb{N}_\tau$ has the following properties: $h(1)=1$, $h(\Pi)=\Pi$, $\Pi_{h(x)}=h(\Pi_x)$, and $h(x^{\mathbb{N}})=h(x)^{\mathbb{N}}$ for all $x\in\mathbb{N}$. Here $x^{\mathbb{N}}:=\{x^n\colon n\in\mathbb{N}\}$ and Π_x denotes the set of prime divisors of x.

Keywords: Golomb space; arithmetic progression; superconnected space; homeomorphism

Classification: 54D05, 11A41

In the AMS Meeting announcement, see [6], M. Brown introduced an amusing topology τ on the set $\mathbb{N}$ of positive integers turning it into a connected Hausdorff space. The topology τ is generated by the base consisting of arithmetic progressions $a+b\,\mathbb{N}_0:=\{a+bn\colon n\in\mathbb{N}_0\}$ with coprime parameters $a,b\in\mathbb{N}$. Here by $\mathbb{N}_0=\{0\}\cup\mathbb{N}$ we denote the set of non-negative integer numbers.

In [18] the topology τ is called the *relatively prime integer topology*. This topology was popularized by S. Golomb, see [13], [14], who observed that the classical Dirichlet theorem on primes in arithmetic progressions is equivalent to the density of the set Π of prime numbers in the topological space $(\mathbb{N},\tau)$. As a by-product of such popularization efforts, the topological space $\mathbb{N}_\tau:=(\mathbb{N},\tau)$ is known in general topology as the *Golomb space*, see [21], [22].

The problem of studying the topological structure of the Golomb space was posed to the first author T. Banakh by the third author S. Turek in 2006. In his turn, S. Turek learned about this problem from the second author J. Mioduszewski who listened to the lecture of S. Golomb on the first Toposym in 1961.

In this paper we study continuous self-maps and homeomorphisms of the Golomb space. In particular, we prove that the Golomb space has continuum many continuous self-maps and is not topologically homogeneous.

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1. Preliminaries and notations

First let us fix notation. For a point $x \in \mathbb{N}$ by $\tau_x = \{U \in \tau \colon x \in U\}$ we denote the family of open neighborhoods of x in the topology τ of the Golomb space $\mathbb{N}_\tau$.

For two numbers x, y by $\gcd(x, y)$ we denote their greatest common divisor, and by $x \dagger y$ the greatest divisor of x, which is coprime with y.

By Π we denote the set of prime numbers. For a number $x \in \mathbb{N}$ by Π_x we denote the set of all prime divisors of x. Two numbers $x, y \in \mathbb{N}$ are *coprime* if and only if $\Pi_x \cap \Pi_y = \emptyset$ (which is equivalent to saying that $\gcd(x, y) = 1$).

A number $q \in \mathbb{N}$ is called *square-free* if it is not divided by the square p^2 of any prime number p.

For a number $x \in \mathbb{N}$ and a prime number p let $l_p(x)$ be the largest integer number such that $p^{l_p(x)}$ divides x. The function $l_p(x)$ plays the role of logarithm with base p. A number x is square-free if and only if $l_p(x) \le 1$ for any prime number p.

A family $\mathcal{F}$ of subsets of a set X is called a *filter* if

- $\emptyset \notin \mathcal{F}$;
- for any $A, B \in \mathcal{F}$ their intersection $A \cap B \in \mathcal{F}$;
- for any sets $F \subset E \subset X$ the inclusion $F \in \mathcal{F}$ implies $E \in \mathcal{F}$.

In the subsequent proofs we shall exploit the following two known results of number theory. The first one is a general version of the Chinese remainder theorem, which can be found in [15, 3.12].

Theorem 1.1 (Chinese remainder theorem)**.** *For any numbers $a_1, \dots, a_n \in \mathbb{Z}$ and $b_1, \dots, b_n \in \mathbb{N}$ the following conditions are equivalent:*

(1) *the intersection $\bigcap_{i=1}^n (a_i + b_i\,\mathbb{N})$ is not empty;*

(2) *the intersection $\bigcap_{i=1}^n (a_i + b_i\,\mathbb{N})$ contains an infinite arithmetic progression;*

(3) *for any i, j the number $a_i - a_j$ is divisible by $\gcd(b_i, b_j)$.*

The second classical result is not elementary and is due to Dirichlet [8, Section VI], see also [1, Chapter 7].

Theorem 1.2 (Dirichlet theorem)**.** *Each arithmetic progression $a + b\,\mathbb{N}$ with $\gcd(a, b) = 1$ contains a prime number.*

2. Superconnectedness of the Golomb space

We define a topological space X to be *superconnected* if for any nonempty open sets $U_1, \dots, U_n \subset X$ the intersection of their closures $\overline{U_1} \cap \dots \cap \overline{U_n}$ is not empty.

The proof of the following proposition is straightforward and is left to the reader as an exercise.

Proposition 2.1. (1) *Each superconnected space is connected.*

(2) *The continuous image of a superconnected space is superconnected.*

(3) *A topological space is superconnected if it contains a dense superconnected subspace.*

In this section we present some examples of superconnected subspaces of the Golomb space $\mathbb{N}_\tau$. But first describe the closures of arithmetic progressions in $\mathbb{N}_\tau$.

Lemma 2.2. *For any $a, b \in \mathbb{N}$*

$$\overline{a + b\,\mathbb{N}_0} = \mathbb{N} \cap \bigcap_{p \in \Pi_b} (p\,\mathbb{N} \cup (a + p^{l_p(b)}\,\mathbb{Z})).$$

Proof: First we prove that $\overline{a + b\,\mathbb{N}_0} \subset p\,\mathbb{N} \cup (a + p^k\,\mathbb{Z})$ for every $p \in \Pi_b$ and $k = l_p(b)$. Take any point $x \in \overline{a + b\,\mathbb{N}_0}$ and assume that $x \notin p\,\mathbb{N}$. Then $x + p^k\,\mathbb{N}_0$ is a neighborhood of x and hence the intersection $(x + p^k\,\mathbb{N}_0) \cap (a + b\,\mathbb{N}_0)$ is not empty. Then there exist $u, v \in \mathbb{N}_0$ such that $x + p^k u = a + bv$. Consequently, $x - a = bv - p^k u \in p^k\,\mathbb{Z}$ and $x \in a + p^k\,\mathbb{Z}$

Next, take any point $x \in \mathbb{N} \cap \bigcap_{p \in \Pi_b} (p\,\mathbb{N} \cup (a + p^{l_p(b)}\,\mathbb{Z}))$. Given any basic neighborhood $x + d\,\mathbb{N}_0$ of x, we should prove that $(x + d\,\mathbb{N}_0) \cap (a + b\,\mathbb{N}_0) \neq \emptyset$.

Our assumption guarantees that $x \in \bigcap_{p \in \Pi_b \setminus \Pi_x} (a + p^{l_p(b)}\,\mathbb{Z}) = a + q\,\mathbb{Z}$ where $q = \prod_{p \in \Pi_b \setminus \Pi_x} p^{l_p(b)}$. Since the numbers x and d are coprime, the greatest common divisor of b and d divides the number q. Since $x - a \in q\,\mathbb{Z}$, the Euclides algorithm yields two numbers $u, v \in \mathbb{Z}$ such that $x - a = bu - dv$, which implies that $(x + d\,\mathbb{Z}) \cap (a + b\,\mathbb{Z}) \neq \emptyset$ and so $(x + d\,\mathbb{N}_0) \cap (a + b\,\mathbb{N}_0) \neq \emptyset$. □

The next proposition generalizes the connectedness of subspaces of the form $(1 + p\,\mathbb{N}_0) \cup p\,\mathbb{N}$, where $p \in \Pi$ (proved in [22, Lemma 3.2]).

Proposition 2.3. *For any sequences $\{a_i\}_{i \in \omega} \subset \mathbb{N}_0$ and $\{b_i\}_{i \in \omega} \subset \mathbb{N}$ with $a_0 = 0$ the subspace*

$$X = \mathbb{N} \cap \bigcup_{i \in \omega} (a_i + b_i\,\mathbb{N}_0)$$

of $\mathbb{N}_\tau$ is superconnected.

Proof: Given nonempty open sets $U_1, \dots, U_n \subset X$, we should prove that $X \cap \overline{U_1} \cap \dots \cap \overline{U_n} \neq \emptyset$. We can assume that each set U_j is of the form $X \cap (c_j + d_j\,\mathbb{N})$ for some coprime numbers c_j, d_j. By the Chinese remainder theorem U_j contains an arithmetic progression $e_j + f_j\,\mathbb{N}$ and by Lemma 2.2, $\overline{U_j}$ contains $\bigcap_{p \in \Pi_{f_j}} p\,\mathbb{N}$. Since $a_0 = 0$ we have

$$\emptyset \neq b_0\,\mathbb{N}_0 \cap \bigcap_{j=1}^{n} \bigcap_{p \in \Pi_{f_j}} p\,\mathbb{N} \subset X \cap \overline{U_1} \cap \dots \cap \overline{U_n}.$$

□

Corollary 2.4. *The Golomb space is superconnected.*

The superconnectedness of the Golomb space can be also derived from the following lemma.

Lemma 2.5. *A subspace $X \subset \mathbb{N}_\tau$ is superconnected if $\Pi \subset X$ and $X \cap q\,\mathbb{N} \neq \emptyset$ for any square-free number q.*

Proof: We need to prove that for any nonempty open sets $U_1, \ldots, U_n \subset X$ the intersection $X \cap \bigcap_{i=1}^n \overline{U_i}$ is not empty. Without loss of generality, we can assume that each set U_i is of basic form $U_i = X \cap (a_i + b_i\,\mathbb{N}_0)$ for some numbers $a_i \in X$ and $b_i \in \mathbb{N}$ with $\gcd(a_i, b_i) = 1$. Let q be the square-free number equal to the product of all prime divisors of the number $\prod_{i=1}^n b_i$. By our assumption, $X \cap q\,\mathbb{N}$ contains some number a. We claim that $a \in X \cap \bigcap_{i=1}^n \overline{U_i}$. We need to show that for any number $b \in \mathbb{N}$ with $\gcd(a, b) = 1$, the basic neighborhood $a + b\,\mathbb{N}_0$ of a intersects each set $U_i = X \cap (a_i + b_i\,\mathbb{N}_0)$. Taking into account that $\Pi_{b_i} \subset \Pi_q \subset \Pi_a$ and b is coprime with a, we conclude that b and b_i are coprime. So, we can apply the Chinese remainder theorem and conclude that the intersection $(a + b\,\mathbb{N}_0) \cap (a_i + b_i\,\mathbb{N}_0)$ is nonempty and being open in $\mathbb{N}_\tau$ contains some prime number p, by the Dirichlet theorem. Then $p \in (a + b\,\mathbb{N}_0) \cap (a_i + b_i\,\mathbb{N}_0) \cap X = (a + b\,\mathbb{N}_0) \cap U_i$ and hence $a \in X \cap \bigcap_{i=1}^n \overline{U_i}$. □

3. Metrizability of the set of prime numbers in the Golomb space

The main result of this section is the following theorem.

Theorem 3.1. *The set Π of prime numbers is a dense metrizable subspace of the Golomb space $\mathbb{N}_\tau$. Moreover, Π is homeomorphic to the space $\mathbb{Q}$ of rational numbers.*

Proof: The density of the set Π in the Golomb space $\mathbb{N}_\tau$ follows from Dirichlet's theorem.

Next, we prove that the subspace Π of $\mathbb{N}_\tau$ is regular. Given any number $x \in \Pi$ and an open neighborhood $O_x \subset \mathbb{N}_\tau$ of x, we should find a neighborhood $U_x \subset \mathbb{N}_\tau$ of x such that $\Pi \cap \overline{U_x} \subset O_x$. We lose no generality assuming that $O_x = x + b\,\mathbb{N}_0$ for some number b such that $x \notin \Pi_b$ and $|\Pi_b| > 1$. Choose $n \in \mathbb{N}$ so large that $b^n > x$ and for any $p, r \in \Pi_b$ the difference $p - x$ is not divisible by r^n. We claim that the neighborhood $U_x := x + b^n\,\mathbb{N}_0$ has the required property: $\Pi \cap \overline{U_x} \subset O_x$.

Indeed, take any number $p \in \Pi \cap \overline{x + b^n\,\mathbb{N}_0}$. By Lemma 2.2,

$$\overline{x + b^n\,\mathbb{N}_0} = \mathbb{N} \cap \bigcap_{r \in \Pi_b} (r\,\mathbb{N} \cup (x + r^{l_r(b^n)}\,\mathbb{Z})).$$

If $p \in \Pi_b$, then for any number $r \in \Pi_b \setminus \{p\}$, the choice of n guarantees that $p \notin r\,\mathbb{N} \cup (x + r^{\,l_r(b^n)}\,\mathbb{Z})$ and hence $p \notin \overline{x + b^n\,\mathbb{N}_0}$, which is a contradiction. So, $p \notin \Pi_b$. In this case $p \in \bigcap_{r \in \Pi_b} (x + r^{\,l_r(b^n)}\,\mathbb{Z}) = x + b^n\,\mathbb{Z}$ and hence $p - x$ is divisible by b^n. Taking into account that $b^n > x$, we conclude that $p \geq x$ and hence $p \in x + b^n\,\mathbb{N}_0 \subset x + b\,\mathbb{N}_0 \subset O_x$. This completes the proof of the regularity of Π.

By the Tychonoff–Urysohn metrization theorem, see [9, 4.2.9], the second-countable regular space Π is metrizable. The Dirichlet theorem implies that the space Π has no isolated points. By the Sierpiński theorem, see [9, 6.2.A (d)], Π is homeomorphic to $\mathbb{Q}$ (being a countable metrizable space without isolated points). $\square$

In contrast to the set of prime numbers, basic open sets in the Golomb space are not regular (but are totally disconnected). We recall that a topological space X is *totally disconnected* if for any distinct points $x, y \in X$ there exists a closed-and-open set $U \subset X$ such that $x \in U$ and $y \notin U$.

Proposition 3.2. *For any coprime numbers $a \in \mathbb{N}$ and $b \in \mathbb{N} \setminus \{1\}$ the subspace $X = a + b\,\mathbb{N}_0$ of $\mathbb{N}_\tau$ is totally disconnected but not regular.*

Proof: First we show that the space $X = a + b\,\mathbb{N}_0$ is not regular. Choose any prime number $q \notin \Pi_b \cup \Pi_a$ and consider the basic neighborhood $V = a + qb\,\mathbb{N}_0 \subset a + b\,\mathbb{N}_0$ of the point a. Each basic neighborhood of a which is contained in V has form $W = a + qbc\,\mathbb{N}_0$ for some number $c \in \mathbb{N}$, coprime with a. By Lemma 2.2 we have

$$\begin{aligned}\overline{W} = \overline{a + qbc\,\mathbb{N}_0} &\supset \bigcap_{p \in \Pi_{qbc}} (p\,\mathbb{N} \cup (a + p^{l_p(qbc)}\,\mathbb{N}_0))\\ &= \bigcap_{p \in \Pi_{qbc} \setminus \Pi_b} (p\,\mathbb{N} \cup (a + p^{l_p(qbc)}\,\mathbb{N}_0)) \cap \bigcap_{p \in \Pi_b} (p\,\mathbb{N} \cup (a + p^{l_p(qbc)}\,\mathbb{N}_0))\\ &\supset \bigcap_{p \in \Pi_{qbc} \setminus \Pi_b} p\,\mathbb{N} \cap \bigcap_{p \in \Pi_b} (a + p^{l_p(qbc)}\,\mathbb{N}_0).\end{aligned}$$

The set

$$\bigcap_{p \in \Pi_{qbc} \setminus \Pi_b} p\,\mathbb{N} \cap \bigcap_{p \in \Pi_b} (a + p^{l_p(qbc)}\,\mathbb{N}_0)$$

is nonempty by the Chinese remainder theorem and is contained in $q\,\mathbb{N} \cap (a + b\,\mathbb{N}_0) = q\,\mathbb{N} \cap X$. However, $V \cap q\,\mathbb{N} = (a + qb\,\mathbb{N}_0) \cap q\,\mathbb{N} = \emptyset$ because a is not divisible by q. So, $\overline{W} \cap X \not\subset V$ and the space X is not regular.

To see that the space $X = a + b\,\mathbb{N}_0$ is totally disconnected, take any distinct points $x, y \in a + b\,\mathbb{N}_0$ and choose $n \in \mathbb{N}$ so large that b^n does not divide $x - y$. Observe that $\mathcal{V} = \{X \cap (z + b^n\,\mathbb{Z}) \colon z \in a + b\,\mathbb{N}_0\}$ is a disjoint open cover of X, which implies that each set $V \in \mathcal{V}$ is open-and-closed in X. Moreover, since b^n does not divide $x - y$, the points x, y belong to distinct sets of the cover $\mathcal{V}$. This implies that X is totally disconnected. $\square$

The last proposition gives another example of a space that is totally disconnected and not zero-dimensional (cf. [10, Example 1.2.15]).

4. Continuous self-maps of the Golomb space

In this section we study the structure of the set $C(\mathbb{N}_\tau)$ of all continuous self-maps of the Golomb space $\mathbb{N}_\tau$. In the following proposition the set $\mathbb{N}^\mathbb{N}$ of all self-maps of $\mathbb{N}$ is endowed with the (Polish) topology of the Tychonoff product of discrete spaces $\mathbb{N}$.

Let us observe that a map $f\colon \mathbb{N}_\tau \to \mathbb{N}_\tau$ is continuous at a point $x \in \mathbb{N}_\tau$ if and only if for every number b coprime with $f(x)$ there is a number a coprime with x such that $f(x + a\,\mathbb{N}_0) \subset f(x) + b\,\mathbb{N}_0$.

Proposition 4.1. *The set $C(\mathbb{N}_\tau)$ is an $F_{\sigma\delta}$-subset of the Polish space $\mathbb{N}^\mathbb{N}$.*

Proof: It is clear that $C(\mathbb{N}_\tau) = \bigcap_{x\in\mathbb{N}} C_x(\mathbb{N}_\tau)$ where $C_x(\mathbb{N}_\tau)$ denotes the set of all functions $f\colon \mathbb{N}_\tau \to \mathbb{N}_\tau$ which are continuous at x. In its turn, for every $x \in \mathbb{N}$ the set $C_x(\mathbb{N}_\tau) = \bigcap_{b\in\mathbb{N}} C_{x,b}$ where

$$C_{x,b} = \{f \in \mathbb{N}^\mathbb{N}\colon\ \gcd(f(x), b) \neq 1\} \cup C'_{x,b}$$

and

$$C'_{x,b} := \{f \in \mathbb{N}^\mathbb{N}\colon \exists\, a \in \mathbb{N}\ (\gcd(a, x) = 1 \wedge (f(x + a\,\mathbb{N}_0) \subset f(x) + b\,\mathbb{N}_0))\}.$$

Put $A_x := \{a \in \mathbb{N}\colon\ \gcd(a, x) = 1\}$ and observe that

$$C'_{x,b} = \bigcup_{a\in A_x} \{f \in \mathbb{N}^\mathbb{N}\colon f(x + a\,\mathbb{N}_0) \subset f(x) + b\,\mathbb{N}_0\}$$

is a set of type F_σ in $\mathbb{N}^\mathbb{N}$ and so is the set $C_{x,b}$. Then $C_x(\mathbb{N}_\tau) = \bigcap_{b\in\mathbb{N}} C_{x,b}$ is of type $F_{\sigma\delta}$ and so is the set $C(\mathbb{N}_\tau) = \bigcap_{x\in\mathbb{N}} C_x(\mathbb{N}_\tau)$. □

Now we give a simple sufficient condition of continuity of a self-map of the Golomb space.

Definition 4.2. A function $f\colon \mathrm{dom}(f) \to \mathbb{N}$ defined on a subset $\mathrm{dom}(f)$ of $\mathbb{N}$ is called *progressive* if

(1) $\Pi_x \subset \Pi_{f(x)}$ for every $x \in \mathrm{dom}(f)$;
(2) for any $x < y$ in $\mathrm{dom}(f)$ the number $f(y) - f(x)$ is divisible by $(y-x) \dagger f(x)$.

We recall that for two numbers x, y by $x \dagger y$ we denote the greatest divisor of x which is coprime with y.

Proposition 4.3. *Each progressive function $f\colon \mathbb{N}_\tau \to \mathbb{N}_\tau$ is continuous.*

Proof: Given a point $x \in \mathbb{N}_\tau$ and a neighborhood $O_{f(x)} \in \tau$ of $f(x)$, we need to find a neighborhood $O_x \in \tau$ of x such that $f(O_x) \subset O_{f(x)}$. We lose no generality assuming that $O_{f(x)}$ is of basic form $O_{f(x)} = f(x) + d\,\mathbb{N}_0$, where the number $d > f(x)$ is coprime with $f(x)$. Then $f(x) + d\,\mathbb{N}_0 = \mathbb{N} \cap (f(x) + d\,\mathbb{Z})$. Since $\Pi_x \subset \Pi_{f(x)}$, the numbers x and d are coprime and hence $O_x := x + d\,\mathbb{N}_0$ is a neighborhood of x in $\mathbb{N}_\tau$. It remains to prove that $f(O_x) \subset O_{f(x)}$. Given any $y \in O_x$, we need to show that $f(y) \in f(x) + d\,\mathbb{Z}$. This is trivially true if $y = x$. So, we assume that $y \neq x$ and hence $y \in x + d\,\mathbb{N}$. Then d divides $y - x$. Since d and $f(x)$ are coprime, d divides the number $b = (y - x) \dagger f(x)$. Taking into account that the function f is progressive, we conclude that $f(y) - f(x) \in b\,\mathbb{Z} \subset d\,\mathbb{Z}$. □

For polynomials with nonnegative integer coefficients the equivalence of conditions (1)–(3) in the following theorem was proved by P. Szczuka [22, Theorem 4.3].

Theorem 4.4. *For a non-constant polynomial $f\colon \mathbb{N} \to \mathbb{N}$, $f\colon x \mapsto a_0 + a_1x + \dots + a_nx^n$ with integer coefficients the following conditions are equivalent:*

(1) *$a_0 = 0$;*
(2) *f is a continuous self-map of the Golomb space $\mathbb{N}_\tau$;*
(3) *for any connected subspace $C \subset \mathbb{N}_\tau$ the image $f(C)$ is connected;*
(4) *for any superconnected subspace $C \subset \mathbb{N}_\tau$ the image $f(C)$ is superconnected;*
(5) *for any superconnected subspace $C \subset \mathbb{N}_\tau$ the image $f(C)$ is connected.*

Proof: To prove the implication (1) ⇒ (2), assume that $a_0 = 0$. In this case the polynomial $f(x)$ is divisible by x and $f(x)-f(y)$ is divisible by $x-y$. These observations imply that the function f is progressive and hence continuous, according to Proposition 4.3.

The implications (3) ⇐ (2) ⇒ (4) and (3) ⇒ (5) ⇐ (4) follow from Proposition 2.1. So, it remains to prove that (5) ⇒ (1). To derive a contradiction, assume that $a_0 \neq 0$. Since f is not constant, there exists $x \in \mathbb{N}$ such that $f(x) \neq a_0$. Choose any prime number $p > \max\{a_0, x, f(x)\}$. By Proposition 2.3, the subspace $X = p\,\mathbb{N} \cup (x + p\,\mathbb{N}_0)$ is superconnected. On the other hand, its image $f(X)$ can be written as the union $f(X) = U \cup V$ of two nonempty disjoint open subspaces $U = f(p\,\mathbb{N}) = f(X) \cap (a_0 + p\,\mathbb{Z})$ and $V = f(x + p\,\mathbb{N}) = f(X) \cap (f(x) + p\,\mathbb{Z})$ of $f(X)$. □

The following example shows that Theorem 4.4 cannot be extended to polynomials with rational coefficients.

Example 4.5. The polynomial $f\colon \mathbb{N}_\tau \to \mathbb{N}_\tau$, $f\colon x \mapsto (x + x^2)/2$, is discontinuous.

Proof: It suffices to check that f is discontinuous at $x = 2$. Assuming that f is continuous at 2 for the neighborhood $3 + 2\,\mathbb{N}_0$ of $3 = f(2)$, we can find a basic neighborhood $2 + b\,\mathbb{N}_0$ of 2 such that $f(2 + b\,\mathbb{N}_0) \subset 3 + 2\,\mathbb{N}_0$. The number b, being coprime with 2, is odd and hence can be written as $b = 2n - 1$ for some $n \in \mathbb{N}$. Consider the number $4n = 2 + 2(2n - 1) = 2 + 2b \in 2 + b\,\mathbb{N}_0$ and observe that $f(4n) = 2n(4n + 1) \notin 3 + 2\,\mathbb{N}_0$, which contradicts the choice of b. □

Let us remind that a connected space is called *biconnected* if it cannot be decomposed into a sum of two disjoint connected subsets that contain more than one point. It is a consequence of Theorem XI of [16] that a space is biconnected if and only if it is connected and does not contain two disjoint nondegenerate connected sets. It is known that Golomb space is not biconnected (see Problem 86 in [18, page 185]). It turns out that the Golomb space has a much stronger property.

Corollary 4.6. *The Golomb space contains a countable family of pairwise disjoint closed infinite superconnected subspaces.*

Proof: By Theorem 4.4, for every $n \in \mathbb{N}$ the polynomial $f_n \colon \mathbb{N}_\tau \to \mathbb{N}_\tau$, $f_n \colon x \mapsto x^2 + nx$, is a continuous self-map of the Golomb space $\mathbb{N}_\tau$.

By Lemma 4.7 proved below, the set $f_n(\mathbb{N})$ is closed in the Golomb space $\mathbb{N}_\tau$. Choose any prime number $p_n > n^2 + n$ and observe that the set $p_n \mathbb{N}$ is closed in the Golomb space $\mathbb{N}_\tau$. Then the intersection $X_n = p_n \mathbb{N} \cap f_n(\mathbb{N})$ is closed in $\mathbb{N}_\tau$, too. By Proposition 2.3, the set

$$f_n^{-1}(X_n) = f_n^{-1}(p_n \mathbb{N}) = \{x \in \mathbb{N} \colon x(x+n) \in p_n \mathbb{N}\} = p_n \mathbb{N} \cup (p_n - n + p_n \mathbb{N}_0)$$

is superconnected and so is its image $f_n(f_n^{-1}(X_n)) = X_n$.

It remains to prove that for any numbers $n < m$ the sets X_n and X_m are disjoint. Assuming that $X_n \cap X_m$ contains some number z, we conclude that $z \in X_n \cap X_m \subset p_n \mathbb{N} \cap p_m \mathbb{N} \subset p_m \mathbb{N}$. Find numbers $x, y \in \mathbb{N}$ such that $x^2 + nx = f_n(x) = z = f_m(y) = y^2 + my$ and observe that

$$(2x+n)^2 - n^2 = 4x^2 + 4nx = 4y^2 + 4my = (2y+m)^2 - m^2.$$

Then $(2(y-x) + m - n)(2(y+x) + m + n) = (2y+m)^2 - (2x+n)^2 = m^2 - n^2$ and hence

$$y \le 2(y+x) + m + n \le m^2 - n^2 < m^2.$$

On the other hand, $y(y+m) = z \in p_m \mathbb{N}$ implies that y or $y+m$ is divisible by p_m and hence $y \ge p_m - m > m^2$ by the choice of prime number p_m. This contradiction shows that $X_n \cap X_m = \emptyset$. $\square$

Lemma 4.7. *For any $n \in \mathbb{N}_0$ the set $X := \{x^2 + nx \colon x \in \mathbb{N}\}$ is closed in the Golomb space $\mathbb{N}_\tau$.*

Proof: Given any $a \in \mathbb{N} \setminus X$, it suffices to find a prime number $p > a$ such that $a + p\,\mathbb{N}_0$ is disjoint with X. Consider the polynomial $f(x) := x^2 + nx - a$ and observe that its positive root $(-n + \sqrt{n^2 + 4a})/2$ is not integer (as $a \notin X$). This implies that f has no rational roots and hence f is irreducible over the field $\mathbb{Q}$. By the classical Frobenius density theorem (see [20] or [19]), there exist infinitely many prime numbers p such that the polynomial f has no roots in $\mathbb{N}/p\,\mathbb{Z}$. For any such prime number p we have $(a + p\,\mathbb{N}) \cap X = \emptyset$. $\square$

Lemma 4.7 implies that the set $\{x^2 \colon x \in \mathbb{N}\}$ is closed in $\mathbb{N}_\tau$. On the other hand, we have the following fact.

Proposition 4.8. *For any $n \in \mathbb{N}$ the set $X_{8n} := \{x^{8n} \colon x \in \mathbb{N}\}$ is not closed in the Golomb space $\mathbb{N}_\tau$.*

Proof: First we show that the set $X_8 = \{x^8 \colon x \in \mathbb{N}\}$ is not closed in $\mathbb{N}_\tau$. For this purpose we shall exploit a well-known Wang counterexample, see [23], saying that the equation $x^8 = 16$ has no integer solutions but has solutions in any field of odd prime order.

We shall show that $16 \in \overline{X_8} \setminus X_8$. Given any neighborhood $O_{16} \subset \mathbb{N}_\tau$ of 16, we should prove that $O_{16} \cap X_8 \neq \emptyset$. By the definition of the topology τ, there exists an odd number $b \in \mathbb{N}$ such that $16 + b\,\mathbb{N}_0 \subset O_{16}$. Observe that

$$x^8 - 16 = (x^2-2)(x^2+2)(x^2-2x+2)(x^2+2x+2).$$

Theorems 9.3, 9.4 and 9.5 in [1] imply that for every odd prime number p one of the numbers $2, -2, -1$ is a square in field $\mathbb{Z}/p\,\mathbb{Z}$. If some $x \in \mathbb{Z}/p\,\mathbb{Z}$ has $x^2 = \pm 2$, then $x^8 = 16$. If $x^2 = -1$, then $(1+x)^2 - 2(1+x) + 2 = 0$ and hence $(1+x)^8 = 16$. In any case, for any odd prime number p the polynomial $f(x) := x^8 - 16$ has a root in the field $\mathbb{Z}/p\,\mathbb{Z}$. By induction we shall show that this polynomial has a root in the residue rings $\mathbb{Z}/p^k\,\mathbb{Z}$ for all $k \in \mathbb{N}$. Assume that for some $k \in \mathbb{N}$ we have found a number $s \in \mathbb{Z}$ such that $f(s) \in p^k\,\mathbb{Z}$. We claim that $f'(s) = 8s^7 \notin p\,\mathbb{Z}$. Otherwise s would be divisible by p and then $s^8 - 16$ cannot be divisible by p^k, which is a contradiction. So, $f'(s) \notin p\,\mathbb{Z}$ and we can apply Theorem 5.30 in [1] to find a number $r \in \mathbb{Z}$ such that $f(r) \in p^{k+1}\,\mathbb{Z}$, which implies that the equation $x^8 - 16 = 0$ has a solution in the residue ring $\mathbb{Z}/p^{k+1}\,\mathbb{Z}$.

So, for every prime divisor p of b we can find a number $x_p \in \mathbb{N}$ such that $x_p^8 - 16 \in p^{\,l_p(b)}\mathbb{N}_0$. Using the Chinese remainder theorem, find a number $x \in \mathbb{N}$ such that $x \geq 16$ and $x \in x_p + p^{\,l_p(b)}\,\mathbb{Z}$ for every $p \in \Pi_b$. Then

$$x^8 - 16 \in \bigcap_{p \in \Pi_b} x^8 - 16 + p^{\,l_p(b)}\,\mathbb{Z} = \bigcap_{p \in \Pi_b} x_p^8 - 16 + p^{\,l_p(b)}\,\mathbb{Z} = \bigcap_{p \in \Pi_b} p^{\,l_p(b)}\,\mathbb{Z} = b\,\mathbb{Z}$$

and hence $x^8 \in X_8 \cap (16 + b\,\mathbb{N}_0)$. So, $(16 + b\,\mathbb{N}_0) \cap X_8 \neq \emptyset$.

Now we prove that for any $n \in \mathbb{N}$ the set $X_{8n} = \{x^{8n} \colon x \in \mathbb{N}\}$ is not closed in $\mathbb{N}_\tau$. By Theorem 4.4, the polynomial map $f \colon \mathbb{N}_\tau \to \mathbb{N}_\tau$, $f \colon x \mapsto x^n$, is continuous. Taking into account that this map is injective and $f(X_8) = X_{8n}$, we conclude that

$$16^n = f(16) \in f(\overline{X_8} \setminus X_8) \subset \overline{f(X_8)} \setminus f(X_8) = \overline{X_{8n}} \setminus X_{8n}.$$

$\square$

Now we show that the set $C(\mathbb{N}_\tau)$ of all continuous self-maps of the Golomb space has cardinality of continuum.

Theorem 4.9. *The set $C(\mathbb{N}_\tau)$ contains a subset ∂T of cardinality continuum which is closed in $\mathbb{N}^{\mathbb{N}}$.*

Proof: For every $n \in \mathbb{N}$ let T_n be the family of increasing progressive functions $f \colon [1,n] \to \mathbb{N}$ defined on the interval $[1,n] := \{1, \dots, n\}$. Let T_0 be the singleton consisting of the unique function $f_0 \colon \emptyset \to \mathbb{N}$. On the union $T = \bigcup_{n \in \omega} T_n$ consider the partial order "$\leq$" defined by $f \leq g$ if and only if $\mathrm{dom}(f) \subset \mathrm{dom}(g)$ and $f = g \restriction \mathrm{dom}(f)$. It is clear that this partial order turns T into a tree. The set $\partial T = \{f \in \mathbb{N}^{\mathbb{N}} \colon \forall n \in \mathbb{N} \;\; f \restriction [1,n] \in T_n\}$ is closed in $\mathbb{N}^{\mathbb{N}}$ and can be identified with the set of branches of the tree T. Since each function $f \in \partial T$ is increasing and progressive, Proposition 4.3 guarantees that f is continuous and hence $\partial T \subset C(\mathbb{N}_\tau)$.

It remains to check that $|\partial T| = \mathfrak{c}$. This equality will follow as soon as for any $n \in \mathbb{N}$ and $f \in T_n$ we show that the set $\mathrm{succ}(f) = \{g \in T_{n+1} \colon g \restriction [1,n] = f\}$ of successors of f in the tree T contains more than one point. We shall prove more: the set $\mathrm{succ}(f)$ is infinite.

Observe that a function $g \colon [1, n+1] \to \mathbb{N}$ belongs to $\mathrm{succ}(f)$ if and only if $g \restriction [1,n] = f$, $g(n+1) > f(n)$ and $g(n+1)$ belongs to the set

$$Y_f := \bigcap_{p \in \Pi_{n+1}} p\,\mathbb{N} \cap \bigcap_{k \in [1,n]} (f(k) + ((n+1-k) \dagger f(k))\,\mathbb{Z}).$$

So, it suffices to prove that the set Y_f is infinite. By the Chinese remainder theorem, the set Y_f is infinite if and only if

(1) for every $p \in \Pi_{n+1}$ and $k \in [1,n]$ the number $f(k)$ is divisible by $\gcd(p, (n+1-k) \dagger f(k))$;
(2) for any numbers $k < l$ in $[1,n]$ the number $f(l) - f(k)$ is divisible by $\gcd((n+1-k) \dagger f(k), (n+1-l) \dagger f(l))$.

To verify the first condition, fix any prime number $p \in \Pi_{n+1}$ and any $k \in [1,n]$. We claim that $\gcd(p, (n+1-k) \dagger f(k)) = 1$. If $p \notin \Pi_k$, then p does not divide $n+1-k$ and hence $\gcd(p, (n+1-k) \dagger f(k)) = 1$. If $p \in \Pi_k$, then $p \in \Pi_{f(k)}$ and hence $\gcd(p, (n+1-k) \dagger f(k)) = 1$. In both cases $\gcd(p, (n+1-k) \dagger f(k)) = 1$ divides $f(k)$.

To verify the second condition, fix any numbers $k < l$ in $[1,n]$. Let d be the largest common divisor of $(n+1-k) \dagger f(k)$ and $(n+1-l) \dagger f(l)$. Then d is coprime with $f(k)$ and divides both the numbers $n+1-k$ and $n+1-l$, so d divides their difference $(n+1-k) - (n+1-l) = l - k$. Consequently, d divides $(l-k) \dagger f(k)$. Since f is progressive, $f(l) - f(k)$ is divisible by $(l-k) \dagger f(k)$ and hence is divisible by d. $\square$

Problem 4.10. Is the set $C(\mathbb{N}_\tau)$ dense in $\mathbb{N}^{\mathbb{N}}$ (or in $\mathbb{N}_\tau^{\mathbb{N}}$)?

5. Homeomorphisms of the Golomb space

In this section we study homeomorphisms of the Golomb space $\mathbb{N}_\tau$ and prove the following main theorem.

Theorem 5.1. *Each homeomorphism $h \colon \mathbb{N}_\tau \to \mathbb{N}_\tau$ of the Golomb space has the following properties:*

(1) $h(1) = 1$;
(2) $h(\Pi) = \Pi$;
(3) $\Pi_{h(x)} = h(\Pi_x)$ *for every* $x \in \mathbb{N}$;
(4) *there exists a multiplicative bijection μ of $\mathbb{N}$ such that $h(x^n) = h(x)^{\mu(n)}$ for all $x, n \in \mathbb{N}$.*

We recall that for a number $x \in \mathbb{N}$ by Π_x the set of all prime divisors of x is denoted. Also by τ we denote the topology of the Golomb space $\mathbb{N}_\tau$ and by τ_x the family of open neighborhoods of a point $x \in \mathbb{N}$ in the Golomb space $\mathbb{N}_\tau$. A function $\mu\colon \mathbb{N} \to \mathbb{N}$ is *multiplicative* if $\mu(xy) = \mu(x)\mu(y)$ for any $x, y \in \mathbb{N}$.

The four items of Theorem 5.1 are proved in Lemmas 5.4, 5.11, 5.12, and 5.15, respectively. Moreover, in Lemma 5.16 we shall prove that the multiplicative bijection μ appearing in Theorem 5.1 (4) is a homeomorphism of $\mathbb{N}$ endowed with the Furstenberg topology (generated by the base consisting of all possible arithmetic sequences $a + b\,\mathbb{N}_0$).

The superconnectedness of the Golomb space implies that the family

$$\mathcal{F}_0 = \Big\{F \subset \mathbb{N}\colon \exists\, U_1, \dots, U_n \in \tau \setminus \{\emptyset\} \text{ with } \bigcap_{i=1}^{n} \overline{U_i} \subset F\Big\}$$

is a filter on $\mathbb{N}$.

The definition of $\mathcal{F}_0$ implies that this filter is preserved by any homeomorphism h of $\mathbb{N}_\tau$ (which means that the filter $h[\mathcal{F}_0] := \{h(F)\colon F \in \mathcal{F}_0\}$ coincides with $\mathcal{F}_0$).

Lemma 5.2. *The filter $\mathcal{F}_0$ is generated by the base consisting of the sets $q\,\mathbb{N}$ for a square-free number $q \in \mathbb{N}$.*

Proof: Lemma 2.2 implies that each element $F \in \mathcal{F}_0$ contains the set $q\,\mathbb{N}$ for some square-free number q. It remains to show that for each square-free number $q > 1$ the set $q\,\mathbb{N}$ is contained in the filter $\mathcal{F}_0$. This is proved in the following lemma. $\square$

Lemma 5.3. *For two distinct numbers $x, y \in \mathbb{N}$ and a square-free number q the following conditions are equivalent:*

(1) *q is coprime with x and y;*
(2) *there are open sets $U_x \in \tau_x$ and $U_y \in \tau_y$ such that $\overline{U_x} \cap \overline{U_y} = q\,\mathbb{N}$;*
(3) *there are open sets $U_x \in \tau_x$ and $U_y \in \tau_y$ such that $\overline{U_x} \cap \overline{U_y} \subset q\,\mathbb{N}$.*

Proof: To prove the implication (1) $\Rightarrow$ (2), assume that a square-free number q is coprime with x and y. Choose $n \in \mathbb{N}$ so large that for any prime number $p \in \Pi_q$ the difference $x - y$ is not divided by p^n. Then for the neighborhoods $U_x = x + q^n\,\mathbb{N}_0$ and $U_y = y + q^n\,\mathbb{N}_0$ we get

$$\begin{aligned}\overline{U_x} \cap \overline{U_y} &= \overline{x + q^n\,\mathbb{N}_0} \cap \overline{y + q^n\,\mathbb{N}_0} \\ &= \mathbb{N} \cap \bigcap_{p \in \Pi_q} (p\,\mathbb{N} \cup (x + p^n\,\mathbb{Z})) \cap (p\,\mathbb{N} \cup (y + p^n\,\mathbb{Z})) \\ &= \mathbb{N} \cap \bigcap_{p \in \Pi_q} p\,\mathbb{N} = q\,\mathbb{N}.\end{aligned}$$

The implication (2) $\Rightarrow$ (3) is trivial. To prove the implication (3) $\Rightarrow$ (1), choose two open sets $U_x \in \tau_x$ and $U_y \in \tau_y$ with $\overline{U_x} \cap \overline{U_y} \subset q\,\mathbb{N}$. We can assume that the open sets U_x and U_y are of basic form $U_x = x + b\,\mathbb{N}_0$ and $U_y = y + d\,\mathbb{N}$. To derive a contradiction, assume that q has a common prime divisor p with x or y. Without loss of generality $p \in \Pi_x$ and hence $p \notin \Pi_b$. If also $p \notin \Pi_d$, then by the Chinese remainder theorem and Lemma 2.2,

$$\emptyset \neq (1 + p\,\mathbb{N}) \cap \bigcap_{r \in \Pi_b \cup \Pi_d} r\,\mathbb{N} \subset \overline{U_x} \cap \overline{U_y} \setminus q\,\mathbb{N}.$$

If $p \in \Pi_d$, then $p \notin \Pi_y$ and by the Chinese remainder theorem and Lemma 2.2,

$$\emptyset \neq (y + p^{l_p(d)}\,\mathbb{N}) \cap \bigcap_{r \in \Pi_b \cup \Pi_d \setminus \{p\}} r\,\mathbb{N} \subset \overline{U_x} \cap \overline{U_y} \setminus q\,\mathbb{N}.$$

In both cases we obtain a contradiction with our assumption $\overline{U_x} \cap \overline{U_y} \subset q\,\mathbb{N}$. □

Lemma 5.4. *The number* 1 *is a fixed point of any homeomorphism h of the Golomb space $\mathbb{N}_\tau$.*

Proof: To derive a contradiction, assume that $x = h(1)$ is not equal to 1. Fix any $p \in \Pi_x$. By Lemma 5.2, the set $p\,\mathbb{N}$ belongs to the filter $\mathcal{F}_0$. Since the filter $\mathcal{F}_0$ is invariant under homeomorphisms of $\mathbb{N}_\tau$, $h^{-1}(p\,\mathbb{N}) \in \mathcal{F}_0$. By Lemma 5.2, there exists a square-free number q such that $q\,\mathbb{N} \subset h^{-1}(p\,\mathbb{N})$. Choose any $z \in \mathbb{N} \setminus \{1\}$ coprime with q. By Lemma 5.3, the Golomb space $\mathbb{N}_\tau$ contains open sets $U_z \ni z$ and $U_1 \ni 1$ such that $\overline{U_z} \cap \overline{U_1} \subset q\,\mathbb{N}$. Then for the point $y = h(z)$, the sets $V_y = h(U_z) \ni y$ and $V_x = h(U_1) \ni x$ have the property $\overline{V_y} \cap \overline{V_x} \subset h(q\,\mathbb{N}) \subset p\,\mathbb{N}$. Now Lemma 5.3 implies that $p \notin \Pi_x$, which contradicts the choice of p. □

For any point $x \in \mathbb{N}$ consider the filter

$$\mathcal{F}_x = \{F \subset \mathbb{N} \colon \exists\, U_x \in \tau_x\ \exists\, U_1 \in \tau_1 \text{ such that } \overline{U_x} \cap \overline{U_1} \subset F\}.$$

The definition of the filter $\mathcal{F}_x$ and Lemma 5.3 imply

Lemma 5.5. *For any point $x \in \mathbb{N} \setminus \{1\}$ we get*

(1) $\Pi_x = \{p \in \Pi \colon p\,\mathbb{N} \notin \mathcal{F}_x\}$;
(2) $\mathcal{F}_x = \{F \subset \mathbb{N} \colon \exists\, a, b \in \mathbb{N}$ *such that* $\Pi_b \cap \Pi_x = \emptyset$ *and* $\overline{1 + a\,\mathbb{N}_0} \cap \overline{x + b\,\mathbb{N}_0} \subset F\}$.

The interplay between Π_x and $\mathcal{F}_x$ from Lemma 5.5 implies

Lemma 5.6. *For two numbers $x, y \in \mathbb{N} \setminus \{1\}$ the following conditions are equivalent:*

(1) $\mathcal{F}_x \subset \mathcal{F}_y$;
(2) $\Pi_y \subset \Pi_x$.

Lemma 5.7. *For every homeomorphism h of the Golomb space $\mathbb{N}_\tau$, and any numbers $x, y \in \mathbb{N} \setminus \{1\}$ with $\Pi_x \subset \Pi_y$ we get $\Pi_{h(x)} \subset \Pi_{h(y)}$.*

Proof: Assume that $\Pi_x \subset \Pi_y$. By Lemma 5.6, $\mathcal{F}_y \subset \mathcal{F}_x$. Lemma 5.4 and the (topological) definition of the filters $\mathcal{F}_x$ and $\mathcal{F}_{h(x)}$ imply that $h[\mathcal{F}_x] = \mathcal{F}_{h(x)}$ where

$h[\mathcal{F}_x] = \{h(F)\colon F \in \mathcal{F}_x\}$. By analogy we can show that $h[\mathcal{F}_y] = \mathcal{F}_{h(y)}$. Then $\mathcal{F}_{h(y)} = h[\mathcal{F}_y] \subset h[\mathcal{F}_x] = \mathcal{F}_{h(x)}$ and by Lemma 5.6, $\Pi_{h(x)} \subset \Pi_{h(y)}$. □

Lemma 5.8. *For every homeomorphism $h\colon \mathbb{N}_\tau \to \mathbb{N}_\tau$ there exists a unique bijective map $\sigma\colon \Pi \to \Pi$ of the set Π of all prime numbers such that $\Pi_{h(p)} = \{\sigma(p)\}$ and $\Pi_{h^{-1}(q)} = \{\sigma^{-1}(q)\}$ for any $p, q \in \Pi$.*

Proof: By Lemma 5.4, for every prime number p the image $h(p)$ is not equal to 1, which implies that the set $\Pi_{h(p)}$ is not empty and hence contains some prime number $\sigma(p)$. We claim that $\Pi_{h(p)} = \{\sigma(p)\}$. Since $\Pi_{\sigma(p)} = \{\sigma(p)\} \subset \Pi_{h(p)}$, we can apply Lemma 5.7 and conclude that $\Pi_{h^{-1}(\sigma(p))} \subset \Pi_p = \{p\}$ and hence $\Pi_{h^{-1}(\sigma(p))} = \{p\}$. Since $\Pi_p = \{p\} \subset \Pi_{h^{-1}(\sigma(p))}$, we can apply Lemma 5.7 once more and conclude that $\Pi_{h(p)} \subset \Pi_{\sigma(p)} = \{\sigma(p)\}$ and hence $\Pi_{h(p)} = \{\sigma(p)\}$.

Next, we show that the map σ is bijective. The injectivity of σ follows from the equality $\Pi_{h^{-1}(\sigma(p))} = \{p\}$ holding for every $p \in \Pi$. To see that σ is surjective, take any prime number q and choose any prime number $p \in \Pi_{h^{-1}(q)}$. Since $\Pi_p \subset \Pi_{h^{-1}(q)}$, we can apply Lemma 5.7 and conclude that $\{\sigma(p)\} = \Pi_{h(p)} \subset \Pi_q = \{q\}$ and hence $q = \sigma(p)$. Then

$$\Pi_{h^{-1}(q)} = \Pi_{h^{-1}(\sigma(p))} = \{p\} = \{\sigma^{-1}(q)\}.$$

The equality $\Pi_{h(p)} = \{\sigma(p)\}$ holding for every $p \in \Pi$ witnesses that the map $\sigma\colon \Pi \to \Pi$ is uniquely determined by the homeomorphism h. □

Lemma 5.8 admits a self-improvement:

Lemma 5.9. *For every homeomorphism $h\colon \mathbb{N}_\tau \to \mathbb{N}_\tau$ there exists a unique bijective map $\sigma\colon \Pi \to \Pi$ such that $\Pi_{h(x)} = \sigma(\Pi_x)$ and $\Pi_{h^{-1}(x)} = \sigma^{-1}(\Pi_x)$ for any $x \in \mathbb{N}$.*

Proof: By Lemma 5.8 for every homeomorphism $h\colon \mathbb{N}_\tau \to \mathbb{N}_\tau$ there exists a unique bijective map $\sigma\colon \Pi \to \Pi$ of such that $\Pi_{h(p)} = \sigma(p)$ and $\Pi_{h^{-1}(p)} = \sigma^{-1}(p)$ for any $p \in \Pi$.

We claim that $\Pi_{h(x)} = \sigma(\Pi_x)$ for any number $x \in \mathbb{N}$. If $x = 1$, then this follows from Lemma 5.4. So, we assume that $x \neq 1$. For every prime number $p \in \Pi_x$ the inclusion $\Pi_p = \{p\} \subset \Pi_x$ and Lemma 5.7 imply $\{\sigma(p)\} = \Pi_{h(p)} \subset \Pi_{h(x)}$. So, $\sigma(\Pi_x) \subset \Pi_{h(x)}$.

On the other hand, for any prime number $q \in \Pi_{h(x)}$, we can apply Lemma 5.8 to the homeomorphism h^{-1} and conclude that the set $\Pi_{h^{-1}(q)}$ coincides with the singleton $\{p\}$ of some prime number p. Taking into account that $\Pi_p = \{p\} = \Pi_{h^{-1}(q)}$ and applying Lemma 5.7, we conclude that $\{\sigma(p)\} = \Pi_{h(p)} = \Pi_q = \{q\} \subset \Pi_{h(x)}$. Applying Lemma 5.7 to the inclusion $\Pi_q \subset \Pi_{h(x)}$, we get the inclusion $\{p\} = \Pi_{h^{-1}(q)} \subset \Pi_{h^{-1}(h(x))} = \Pi_x$ and finally $q = \sigma(p) \in \sigma(\Pi_x)$. So, $\Pi_{h(x)} \subset \sigma(\Pi_x)$ and hence $\Pi_{h(x)} = \sigma(\Pi_x)$. Also $\Pi_{h^{-1}(q)} = \{p\} = \{\sigma^{-1}(q)\}$. □

For a number $x \in \mathbb{N}$ we shall denote by $x^{\mathbb{N}} := \{x^n\colon n \in \omega\}$ the multiplicative semigroup in $\mathbb{N}$ generated by x. In this case we say that $x^{\mathbb{N}}$ is the *monogenic* *semigroup generated by* x. For $x, k \in \mathbb{N}$ the monogenic semigroup $(x^k)^{\mathbb{N}}$ generated by x^k will be denoted by $x^{k\,\mathbb{N}}$.

Lemma 5.10. *Each homeomorphism* h *of the Golomb space* $\mathbb{N}_\tau$ *preserves monogenic semigroups in the sense that*

$$h(a^{\mathbb{N}}) = h(a)^{\mathbb{N}} \qquad \textit{for any } a \in \mathbb{N}.$$

Proof: Fix any point $a \in \mathbb{N}$ and put $b := h(a)$. First we show that $h(a^{\mathbb{N}}) \subset b^{\mathbb{N}}$. To derive a contradiction, assume that $h(a^n) \notin b^{\mathbb{N}}$ for some $n \geq 2$.

By Lemma 5.9, there exists a bijective function $\sigma\colon \Pi \to \Pi$ such that $\Pi_{h(x)} = \sigma(\Pi_x)$ for all $x \in \mathbb{N}$. By Theorem 4.4, the polynomial $f\colon \mathbb{N}_\tau \to \mathbb{N}_\tau$, $f\colon x \mapsto x^n$, is continuous. Then the map $\varphi = h \circ f \circ h^{-1}\colon \mathbb{N}_\tau \to \mathbb{N}_\tau$ is continuous, too. Observe that $\varphi(b) = h \circ f(a) = h(a^n)$ and

$$\Pi_{\varphi(b)} = \Pi_{h(a^n)} = \sigma(\Pi_{a^n}) = \sigma(\Pi_a) = \Pi_{h(a)} = \Pi_b.$$

Then for every $k \in \mathbb{N}$ the numbers $h(a^n)$ and $b^k - 1$ are coprime. Choose a number $k \in \mathbb{N}$ such that $b^k - 1 > h(a^n)$ and consider the neighborhood $h(a^n) + (b^k - 1)\,\mathbb{N}_0$ of $h(a^n) = \varphi(b)$. By the continuity of the map φ, the point b has a neighborhood $b + d\,\mathbb{N}_0 \in \tau_b$ such that $\varphi(b + d\,\mathbb{N}_0) \subset h(a^n) + (b^k - 1)\,\mathbb{N}_0$.

By Dirichlet theorem the arithmetic progression $b+(b^k-1)d\,\mathbb{N}_0$ contains a prime number p. Lemma 5.9 implies that $\varphi(p) = p^{\,l}$ for some $l \geq 1$. Then

$$\varphi(p) = p^{\,l} \in \varphi(b+d\,\mathbb{N}_0) \cap (b+(b^k-1)\,\mathbb{N}_0)^l \subset (h(a^n)+(b^k-1)\,\mathbb{N}_0) \cap (b^{\,l}+(b^k-1)\,\mathbb{N}_0)$$

and hence $h(a^n) \in b^{\,l} + (b^k - 1)\,\mathbb{Z}$. Write l as $l = ki + j$ where $j \in [0, k)$. If $i = 0$, then $l = j$ and hence $h(a^n) \in b^{\,l} + (b^k - 1)\,\mathbb{Z} = b^{\,j} + (b^k - 1)\,\mathbb{Z}$. If $i > 0$, then $b^{\,l} - b^{\,j} = b^{\,j}((b^k)^i - 1) \in (b^k - 1)\,\mathbb{Z}$ and again $h(a^n) \in b^{\,l} + (b^k - 1)\,\mathbb{Z} = b^{\,j} + (b^k - 1)\,\mathbb{Z}$. In both cases we obtain the inclusion $h(a^n) \in b^{\,j} + (b^k - 1)\,\mathbb{Z}$, which is not possible as $0 < |b^{\,j} - h(a^n)| \leq \max\{h(a^n), b^{k-1}\} < b^k - 1$. This contradiction completes the proof of the inclusion $h(a^{\mathbb{N}}) \subset h(a)^{\mathbb{N}}$.

By analogy, for $x := h(a)$ we can prove that $h^{-1}(x^{\mathbb{N}}) \subset (h^{-1}(x))^{\mathbb{N}}$ and hence $h(a)^{\mathbb{N}} = x^{\mathbb{N}} \subset h(a^{\mathbb{N}})$, which, combined with $h(a^{\mathbb{N}}) \subset h(a)^{\mathbb{N}}$, yields the equality $h(a^{\mathbb{N}}) = h(a)^{\mathbb{N}}$. □

Lemma 5.11. *The set* $h(\Pi) = \Pi$ *for any homeomorphism* h *of the Golomb space* $\mathbb{N}_\tau$*.*

Proof: It suffices to show that $h(p) \in \Pi$ for every $p \in \Pi$. By Lemma 5.9, there exists a bijective map $\sigma\colon \Pi \to \Pi$ such that $\Pi_{h(x)} = \sigma(\Pi_x)$ and $\Pi_{h^{-1}(x)} = \sigma^{-1}(\Pi_x)$ for all $x \in \mathbb{N}$. In particular, $\Pi_{h(p)} = \{q\}$ for $q = \sigma(p)$ and $\Pi_{h^{-1}(q)} = \{p\}$. This implies that $h(p) = q^n$ and $h^{-1}(q) = p^m$ for some $n, m \in \mathbb{N}$. Applying Lemma 5.10, we obtain

$$q^{\mathbb{N}} = h(p^m)^{\mathbb{N}} = h(p^{m\,\mathbb{N}}) \subset h(p^{\mathbb{N}}) = h(p)^{\mathbb{N}} = q^{n\,\mathbb{N}}$$

and hence $n = 1$. □

Lemma 5.12. *For any homeomorphism h of the Golomb space $\mathbb{N}_\tau$ we have $\Pi_{h(x)} = h(\Pi_x)$ for every $x \in \mathbb{N}$.*

Proof: By Lemma 5.9, there exists a bijective function $\sigma\colon \Pi \to \Pi$ such that $\Pi_{h(x)} = \sigma(\Pi_x)$ for all $x \in \mathbb{N}$. By Lemma 5.11 for every prime number p we get $\{h(p)\} = \Pi_{h(p)} = \{\sigma(p)\}$. Therefore, $\sigma = h \restriction \Pi$ and $\Pi_{h(x)} = \sigma(\Pi_x) = h(\Pi_x)$ for every $x \in \mathbb{N}$. □

Next, we investigate the restrictions of homeomorphisms of $\mathbb{N}_\tau$ to monogenic subsemigroups of $\mathbb{N}$.

A function $\mu\colon \mathbb{N} \to \mathbb{N}$ is called *multiplicative* if $\mu(xy) = \mu(x)\mu(y)$ for all $x, y \in \mathbb{N}$. It is easy to see that a function is multiplicative if and only if

- ∘ $\mu(p^n) = \mu(p)^n$ for every prime number $p \in \Pi$ and every $n \in \mathbb{N}$;
- ∘ $\mu(xy) = \mu(x)\mu(y)$ for any coprime numbers $x, y \in \mathbb{N}$.

This implies that each multiplicative function μ is uniquely determined by its restriction $\mu \restriction \Pi$ to the set Π of prime numbers. If a multiplicative function $\mu\colon \mathbb{N} \to \mathbb{N}$ is bijective, then $\mu(\Pi) = \Pi$ and the inverse function μ^{-1} is multiplicative, too.

Lemma 5.13. *Let h be a homeomorphism of the Golomb space $\mathbb{N}_\tau$. For every $a \in \mathbb{N} \setminus \{1\}$ there exists a multiplicative bijection μ_a of $\mathbb{N}$ such that $h(a^n) = h(a)^{\mu_a(n)}$ for any $n \in \omega$.*

Proof: Given any $a \in \mathbb{N}$, let $b := h(a)$. By Lemma 5.10, we have a function $\mu_a\colon \mathbb{N} \to \mathbb{N}$ such that $h(a^n) = b^{\mu_a(n)}$ for every $n \in \mathbb{N}$. Let us consider the families $X := \{a^{n\,\mathbb{N}}\colon n \in \mathbb{N}\}$ and $Y := \{b^{n\,\mathbb{N}}\colon n \in \mathbb{N}\}$ of monogenic subsemigroups of $\mathbb{N}$. We endow $\mathbb{N}$ with the partial order of divisibility and X, Y with the partial order of inclusion. The maps $n \mapsto a^{n\,\mathbb{N}}$ and $n \mapsto b^{n\,\mathbb{N}}$ are lattice-isomorphisms. We consider the map $\widetilde{h}\colon X \to Y$, $\widetilde{h}\colon a^{n\,\mathbb{N}} \mapsto b^{\mu_a(n)\,\mathbb{N}} = h(a^n)^{\mathbb{N}} = h(a^{n\,\mathbb{N}})$. Since $\widetilde{h}$ is taking the image under the homeomorphism h and since the same construction may be performed for h^{-1}, we have that $\widetilde{h}$ and so μ_a are lattice isomorphisms. We are done since a bijection of $\mathbb{N}$ is multiplicative if and only if it is a lattice isomorphism. The implication we need holds since any lattice isomorphism preserves coprimeness and the product and the join of coprime numbers are the same, and since the chains $\{p^n\colon n \in \mathbb{N}_0\}$ for $p \in \Pi$ are the only maximal chains made of join-irreducible elements in $\mathbb{N}$. □

Next, we shall prove that $\mu_a = \mu_b$ for any elements $a, b \in \mathbb{N}$. For this we use the following lemma, proved by joint efforts of MathOverflow users François Brunault and so-called friend Don, see [4].

Lemma 5.14. *For any numbers $b \in \Pi$ and $a \in \mathbb{N} \setminus \{x^b\colon x \in \mathbb{N}\}$ there exist infinitely many prime numbers $p \in 1 + b\,\mathbb{N}$ such that $a^{(p-1)/b} \neq 1 \mod p$.*

Proof: The choice of a ensures that the equation $x^b = a$ has no solutions in $\mathbb{Q}$. Then we can apply the Grunwald–Wang theorem, see [2, Chapter X], and conclude that the set P of prime numbers p for which the equation $x^b = a \mod p$ has no solutions is infinite. We claim that $P \setminus \Pi_a \subset 1 + b\,\mathbb{Z}$. In the opposite case, we could find a prime number $p \in P$ with $p \notin \Pi_a \cup (1 + b\,\mathbb{Z})$ and conclude that $p-1$ is not divisible by the prime number b and hence b is coprime with $p-1$. It is well-known that the multiplicative group $\mathbb{Z}_p^*$ of the finite field $\mathbb{Z}_p := \mathbb{Z}/p\,\mathbb{Z}$ is cyclic of order $p-1$. Since b is coprime with $p-1$, the map $\mathbb{Z}_p^* \to \mathbb{Z}_p^*$, $x \mapsto x^b$, is bijective, which implies that the equation ($x^b = a \mod p$) has a solution. But this contradicts the choice of $p \in P$. This contradiction completes the proof of the inclusion $P \setminus \Pi_a \subset 1 + b\,\mathbb{Z}$.

It remains to prove that for each $p \in P \cap (1 + b\,\mathbb{Z})$ we have $a^{(p-1)/b} \neq 1 \mod p$. Since the group $\mathbb{Z}_p^*$ is cyclic, there exists a positive number $g < p$ such that the coset $g + p\,\mathbb{Z}$ is a generator of $\mathbb{Z}_p^*$. Then $a = g^k \mod p$ for some $k < p$. Assuming that $a^{(p-1)/b} = 1 \mod p$, we would conclude that $g^{k(p-1)/b} = 1 \mod p$ and hence $k(p-1)/b \in (p-1)\,\mathbb{Z}$. Then k/b is integer and hence $a = g^k = (g^{k/b})^b \mod p$, which contradicts the choice of $p \in P$. □

Lemma 5.15. *For any homeomorphism h of the Golomb space $\mathbb{N}_\tau$ there exists a multiplicative bijection $\mu\colon \mathbb{N} \to \mathbb{N}$ such that $h(a^n) = h(a)^{\mu(n)}$ for any $a, n \in \mathbb{N}$.*

Proof: By Lemma 5.13 for every $a \in \mathbb{N}\setminus\{1\}$ there exists a multiplicative bijection μ_a of $\mathbb{N}$ such that $h(a^n) = h(a)^{\mu_a(n)}$ for all $n \in \mathbb{N}$. Let $P := \{n^{k+1} : n, k \in \mathbb{N}\}$ be the set of all nontrivial powers of natural numbers.

First we prove that for every $a \in \mathbb{N} \setminus P$ and $n \in \Pi$ there exists a neighborhood $U_a \subset \mathbb{N}_\tau$ of a such that $\mu_a(n) = \mu_r(n)$ for every prime number $r \in U_a$.

Observe that the image $q := \mu_a(n)$ of the prime number n under the multiplicative bijection μ_a of $\mathbb{N}$ is prime. Taking into account that $a \notin P$, we can apply Lemma 5.13 and conclude that $h(a) \notin P$.

By Lemma 5.14, there exists a prime number $p \in (1+q\,\mathbb{N})\setminus(\Pi_{h(a)} \cup \Pi_{h(a^n)})$ such that $h(a)^{(p-1)/q} \neq 1 \mod p$. By the continuity of the maps h and $f\colon \mathbb{N}_\tau \to \mathbb{N}_\tau$, $f\colon x \mapsto h(x^n)$, the point a has a neighborhood $U_a \in \tau_a$ such that $h(U_a) \subset h(a) + p\,\mathbb{N}_0$ and $f(U_a) \subset f(a) + p\,\mathbb{N}_0 = h(a^n) + p\,\mathbb{N}_0$. We claim that the neighborhood U_a has the required property.

For every prime number $r \in U_a$ we have $h(r) = h(a) \mod p$ and hence $h(r)^{\mu_r(n)} = h(a)^{\mu_r(n)} \mod p$. Also, $h(a)^{\mu_a(n)} = h(a^n) = f(a) = f(r) = h(r^n) = h(r)^{\mu_r(n)} \mod p$. Together, we have $h(a)^{\mu_a(n)} = h(a)^{\mu_r(n)} \mod p$. Since $h(a)$ is coprime with p, we have $h(a) \in \mathbb{Z}_p^*$, and hence $h(a)^{|\mu_a(n)-\mu_r(n)|} = 1$ in $\mathbb{Z}_p^*$. Let $k \in \mathbb{N}$ be the smallest number such that $h(a)^k = 1$ in $\mathbb{Z}_p^*$. Clearly, k divides $p-1$. Since $h(a)^{(p-1)/q} \neq 1$ in $\mathbb{Z}_p^*$ and q is prime, we have that $\mu_a(n) = q$ divides k, which divides $\mu_a(n) - \mu_r(n)$. Hence, $\mu_a(n)$ divides $\mu_r(n)$, and since both numbers are prime, they are equal.

Now we can prove that $\mu_a(n) = \mu_b(n)$ for arbitrary $n \in \Pi$ and $a, b \in \mathbb{N} \setminus P$. By Lemma 2.5, the subspace $\mathbb{N} \setminus P$ of $\mathbb{N}_\tau$ is connected. Then there exists a chain of points $a = a_0, a_1, \ldots, a_m = b$ in $\mathbb{N} \setminus P$ such that for every $i < m$ the intersection $U_{a_i} \cap U_{a_{i+1}}$ is not empty and hence contains some prime number r_i. Then $\mu_{a_i}(n) = \mu_{r_i}(n) = \mu_{a_{i+1}}(n)$ for all $i < m$ and hence $\mu_a(n) = \mu_{a_0}(n) = \mu_{a_n}(n) = \mu_b(n)$.

Since the functions μ_a and μ_b are multiplicative, the equality $\mu_a(n) = \mu_b(n)$ holding for all prime $n \in \Pi$ implies the equality $\mu_a = \mu_b$. Choose any number $c \in \mathbb{N} \setminus P$ and put $\mu := \mu_c$.

Finally, we prove that $\mu_a = \mu$ for any $a \in \mathbb{N}$. This equality has been proved for any $a \in \mathbb{N} \setminus P$. So, assume that $a \in P$. In this case $a = \alpha^k$ for some $\alpha \in \mathbb{N} \setminus P$ and some $k \in \mathbb{N}$. Lemma 5.13 implies that $h(\alpha) \notin P$. Then for any $n \in \mathbb{N}$ we get

$$\begin{aligned} h(a)^{\mu_a(n)} &= h(a^n) = h(\alpha^{kn}) = h(\alpha)^{\mu_\alpha(kn)} = h(\alpha)^{\mu(kn)} \\ &= h(\alpha)^{\mu(k)\mu(n)} = (h(\alpha)^{\mu(k)})^{\mu(n)} = h(\alpha^k)^{\mu(n)} = h(a)^{\mu(n)}. \end{aligned}$$

□

Finally, we show that the multiplicative bijection μ appearing in Lemma 5.15 is continuous in the *Furstenberg topology* on $\mathbb{N}$. This topology is generated by the base consisting of all possible arithmetic progressions $a + b\,\mathbb{N}_0$ where $a, b \in \mathbb{N}$. This topology was introduced by H. Furstenberg, see [11], in his famous topological proof of the Euclides theorem on infinitude of prime numbers. It is clear that the Furstenberg topology on $\mathbb{N}$ is stronger than the Golomb topology. It is easy to see that the space $\mathbb{N}$ endowed with the Furstenberg topology is regular, second-countable and has no isolated points. By Sierpiński theorem [9, 6.2.A (d)], it is homeomorphic to the space $\mathbb{Q}$ of rational numbers.

A bijective function $f\colon \mathbb{N} \to \mathbb{N}$ will be called a *Furstenberg homeomorphism* if f is a homeomorphism of the space $\mathbb{N}$ endowed with the Furstenberg topology.

Lemma 5.16. *For any homeomorphism h of the Golomb space $\mathbb{N}_\tau$ there exists a unique multiplicative Furstenberg homeomorphism $\mu\colon \mathbb{N} \to \mathbb{N}$ such that $h(a^n) = h(a)^{\mu(n)}$ for any $a, n \in \mathbb{N}$.*

Proof: Fix a homeomorphism h of the Golomb space $\mathbb{N}_\tau$. By Lemma 5.15, there exists a multiplicative bijection $\mu\colon \mathbb{N} \to \mathbb{N}$ such that $h(a^n) = h(a)^{\mu(n)}$ for all $a, n \in \mathbb{N}$. It is clear that the latter formula uniquely determines μ. So, it remains to prove that the map $\mu\colon \mathbb{N} \to \mathbb{N}$ is continuous with respect to the Furstenberg topology on $\mathbb{N}$. It is clear that the Furstenberg topology is generated by a subbase consisting of the arithmetic progressions $\mathbb{N} \cap (a + p^k\,\mathbb{Z})$ where $a, k \in \mathbb{N}$ and p is prime. So, fix any $n \in \mathbb{N}$ and a subbasic neighborhood $\mathbb{N} \cap (\mu(n) + p^k\,\mathbb{Z})$ for some $p \in \Pi$ and $k \in \mathbb{N}$. Consider the group $\mathbb{Z}^*_{p^{k+2}}$ of invertible elements of the ring $\mathbb{Z}_{p^{k+2}} := \mathbb{Z}/p^{k+2}\,\mathbb{Z}$ and fix an element $b + p^{k+2}\,\mathbb{Z} \in \mathbb{Z}^*_{p^{k+2}}$ of the highest possible order. By the classical result of C. F. Gauss, see [12, pages 52–56], if p is odd, then the group $\mathbb{Z}^*_{p^{k+2}}$ is cyclic of order $p^{k+1}(p-1)$. If $p = 2$ then the group $\mathbb{Z}^*_{p^{k+2}}$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2^k}$. This implies that $b + p^{k+2}\,\mathbb{Z}$ has order $p^{k+1}(p-1)$ or p^k. Let $a := h^{-1}(b)$ and observe that $h(a)^{\mu(n)} + p^{k+2}\,\mathbb{N}_0$ is a neighborhood of $b^{\mu(n)} = h(a)^{\mu(n)} = h(a^n)$ in the Golomb topology.

By the continuity of the map h at a^n, the point a^n has a neighborhood $a^n + d\,\mathbb{N}_0$ in the Golomb topology such that $h(a^n + d\,\mathbb{N}_0) \subset h(a^n) + p^{k+2}\,\mathbb{N}_0$. Let $\varphi(d)$ denote the cardinality of the group $\mathbb{Z}^*_d$ of invertible elements of the ring $\mathbb{Z}_d$. We claim that

$\mu(n+\varphi(d)\,\mathbb{N}_0) \subset \mu(n)+p^k\,\mathbb{Z}$, (so $n+\varphi(d)\,\mathbb{N}_0$ is a required neighborhood of n in the Furstenberg topology, witnessing that the map μ is continuous at n). Given any number $m \in n + \varphi(d)\,\mathbb{N}_0$, observe that $a^m - a^n = a^n(a^{m-n} - 1) = 0 \mod d$ and hence $a^m \in a^n + d\,\mathbb{N}_0$. Then choice of d guarantees that $h(a^m) \in h(a^n) + p^{k+2}\,\mathbb{N}_0$ and hence

$$\begin{aligned} h(a^m) - h(a^n) &= h(a)^{\mu(m)} - h(a)^{\mu(n)} \\ &= h(a)^{\min\{\mu(n),\mu(m)\}}(h(a)^{|\mu(m)-\mu(n)|} - 1) \\ &\in p^{k+2}\,\mathbb{N}_0. \end{aligned}$$

Since $h(a) = b$ is coprime with p, this implies that $h(a)^{|\mu(m)-\mu(n)|} = 1 \mod p^{k+2}$. Taking into account that $b+p^{k+2}\,\mathbb{Z}$ has order $p^{k+1}(p-1)$ or p^k in the group $\mathbb{Z}^*_{p^{k+2}}$, we conclude that $\mu(m) - \mu(n) \in p^k\,\mathbb{Z}$. □

Remark 5.17. Answering a problem, see [5], posed by the first author on the MathOverlow, Y. de Cornulier proved that any multiplicative bijection $\mu\colon \mathbb{N} \to \mathbb{N}$ with finite support $\mathrm{supp}_\Pi(\mu) := \{p \in \Pi\colon \mu(p) \neq p\}$ is a Furstenberg homeomorphism, and also that there exist multiplicative Furstenberg homeomorphisms of $\mathbb{N}$ which have infinite support. Also he constructed a multiplicative bijection of $\mathbb{N}$ which is not a Furstenberg homeomorphism. On the other hand, the density of the set Π in $\mathbb{N}_\tau$ implies that any non-identity homeomorphism h of the Golomb space $\mathbb{N}_\tau$ has infinite support $\mathrm{supp}_\Pi(h)$.

We do not know if the multiplicative bijection μ in Lemmas 5.15 and 5.16 is always equal to the identity map of $\mathbb{N}$. So, we ask

Question 5.18. Let h be a homeomorphism of the Golomb space $\mathbb{N}_\tau$. Is it true that $h(x^n) = h(x)^n$ for any $x \in \mathbb{N}$?

Theorem 5.1 implies that the Golomb space $\mathbb{N}_\tau$ is not topologically homogeneous. On the other hand, we do not know the answer to the following intriguing problem (posed also in [3]).

Problem 5.19. Is the Golomb space rigid?

We recall that a topological space X is *rigid* if each homeomorphism $h\colon X \to X$ is equal to the identity map of X.

The affirmative answer to the following problem would imply a negative answer to Problem 5.19.

Problem 5.20. Let μ be a multiplicative bijection of $\mathbb{N}$. Is μ a homeomorphism of $\mathbb{N}_\tau$ if the restriction $\mu \restriction \Pi$ is a homeomorphism of the (metrizable zero-dimensional) subspace Π of $\mathbb{N}_\tau$?

Remark 5.21. A counterpart of the Golomb topology on domains (= commutative rings without zero divisors) was introduced and studied by J. Knopfmacher and Š. Porubský, see [17]. In their recent preprint [7] P.L. Clark, N. Lebowitz-Lockard and P. Pollack extended some results of this paper to the Golomb topology on domains.

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References

[1] Apostol T. M., *Introduction to Analytic Number Theory*, Undergraduate Texts in Mathematics, Springer, New York, 1976.

[2] Artin E., Tate J., *Class Field Theory*, Advanced Book Classics, Addison-Wesley Publishing Company, Advanced Book Program, Redwood City, 1990.

[3] Banakh T., *Is the Golomb countable connected space topologically rigid?*, available at `https://mathoverflow.net/questions/285557`.

[4] Banakh T., *A simultaneous generalization of the Grunwald-Wang and Dirichlet theorems on primes*, available at `https://mathoverflow.net/questions/310130`.

[5] Banakh T., *Is the identity function a unique multiplicative homeomorphism of* $\mathbb{N}$*?*, available at `https://mathoverflow.net/questions/310163`.

[6] Brown M., *A countable connected Hausdorff space*, Bull. Amer. Math. Soc. **59** (1953), Abstract 423, 367.

[7] Clark P. L., Lebowitz-Lockard N., Pollack P., *A note on Golomb topologies*, Quaest. Math. published online, available at `https://doi.org/10.2989/16073606.2018.1438533`.

[8] Dirichlet P. G. L., *Lectures on Number Theory*, History of Mathematics, 16, American Mathematical Society, Providence, London, 1999.

[9] Engelking R., *General Topology*, Sigma Series in Pure Mathematics, 6, Heldermann Verlag, Berlin, 1989.

[10] Engelking R., *Theory of Dimensions Finite and Infinite*, Sigma Series in Pure Mathematics, 10, Heldermann Verlag, Lemgo, 1995.

[11] Furstenberg H., *On the infinitude of primes*, Amer. Math. Monthly **62** (1955), 353.

[12] Gauss C. F., *Disquisitiones Arithmeticae*, Springer, New York, 1986.

[13] Golomb S. W., *A connected topology for the integers*, Amer. Math. Monthly **66** (1959), 663–665.

[14] Golomb S. W., *Arithmetica topologica*, General Topology and Its Relations to Modern Analysis and Algebra, Proc. Symp., Prague, 1961, Academic Press, New York; Publ. House Czech. Acad. Sci., Praha (1962), 179–186.

[15] Jones G. A., Jones J. M., *Elementary Number Theory*, Springer Undergraduate Mathematics Series, Springer, London, 1998.

[16] Knaster B., Kuratowski K., *Sur les ensembles connexes*, Fund. Math. **2** (1921), no. 1, 206–256 (French).

[17] Knopfmacher J., Porubský Š., *Topologies related to arithmetical properties of integral domains*, Exposition Math. **15** (1997), no. 2, 131–148.

[18] Steen L. A., Seebach J. A. Jr., *Counterexamples in Topology*, Dover Publications, Mineola, 1995.

[19] Stevenhagen P., Lenstra H. W. Jr., *Chebotarëv and his density theorem*, Math. Intelligencer **18** (1996), no. 2, 26–37.

[20] Sury B., *Frobenius and his density theorem for primes*, Resonance **8** (2003), no. 12, 33–41.

[21] Szczuka P., *The connectedness of arithmetic progressions in Furstenberg's, Golomb's, and Kirch's topologies*, Demonstratio Math. **43** (2010), no. 4, 899–909.
[22] Szczuka P., *The Darboux property for polynomials in Golomb's and Kirch's topologies*, Demonstratio Math. **46** (2013), no. 2, 429–435.
[23] Wang S., *A counter-example to Grunwald's theorem*, Ann. of Math. (2) **49** (1948), 1008–1009.

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