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GENERALIZED MORREY SPACES ASSOCIATED TO SCHRÖDINGER OPERATORS AND APPLICATIONS

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Abstract. We first introduce new weighted Morrey spaces related to certain non-negative potentials satisfying the reverse Hölder inequality. Then we establish the weighted strong-type and weak-type estimates for the Riesz transforms and fractional integrals associated to Schrödinger operators. As an application, we prove the Calderón-Zygmund estimates for solutions to Schrödinger equation on these new spaces. Our results cover a number of known results.

Keywords: Morrey space; Schrödinger operator; Riesz transform; fractional integral; Calderón-Zygmund estimate

MSC 2010: 42B20, 42B35

1. Introduction

It is well known that Morrey first introduced the classical Morrey spaces to investigate the local behavior of solutions to second-order elliptic partial differential equations in [21]. In recent years there has been an explosion of interest in studying the boundedness of operators on Morrey-type spaces. It was found that many properties of solutions to PDEs are concerned with the boundedness of some operators on Morrey-type spaces. In fact, the better inclusion between Morrey and Hölder spaces permits to obtain higher regularity of the solutions to different elliptic and parabolic boundary problems.

Weighted inequalities arise naturally in Fourier analysis, but their use is best justified by the variety of applications in which they appear. For example, the theory of weights is of great importance in the study of boundary value problems for Laplace equations on Lipschitz domains. Other applications of weighted inequalities include extrapolation theory, vector-valued inequalities, and estimates for certain

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classes of nonlinear mathematical physics equations. Therefore, it is worth pointing out that many authors are interested in the weighted norm inequalities when the weight function belongs to the Muckenhoupt classes.

In the last decade many results in classical harmonic analysis have been devoted to norm inequalities involving classical and non-classical operators in the setting of weighted Morrey spaces. The results obtained are mostly the extensions of well-known analogues in the weighted Lebesgue spaces. It has been proved by many authors that most of the operators are bounded on not only a weighted Lebesgue space but also an appropriate weighted Morrey space. In this paper we show that these results are valid on a larger family of functional spaces including weighted Morrey spaces.

In recent years the problem related to Schrödinger operator has attracted a great deal of attention of many mathematicians; see [3], [4], [5], [6], [8], [11], [12], [20], [27], [34] and references therein. In this paper we consider a Schrödinger operator $\mathcal{L} = -\Delta + \mathcal{V}$ on $\mathbb{R}^n$, $n \geqslant 3$, where the potential $\mathcal{V}$ belongs to RH_q for some $q > n/2$ and RH_q is the reverse Hölder class defined in Section 2.

For $x \in R^n$, we define the functions $\varrho_{\mathcal{V}}(x)$, $m_{\mathcal{V}}(x)$ by:

$$\varrho_{\mathcal{V}}(x) = \frac{1}{m_{\mathcal{V}}(x)} = \sup_{r>0}\biggl\{r\colon\ \frac{1}{r^{n-2}}\int_{B(x,r)} \mathcal{V}(y)\,\mathrm{d}y \leqslant 1\biggr\}.$$

It follows from the above conditions on $\mathcal{V}$ that $0 < m_{\mathcal{V}}(x) < \infty$.

Throughout the paper, for simplicity, we denote by B the ball with the center x_B and radius r_B. Next, we will introduce our new Morrey type spaces.

Let w, v be two weights and $\theta \in [0,1)$, $1 \leqslant p < s \leqslant \infty$, $\alpha \in (-\infty,\infty)$. We define the space $M^{p,s}_{\alpha,\theta}(w,v)$ as the space of all measurable functions f satisfying $\|f\|_{M^{p,s}_{\alpha,\theta}(w,v)} < \infty$, where

$$\|f\|_{M^{p,s}_{\alpha,\theta}(w,v)} = \sup_{r>0}\biggl[\int_{\mathbb{R}^n} ((1+rm_{\mathcal{V}}(x))^{\alpha} v(B(x,r))^{-\theta}\|f\chi_{B(x,r)}\|_{L^p(w)})^s\,\mathrm{d}x\biggr]^{1/s} < \infty.$$

If $w \equiv v$, then we will denote $M^{p,s}_{\alpha,\theta}(w,v)$ by $M^{p,s}_{\alpha,\theta}(w)$ for brevity.

In particular, when $\alpha = 0$ or $\mathcal{V} = 0$ and $w \equiv v \equiv 1$, we recover the space $(L^p, L^s)^{\theta}$ defined in [15] by Fofana (see also [13], [14]). In the case $\alpha = 0$ or $\mathcal{V} = 0$ and $w \equiv v \equiv 1$, $s = \infty$, the space $M^{p,s}_{\alpha,\theta}(w,v)$ is the Morrey space $M^p_{\theta/p}(\mathbb{R}^n)$ first introduced in [23]. Some new properties of $M^p_q(\mathbb{R}^n)$ have been studied in [1], [10], [32].

Komori and Shirai (in [19]) introduced a version of the weighted Morrey space $M^p_q(w,v)$, which is a natural generalization of the weighted Lebesgue space $L^p(w)$. The space $M^{p,s}_{\alpha,\theta}(w,v)$ could be viewed as an extension of the weighted Morrey space $M^p_q(w,v)$ when $\alpha = 0$ or $\mathcal{V} = 0$ and $s = \infty$, $\theta = 1/q - 1/p$. Meanwhile, when

$v = \mathrm{d}x$, $s = \infty$ and $\alpha = 0$, $\theta = 1/q - 1/p$, we have $M^{p,s}_{\alpha,\theta}(w,v)$ corresponds to the weighted Morrey space $M^p_q(w, \mathrm{d}x)$ introduced by Samko in [25]. In the case $s = \infty$ and $w \equiv v \equiv 1$, the space $M^{p,s}_{\alpha,\theta}(w,v)$ is the Morrey space $L^{p,n\theta p}_{\alpha p,V}(\mathbb{R}^n)$ defined in [30] by Tang and Dong.

The main purpose of this article is to study the following operators associated to $\mathcal{L}$:

▷ the Riesz transforms associated to $\mathcal{L}$ given by: $\nabla\mathcal{L}^{-1}\nabla$, $\nabla\mathcal{L}^{-1/2}$ and $\mathcal{L}^{-1/2}\nabla$;

▷ the $\mathcal{L}$-fractional integral operator defined by:

$$I_\beta f(x) = \mathcal{L}^{-\beta/2} f(x) = \int_0^\infty \mathrm{e}^{-t\mathcal{L}} f(x) t^{\beta/2-1}\,\mathrm{d}t \quad \text{for } 0 < \beta < n.$$

The following theorems are the main results of this paper.

Theorem 1.1. *Suppose that $\mathcal{V} \in RH_n$. Let T be one of the Riesz transforms $\nabla\mathcal{L}^{-1}\nabla$, $\nabla\mathcal{L}^{-1/2}$ and $\mathcal{L}^{-1/2}\nabla$. Let $1 \leqslant p < s \leqslant \infty$, $\alpha \in (-\infty,\infty)$, $\theta \in [0, 1/p)$ and $w \in A_p$.*

(i) *If $p > 1$, then T is bounded on $M^{p,s}_{\alpha,\theta}(w)$.*

(ii) *If $p = 1$, then for every $t > 0$,*

$$w(B)^{-\theta}(1 + r_B m_{\mathcal{V}}(x_B))^\alpha w(\{x \in B\colon\ |Tf(x)| > t\}) \leqslant \frac{C}{t}\|f\|_{M^{1,s}_{\alpha,\theta}(w)}$$

holds for all balls B.

Theorem 1.2. *Suppose that $\mathcal{V} \in RH_n$ and $b \in \mathrm{BMO}$. Let T be one of the Riesz transforms $\nabla\mathcal{L}^{-1}\nabla$, $\nabla\mathcal{L}^{-1/2}$ and $\mathcal{L}^{-1/2}\nabla$. Let $1 \leqslant p < s \leqslant \infty$, $\alpha \in (-\infty,\infty)$, $\theta \in [0, 1/p)$ and $w \in A_p$.*

(i) *If $p > 1$, then $[b,T]$ is bounded on $M^{p,s}_{\alpha,\theta}(w)$.*

(ii) *If $p = 1$, then for every $t > 0$,*

$$w(B)^{-\theta}(1 + r_B m_{\mathcal{V}}(x_B))^\alpha w(\{x \in B\colon\ [b,T]f(x) > t\}) \leqslant C\Big\|\Phi\Big(\frac{|f|}{t}\Big)\Big\|_{M^{1,\theta,\alpha}_{L\log L}(w)}$$

holds for all balls $B \subset \mathbb{R}^n$.

Theorem 1.3. *Suppose that $\mathcal{V} \in RH_{n/2}$. Let $\beta \in (0,n)$, $1 < p < n/\beta$, $\alpha \in (-\infty,\infty)$ and $1 < s \leqslant \infty$.*

(i) *If $1/q = 1/p - \beta/n$, $\theta \in [0, 1/q)$ and $w \in A_{p,q}$, then I_β is bounded from $M^{p,s}_{\alpha,\theta}(w^p, w^q)$ to $M^{q,s}_{\alpha,\theta}(w^q)$.*

(ii) *If* $q = n/(n-\beta)$, $\theta \in [0, 1/q)$ *and* $w^q \in A_1$, *then for every* $t > 0$,

$$w^q(B)^{-\theta}(1 + r_B m_{\mathcal{V}}(x_B))^{\alpha} w^q(\{x \in B\colon\ |I_\beta f(x)| > t\})^{1/q} \leqslant \frac{C}{t}\|f\|_{M^{1,s}_{\alpha,\theta}(w,w^q)}$$

holds for all balls B.

Theorem 1.4. *Suppose that* $\mathcal{V} \in RH_{n/2}$ *and* $b \in \mathrm{BMO}$. *Let* $\beta \in (0, n)$, $1 < p < n/\beta$, $\alpha \in (-\infty, \infty)$ *and* $1 < s \leqslant \infty$.

(i) *If* $1/q = 1/p - \beta/n$, $\theta \in [0, 1/q)$ *and* $w \in A_{p,q}$, *then* $[b, I_\beta]$ *is bounded from* $M^{p,s}_{\alpha,\theta}(w^p, w^q)$ *to* $M^{q,s}_{\alpha,\theta}(w^q)$.

(ii) *If* $q = n/(n-\beta)$, $\theta \in [0, 1/q)$ *and* $w^q \in A_1$, *then for every* $t > 0$,

$$w^q(B)^{-\theta}(1 + r_B m_{\mathcal{V}}(x_B))^{\alpha} w^q(\{x \in B\colon\ |[b, I_\beta]f(x)| > t\})^{1/q} \leqslant C\Big\|\Phi\Big(\frac{|f|}{t}\Big)\Big\|_{M^{1,\theta,\alpha}_{L\log L}(w)}$$

holds for all balls B.

In [30] Tang and Dong established the boundedness of Schrödinger type operators as Riesz trasform, fractional integral and their commutators on the weighted Morrey space $L^{p,n\theta p}_{\alpha p,V}(\mathbb{R}^n)$. In [16], [27], [34] the authors proved the boundedness of singular integrals related to Schrödinger operators on $\mathbb{R}^n$ and their commutators with BMO functions. In this sense, our results in Theorems 1.1, 1.2 and 1.3 improve those in [30] for Riesz transforms, $\mathcal{L}$-fractional integral and their commutators in our new space $M^{p,s}_{\alpha,\theta}(w, v)$. Moreover, our results extend those of Samko and Komori-Shirai to new Morrey spaces; see Theorems 1.1, 1.2, 1.3, 1.4. It is worth noticing that our endpoint inequalities are new even in the particular case $w \equiv v \equiv 1$.

The paper is organized as follows. First, we introduce some preliminary results in Section 2. Next, Section 3 and Section 4 are devoted to proving the main theorems. Finally, the Calderón-Zygmund inequalities for Schrödinger type equations are established in Section 5 as the applications of the main theorems.

2. Preliminaries

We begin this section by introducing some notation that will be used in the sequel. We define the set $\mathbb{R}^n \setminus E$ as E^{c} and its characteristic function as χ_E. Throughout the paper, we always use C and c to denote positive constants that are independent of the main parameters involved but their values may differ from line to line. We write $A \lesssim B$ to denote $A \leqslant cB$ for a positive constant c independent of the parameters involved. We say that $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. For each ball $B(x, r) \subset \mathbb{R}^n$ we set

$\lambda B := B(x, \lambda r)$, $S_0(B) = B$ and $S_j(B) = 2^j B \backslash 2^{j-1} B$ for $j \in \mathbb{N}$. For $1 \leqslant p \leqslant \infty$ let p' be the conjugate exponent of p, i.e. $1/p + 1/p' = 1$. Throughout this article, we denote $w(E) := \int_E w(x)\,\mathrm{d}x$ for any measurable set $E \subset \mathbb{R}^n$ and $\log^+ t = \max\{\log t, 0\}$.

We recall the definition of Muckenhoupt weights. For a weight w we mean that w is a non-negative measurable and locally integrable function on $\mathbb{R}^n$. For $1 < p < \infty$ we say that $w \in A_p$ if there is a constant C such that

$$\left(\frac{1}{|B|}\int_B w(x)\,\mathrm{d}x\right)\left(\frac{1}{|B|}\int_B w^{-1/(p-1)}(x)\,\mathrm{d}x\right)^{p-1} \leqslant C$$

for all balls $B \subset X$.

For the case $p = 1$ we say that $w \in A_1$ if there is a constant C such that for all balls $B \subset X$,

$$\frac{1}{|B|}\int_B w(y)\,\mathrm{d}y \leqslant Cw(x)$$

for a.e. $x \in B$.

Moreover, we set $A_\infty = \bigcup\limits_{p \in [1,\infty)} A_p$.

The weight w is said to belong to reverse Hölder classes RH_q, $1 < q < \infty$ if there is a constant C such that for all balls $B \subset X$,

$$\left(\frac{1}{|B|}\int_B w^q(x)\,\mathrm{d}x\right)^{1/q} \leqslant \frac{C}{|B|}\int_B w(x)\,\mathrm{d}x.$$

When $q = \infty$, we say that $w \in RH_\infty$ if there is a constant $C > 0$ such that for all balls $B \subset X$,

$$w(x) \leqslant \frac{C}{|B|}\int_B w(y)\,\mathrm{d}y$$

for a.e. $x \in B$.

For $\sigma \geqslant 1$ and a weight w we say that $w \in D_\sigma$ if there exists a constant $C > 0$ such that

$$w(tB) \leqslant Ct^{n\sigma} w(B) \quad \forall\, t \geqslant 1.$$

Note that $w \in A_p$ implies that $w \in D_p$.

Now, we recall some basic properties on the Muckenhoupt weights.

Lemma 2.1. *The following properties hold:*

(1) $A_1 \subset A_p \subset A_q$ *for* $1 \leqslant p \leqslant q \leqslant \infty$.
(2) $RH_\infty \subset RH_q \subset RH_p$ *for* $1 < p \leqslant q \leqslant \infty$.
(3) *If* $w \in A_p$, $1 < p < \infty$, *then there exists* $\varepsilon > 0$ *such that* $w \in A_{p-\varepsilon}$.
(4) *If* $w \in RH_q$, $1 < q < \infty$, *then there exists* $\varepsilon > 0$ *such that* $w \in RH_{q+\varepsilon}$.

(5) $A_\infty = \bigcup\limits_{1\leqslant p<\infty} A_p \subset \bigcup\limits_{1<p\leqslant\infty} RH_p$.

(6) *There exists* $\delta \in (0,1)$ *such that for any ball* $B \subset \mathbb{R}^n$ *and any measurable subset* E *of all* B,

$$\frac{w(E)}{w(B)} \lesssim \Big(\frac{|E|}{|B|}\Big)^{\delta}. \tag{2.1}$$

To establish the weighted inequality for fractional integrals, we need to introduce class $A_{p,q}$. We say that a weight w belongs to the class $A_{p,q}$ for $1 \leqslant p < \infty$ and $1 \leqslant q < \infty$ if there is a constant C such that for any ball Q,

$$\Big(\frac{1}{|Q|}\int w^q(x)\,\mathrm{d}x\Big)^{1/q}\Big(\frac{1}{|Q|}\int w^{-p'}(x)\,\mathrm{d}x\Big)^{1/p'} \leqslant C.$$

Notice that $w \in A_{p,q}$, then

$$\Big(\frac{1}{|Q|}\int w^{q/p}(x)\,\mathrm{d}x\Big)^{1/q}\Big(\frac{1}{|Q|}\int w^{-p'/p}(x)\,\mathrm{d}x\Big)^{1/p'} \leqslant C$$

for any ball Q.

The connection of $A_{p,q}$ and A_m is showed in the following lemma.

Lemma 2.2. *Suppose that* $0 < \beta < n$, $1 \leqslant p < n/\alpha$ *and* $1/q = 1/p - \beta/n$. *The following statements are true:*

(i) *If* $p > 1$, *then* $w \in A_{p,q}$ *implies* $w^q \in A_q$ *and* $w^{-p'} \in A_{p'}$.

(ii) *If* $p = 1$, *then* $w \in A_{1,q}$ *if and only if* $w^q \in A_1$.

Next, we introduce some notation and recall some properties of the auxiliary function $m_{\mathcal{V}}(x)$. We need the following lemma about $m_{\mathcal{V}}(x)$, see [27].

Lemma 2.3. *Suppose* $\mathcal{V} \in RH_q$ *with* $q \geqslant \frac{n}{2}$. *Then there exist positive constants* C *and* k_0 *such that*

(i) $m_{\mathcal{V}}(x) \sim m_{\mathcal{V}}(y)$ *if* $|x-y| \leqslant C/m_{\mathcal{V}}(x)$,

(ii) $m_{\mathcal{V}}(y) \leqslant C(1+|x-y|m_{\mathcal{V}}(x))^{k_0} m_{\mathcal{V}}(x)$,

(iii) $m_{\mathcal{V}}(y) \geqslant C m_{\mathcal{V}}(x)(1+|x-y|m_{\mathcal{V}}(x))^{-k_0/(k_0+1)}$.

We next recall some definitions and basic facts about Orlicz spaces needed for the proofs of the main results. For further information on this subject we refer the readers to [24]. A function $A\colon [0,\infty) \to [0,\infty)$ is said to be a Young function if it is continuous, convex and strictly increasing, satisfying $A(0) = 0$ and $A(t) \to \infty$ as

$t \to \infty$. Two important examples of Young function are $A(t) = t^p(1+\log^+ t)^p$ with some $1 \leqslant p < \infty$ and $A(t) = \mathrm{e}^t$. Given a Young function A and $w \in A_\infty$, we define the $A(w)$-average of a function f over a ball B by means of the following Luxemburg norm:

$$\|f\|_{A(w),B} := \inf\Big\{\lambda > 0\colon\ \frac{1}{w(B)}\int_B A\Big(\frac{|f(x)|}{\lambda}\Big)w(x)\,\mathrm{d}x \leqslant 1\Big\}.$$

When $A(t) = t(1+\log^+ t)$, this norm is denoted by $\|\cdot\|_{L\log L(w),B}$. The complementary Young function of $A(t)$ is $\overline{A}(t) = \mathrm{e}^t$ with mean Luxemberg norm denoted by $\|\cdot\|_{\exp L(w),B}$. For $w \in A_\infty$ and for every ball B in $\mathbb{R}^n$, the following generalized Hölder inequality in the weighted setting

$$\frac{1}{w(B)}\int_B |f(x)g(x)|w(x)\,\mathrm{d}x \leqslant C\|f\|_{L\log L(w),B}\|g\|_{\exp L(w),B} \tag{2.2}$$

holds (see [33]).

Note that $t \leqslant t(1+\log^+ t)$ for all $t > 0$, then for any ball $B \subset \mathbb{R}^n$ and $w \in A_\infty$, the inequality

$$\frac{1}{w(B)}\int_B |f(x)|w(x)\,\mathrm{d}x \leqslant \|f\|_{L\log L(w),B} \tag{2.3}$$

holds for any ball $B \subset \mathbb{R}^n$.

When $w \equiv 1$, we write $\|\cdot\|_{L\log L(w),B}$ and $\|\cdot\|_{\exp L(w),B}$ for $\|\cdot\|_{L\log L,B}$ and $\|\cdot\|_{\exp L,B}$, respectively for brevity.

We denote by $M^{1,\theta,\alpha}_{L\log L}(w,v)$ the generalized weighted Morrey space of $L\log L$ type, the space of all locally integrable functions f defined on $\mathbb{R}^n$ with finite norm $\|f\|_{M^{1,\theta}_{L\log L}(w,v)}$

$$M^{1,\theta,\alpha}_{L\log L}(w,v) := \{f \in L^1_{\mathrm{loc}}(w)\colon\ \|f\|_{M^{1,\theta,\alpha}_{L\log L}(w,v)} < \infty\},$$

where

$$\|f\|_{M^{1,\theta,\alpha}_{L\log L}(w,v)} := \sup_B\{(1+r_B m_{\mathcal{V}}(x_B))^\alpha w(B)v(B)^{-\theta}\|f\|_{L\log L(w),B}\}.$$

Here the supremum is taken over all balls B in $\mathbb{R}^n$. If $v = w$, then we denote it by $M^{1,\theta}_{L\log L}(w)$ for brevity.

Let $b \in \mathrm{BMO}$, $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, we define the commutator of an operator $A\colon L^1_{\mathrm{loc}}(\mathbb{R}^n) \to L^1_{\mathrm{loc}}(\mathbb{R}^n)$ by $[b,A]f = bA(f) - A(bf)$.

Note that if

$$Af(x) = \int_{\mathbb{R}^n} H(x,y)f(y)\,\mathrm{d}y,$$

then

$$[b,A]f(x) = \int_{\mathbb{R}^n} [b(x)-b(y)]H(x,y)f(y)\,\mathrm{d}y.$$

The bounded mean oscillation function space BMO was first introduced by John and Nirenberg (see [17]) in the study of regular solutions of elliptic PDEs. A locally integrable function f will be said to belong to BMO if

$$\|f\|_{\mathrm{BMO}} = \sup_{B\ni x} \frac{1}{|B|}\int_B |f(y)-f_B|\,\mathrm{d}y < \infty,$$

where $f_B = \frac{1}{|B|}\int_B |f(z)|\,\mathrm{d}z$.

We recall some basic facts on BMO function, which can be easily found in many references.

Lemma 2.4. *Let b be a function in* BMO.

(i) *For every ball B in $\mathbb{R}^n$ and for all $j\in\mathbb{Z}_+$,*

$$|b_{2^{j+1}B} - b_B| \leqslant C(j+1)\|b\|_{\mathrm{BMO}}.$$

(ii) *For $1<q<\infty$ every ball B in $\mathbb{R}^n$ and for all $w\in A_\infty$,*

$$\left(\int_B |b(x)-b_B|^q w(x)\,\mathrm{d}x\right)^{1/q} \leqslant C\|b\|_{\mathrm{BMO}} w(B)^{1/q}.$$

The above lemma is essentially taken from [28], [31].

We will follow the next statements given in [33].

Lemma 2.5. *Let $1\leqslant p<\infty$, $w^p\in A_1$, $b\in\mathrm{BMO}$ and B be a ball. Then for any $y\in B$ and any positive integer m,*

$$\left(\frac{1}{|2^kB|}\int_{2^kB} |b(x)-b_B|^{mp} w^p(x)\,\mathrm{d}x\right)^{1/p} \leqslant C\|b\|^m_{\mathrm{BMO}}(k+1)^m \inf_{y\in B} w(y), \quad \forall\, k=0,1,2,\ldots$$

Lemma 2.6. *For any $b\in\mathrm{BMO}$, B and $w\in A_\infty$,*

$$\|b-b_B\|_{\exp L(w),B} \leqslant \|b\|_{\mathrm{BMO}}.$$

3. Riesz transforms and their commutator

Shen [27] showed that the Schrödinger type operators $\nabla\mathcal{L}^{-1}\nabla$, $\nabla\mathcal{L}^{-1/2}$ and $\mathcal{L}^{-1/2}\nabla$ are the standard Calderón-Zygmund operators provided that $\mathcal{V} \in RH_n$. The following lemma is essentially taken from [27].

Lemma 3.1. *If $V \in B_q$ with $q > n$, then for every k there exists a constant C such that*

$$|K(x,y)| \leqslant \frac{C}{(1+|x-z|/\varrho(x))^k}\frac{1}{|x-z|^n}. \tag{3.1}$$

It is known in [8] that there exists a non-negative potential $\mathcal{V} \in RH_q$ with $q < n$ such that these operators $\nabla\mathcal{L}^{-1}\nabla$, $\nabla\mathcal{L}^{-1/2}$ and $\mathcal{L}^{-1/2}\nabla$ may be not bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$. Hence, in the rest of this paper, we always assume that T is one of the Schrödinger type operators $\nabla\mathcal{L}^{-1}\nabla$, $\nabla\mathcal{L}^{-1/2}$ and $\mathcal{L}^{-1/2}\nabla$ with $\mathcal{V} \in RH_n$.

In 1974, Coifman and Fefferman [7] proved the $L^p(w)$ boundedness for Calderón-Zygmund operator which will be satisfied if $w \in A_p$. When $b \in \mathrm{BMO}$, Alvarez [2] proved that the commutator $[b,T]$ is bounded on the weighted Lebesgue space $L^p(w)$ with $p \in (1,\infty)$ and $w \in A_p$.

Lemma 3.2. *If P is a Calderón-Zygmund operator, then for any $w \in A_p$ $(1 < p < \infty)$, P is bounded on $L^p(w)$. In addition, if $w \in A_1$, then*

$$w(\{x \in \mathbb{R}^n \colon\ |Pf(x)| > \lambda\}) \leqslant \frac{C}{\lambda}\|f\|_{L^1(w)}.$$

The following lemma is the special case of Theorem 4.3 and 4.4 in [29].

Lemma 3.3. *Let $\mathcal{V} \in RH_n$ and $b \in \mathrm{BMO}$.*

(i) *If $w \in A_p$ $(1 < p < \infty)$, the operator $[b,T]$ is bounded on $L^p(w)$.*

(ii) *If $w \in A_1$, then there exists a constant C such that for every $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\lambda > 0$,*

$$w(\{x \in \mathbb{R}^n \colon\ |[b,T]f(x)| > \lambda\}) \leqslant C\int_{\mathbb{R}^n}\Phi\Big(\frac{|f(x)|}{\lambda}\Big)w(x)\,\mathrm{d}x,$$

where $\Phi(t) = t(1+\log^+ t)$.

To prove the main theorems, we need the following technical lemma.

Lemma 3.4. *Let $w \in A_p$ $(1 \leqslant p < \infty)$ and $f \in L^p(w)$. Then we obtain that*

$$\int_Q |f(z)|\,\mathrm{d}z \leqslant C|Q|w(Q)^{-1/p}\|f\chi_Q\|_{L^p(w)}$$

holds for all balls $Q \subset \mathbb{R}^n$.

Proof. We consider two cases.

Case 1: If $p = 1$, then $w \in A_1$ implies that

$$\frac{1}{|Q|}\int_Q |f(z)|\,\mathrm{d}z = \frac{1}{w(Q)}\int_Q \frac{w(Q)}{|Q|} f(z)\,\mathrm{d}z \lesssim \frac{1}{w(Q)}\|f\chi_Q\|_{L^1(w)}$$

for all balls $Q \subset \mathbb{R}^n$.

Case 2: If $p > 1$, then the Hölder inequality and A_p characterization imply that

$$\begin{aligned}\frac{1}{|Q|}\int_Q |f(z)|\,\mathrm{d}z &\leqslant \Big[\frac{1}{|Q|}\Big(\int_Q |f(z)|^p w(z)\,\mathrm{d}z\Big)^{1/p}\Big(\int_Q w^{-1/(p-1)}(z)\,\mathrm{d}z\Big)^{(p-1)/p}\Big]\\ &\lesssim w(Q)^{-1/p}\|f\chi_Q\|_{L^p(w)}\end{aligned}$$

for all balls $Q \subset \mathbb{R}^n$. The proof is finished. □

Next, we will prove the first main theorem.

Proof of Theorem 1.1. Let $f \in M^{p,s}_{\alpha,\theta}(w)$. We fix $y \in \mathbb{R}^n$ and $r > 0$. Let the ball $B = B(x_B, r)$. We write

$$f = f\chi_{2B} + f\chi_{(2B)^c} = f_1 + f_2.$$

By $x \in B$, $z \in S_j(B)$, we obtain $|x - z| \sim 2^j r$. Combining this with (3.1), for all $m \in \mathbb{N}$ we conclude

$$\begin{aligned}|T(f_2)(x)| &\lesssim \int_{(2B)^c} |K(x,z)||f(z)|\,\mathrm{d}z\\ &\lesssim \sum_{j=2}^{\infty}\int_{S_j(B)} \frac{1}{(1+|x-z|m_{\mathcal{V}}(x))^m}\frac{1}{|x-z|^n}|f(z)|\,\mathrm{d}z\\ &\lesssim \sum_{j=2}^{\infty}\frac{1}{(1+2^j r m_{\mathcal{V}}(x))^m}\frac{1}{(2^j r)^n}\int_{2^j B}|f(z)|\,\mathrm{d}z.\end{aligned} \tag{3.2}$$

We invoke Lemma 3.4 to deduce that

$$\int_{(2B)^c} |K(x,z)||f(z)|\,\mathrm{d}z \lesssim \sum_{j=2}^{\infty}(1+2^j r m_{\mathcal{V}}(x))^{-m} w(2^j B)^{-1/q}\|f\chi_{2^j B}\|_{L^q(w)}. \tag{3.3}$$

Thanks to Lemma 2.3, we obtain

$$(3.4)\qquad (1+2^j rm_{\mathcal{V}}(x))^{-m} \lesssim [1+2^j rm_{\mathcal{V}}(x_B)(1+rm_{\mathcal{V}}(x_B))^{-k_0/(k_0+1)}]^{-m} \lesssim (1+2^j rm_{\mathcal{V}}(x_B))^{-m/(k_0+1)}.$$

Thus

$$(3.5)\qquad |T(f_2)(x)| \lesssim \sum_{j=2}^{\infty}(1+2^j rm_{\mathcal{V}}(x_B))^{-m/(k_0+1)} w(2^jB)^{-1/p}\|f\chi_{2^jB}\|_{L^p(w)}.$$

We then obtain that

$$(3.6)\quad |Tf(x)| \lesssim |T(f_1)(x)|+\sum_{j=0}^{\infty}(1+2^j rm_{\mathcal{V}}(x_B))^{-m/(k_0+1)} w(2^jB)^{-1/p}\|f\chi_{2^jB}\|_{L^p(w)}$$

for almost every $x \in B$.

We consider two cases.

Case 1: $p > 1$. Taking the $L^p(w)$-norm on the ball B on both sides of (3.6), we obtain

$$\|T(f)\chi_B\|_{L^p(w)} \lesssim \|f\chi_B\|_{L^p(w)} + \sum_{j=0}^{\infty} w(B)^{1/p}(1+2^j rm_{\mathcal{V}}(x_B))^{-m/(k_0+1)} w(2^jB)^{-1/p}\|f\chi_{2^jB}\|_{L^p(w)}.$$

Hence, multiplying both sides of the above inequality by $w(B)^{-\theta}(1+rm_{\mathcal{V}}(x_B))^{\alpha}$, we obtain

$$\begin{aligned}&w(B)^{-\theta}(1+rm_{\mathcal{V}}(x_B))^{\alpha}\|T(f)\chi_B\|_{L^p(w)}\\&\lesssim \|f\chi_{2B}\|_{L^p(w)} w(B)^{-\theta}(1+rm_{\mathcal{V}}(x_B))^{\alpha}\\&+\sum_{j=0}^{\infty}(1+rm_{\mathcal{V}}(x_B))^{\alpha} w(B)^{1/p-\theta}(1+2^j rm_{\mathcal{V}}(x_B))^{m/(k_0+1)} w(2^jB)^{-1/p}\|f\chi_{2^jB}\|_{L^p(w)}\\&\quad = J_1+J_2.\end{aligned}$$

We estimate J_1. Note that if $\alpha < 0$, then

$$(1+rm_{\mathcal{V}}(x_B))^{\alpha} = \frac{1}{(1+rm_{\mathcal{V}}(x_B))^{-\alpha}} = \frac{2^{-\alpha}}{(2+2rm_{\mathcal{V}}(x_B))^{-\alpha}} \leqslant \frac{2^{-\alpha}}{(1+2rm_{\mathcal{V}}(x_B))^{-\alpha}}.$$

Otherwise, we observe that $(1+rm_{\mathcal{V}}(x_B))^\alpha \leqslant (1+2rm_{\mathcal{V}}(x_B))^\alpha$.

Thus,

$$(1+rm_{\mathcal{V}}(x_B))^\alpha \lesssim (1+2rm_{\mathcal{V}}(x_B))^\alpha \tag{3.7}$$

for all $\alpha \in (-\infty, \infty)$.

This implies that

$$\begin{aligned} J_1 &\lesssim w(2B)^{-\theta}(1+2rm_{\mathcal{V}}(x_B))^\alpha \|f\chi_{2B}\|_{L^p(w)}\Big(\frac{w(2B)}{w(B)}\Big)^\theta \\ &\lesssim w(2B)^{-\theta}(1+2rm_{\mathcal{V}}(x_B))^\alpha \|f\chi_{2B}\|_{L^p(w)}, \end{aligned}$$

where we have used the doubling property of w in the last inequality.

Next, we will consider J_2. If $\alpha < 0$, then

$$(1+rm_{\mathcal{V}}(x_B))^\alpha = \frac{1}{(1+rm_{\mathcal{V}}(x_B))^{-\alpha}} \leqslant 1 \leqslant (1+2^j rm_{\mathcal{V}}(x_B))^{|\alpha|}.$$

Otherwise, $(1+rm_{\mathcal{V}}(x_B))^\alpha \leqslant (1+2^j rm_{\mathcal{V}}(x_B))^{|\alpha|}$.

This implies that

$$(1+rm_{\mathcal{V}}(x_B))^\alpha \leqslant (1+2^j rm_{\mathcal{V}}(x_B))^{|\alpha|}$$

for all $\alpha \in (-\infty, \infty)$, and thus

$$\begin{aligned} \frac{(1+rm_{\mathcal{V}}(x_B))^\alpha}{(1+2^j rm_{\mathcal{V}}(x_B))^{m/(k_0+1)}} &\leqslant \frac{(1+2^j rm_{\mathcal{V}}(x_B))^{|\alpha|}}{(1+2^j rm_{\mathcal{V}}(x_B))^{m/(k_0+1)}} \\ &\leqslant \frac{(1+2^j rm_{\mathcal{V}}(x_B))^{\alpha}}{(1+2^j rm_{\mathcal{V}}(x_B))^{m/(k_0+1)-|\alpha|-\alpha}}. \end{aligned}$$

Let $m \in \mathbb{N}$ such that $m/(k_0+1) - |\alpha| - \alpha > 0$. We have

$$\frac{(1+rm_{\mathcal{V}}(x_B))^\alpha}{(1+2^j rm_{\mathcal{V}}(x_B))^{m/(k_0+1)}} \leqslant (1+2^j rm_{\mathcal{V}}(x_B))^\alpha, \tag{3.8}$$

which implies that

$$\begin{aligned} J_2 &\lesssim \sum_{j=0}^\infty (1+2^j rm_{\mathcal{V}}(x_B))^\alpha w(B)^{1/p-\theta} w(2^j B)^{-1/p} \|f\chi_{2^j B}\|_{L^p(w)} \\ &\lesssim \sum_{j=0}^\infty (1+2^j rm_{\mathcal{V}}(x_B))^\alpha w(2^j B)^{-\theta}\Big(\frac{w(B)}{w(2^j B)}\Big)^{1/p-\theta} \|f\chi_{2^j B}\|_{L^p(w)}. \end{aligned}$$

By combining $1/p-\theta \geqslant 0$ with (2.1), we obtain

$$J_2 \lesssim \sum_{j=0}^{\infty} 2^{-j\delta(1/p-\theta)}(1+2^j r m_{\mathcal{V}}(x_B))^{\alpha} w(2^j B)^{-\theta} \|f\chi_{2^j B}\|_{L^p(w)}$$

and hence

$$\|Tf\|_{M^{p,s}_{\alpha,\theta}(w)} \lesssim \left(1+\sum_{j=0}^{\infty} 2^{-j\delta(1/p-\theta)}\right)\|f\|_{M^{p,s}_{\alpha,\theta}(w)}.$$

It then follows from $1/p-\theta \geqslant 0$ that

$$\|Tf\|_{M^{p,s}_{\alpha,\theta}(w)} \lesssim \|f\|_{M^{p,s}_{\alpha,\theta}(w)}.$$

Case 2: $p=1$. The proof is very similar with a small modification: replacing the above appropriate estimate with the corresponding weak estimate (see Lemma 3.2). □

The strong-type estimate and the weak-type $L\log L$ estimate of the linear commutator $[b,T]$ in our new weighted Morrey type space associated to $\mathcal{L}=-\Delta+V$ will be proved in Theorem 1.2.

Proof of Theorem 1.2. Let $f \in M^{p,s}_{\alpha,\theta}(w)$. We fix $y \in \mathbb{R}^n$ and $r>0$. Set $B := B(x_B, r)$. We write

$$f = f\chi_{2B} + f\chi_{(2B)^c}.$$

We consider two cases.

Case 1: $p>1$. By the linearity of the commutator operator $[b,T]$, we write

$$\begin{aligned} w^q(B)^{-\theta}(1+rm_{\mathcal{V}}(x_B))^{\alpha}\left(\int_B |[b,T](f)(x)|^p w(x)\,\mathrm{d}x\right)^{1/p} \\ \leqslant w^q(B)^{-\theta}(1+rm_{\mathcal{V}}(x_B))^{\alpha}\left(\int_B |[b,T](f_1)(x)|^p w(x)\,\mathrm{d}x\right)^{1/p} \\ + w^q(B)^{-\theta}(1+rm_{\mathcal{V}}(x_B))^{\alpha}\left(\int_B |[b,T](f_2)(x)|^p w(x)\,\mathrm{d}x\right)^{1/p}. \end{aligned}$$

Thus

$$\|[b,T](f)\|_{M^{p,s}_{\alpha,\theta}(w)} \leqslant \|[b,T](f_1)\|_{M^{p,s}_{\alpha,\theta}(w)} + \|[b,T](f_2)\|_{M^{p,s}_{\alpha,\theta}(w)} := J_1 + J_2.$$

We next give the estimates of J_1 and J_2. It follows from the weighted $L^p(w)$-boundedness of $[b,T]$, that

$$\begin{aligned}
&w^q(B)^{-\theta}(1+rm_{\mathcal{V}}(u))^{\alpha}\left(\int_B |[b,T](f_1)(x)|^p w(x)\,\mathrm{d}x\right)^{1/p}\\
&\quad\leqslant w(B)^{-\theta}(1+rm_{\mathcal{V}}(u))^{\alpha}\left(\int_{2B} |f(x)|^p w(x)\,\mathrm{d}x\right)^{1/p}\|b\|_{\mathrm{BMO}}\\
&\quad\leqslant \|b\|_{\mathrm{BMO}}\left(\frac{w(2B)}{w(B)}\right)^{\theta} w(2B)^{-\theta}(1+2rm_{\mathcal{V}}(u))^{\alpha}\left(\int_{2B} |f(x)|^p w(x)\,\mathrm{d}x\right)^{1/p}.
\end{aligned}$$

Using the doubling property of w, we get

$$\begin{aligned}
\text{(3.9)}\qquad &w^q(B)^{-\theta}(1+rm_{\mathcal{V}}(x_B))^{\alpha}\left(\int_B |[b,T](f_1)(x)|^p w(x)\,\mathrm{d}x\right)^{1/p}\\
&\quad\lesssim \|b\|_{\mathrm{BMO}} w(2B)^{-\theta}(1+2rm_{\mathcal{V}}(x_B))^{\alpha}\left(\int_{2B} |f(x)|^p w(x)\,\mathrm{d}x\right)^{1/p}.
\end{aligned}$$

It is not hard to see that

$$\|[b,T](f_1)\|_{M^{p,s}_{\alpha,\theta}(w)} \lesssim \|f\|_{M^{p,s}_{\alpha,\theta}(w)}.$$

For all $m\in\mathbb{N}$,

$$\begin{aligned}
|[b,T](f_2)(x)| &\leqslant \int_{(2B)^{\mathrm{c}}} |b(x)-b(z)||K(x,z)||f(z)|\,\mathrm{d}z\\
&\leqslant |b(x)-b_B|\int_{(2B)^{\mathrm{c}}} |K(x,z)||f(z)|\,\mathrm{d}z\\
&\quad+\int_{(2B)^{\mathrm{c}}} |b(z)-b_B||K(x,z)||f(z)|\,\mathrm{d}z\\
&\lesssim Q_1(x)+Q_2(x)
\end{aligned}$$

and we arrive at

$$\|[b,T](f_2)\|_{M^{p,s}_{\alpha,\theta}(w)} \leqslant \|Q_1\|_{M^{p,s}_{\alpha,\theta}(w)} + \|Q_2\|_{M^{p,s}_{\alpha,\theta}(w)}.$$

As $x\in B$ and $z\in S_j(B)$, $|x-z|\sim 2^j r$. We deduce from (3.5) that

$$Q_1(x)\lesssim |b(x)-b_B|\sum_{j=0}^{\infty}(1+2^j rm_{\mathcal{V}}(x_B))^{-m/(k_0+1)}\frac{1}{w(2^jB)^{1/p}}\|f\chi_{2^jB}\|_{L^p(w)}.$$

Consequently,

$$\begin{aligned}
w^q(B)^{-\theta}(1+rm_{\mathcal{V}}(x_B))^{\alpha}&\left(\int_B |Q_1(x)|^p w(x)\,\mathrm{d}x\right)^{1/p}\\
&\lesssim w^q(B)^{-\theta}(1+rm_{\mathcal{V}}(x_B))^{\alpha}\sum_{j=2}^{\infty}(1+2^j rm_{\mathcal{V}}(x_B))^{-m/(k_0+1)}\frac{1}{w(2^jB)^{1/p}}\\
&\quad\times \|f\chi_{2^jB}\|_{L^p(w)}\left(\int_B |b(x)-b_B|^p w(x)\,\mathrm{d}x\right)^{1/p}.
\end{aligned}$$

By applying the second part of Lemma 2.4, we conclude

$$\begin{aligned}
w(B)^{-\theta}(1+rm_{\mathcal{V}}(x_B))^{\alpha}&\left(\int_B |Q_1(x)|^p w(x)\,\mathrm{d}x\right)^{1/p}\\
&\lesssim w(B)^{-\theta}(1+rm_{\mathcal{V}}(x_B))^{\alpha}\sum_{j=0}^{\infty}(1+2^j rm_{\mathcal{V}}(x_B))^{\alpha}\frac{1}{w(2^jB)^{1/p}}\\
&\quad\times \|f\chi_{2^jB}\|_{L^p(w)}\|b\|_{\mathrm{BMO}}w(B)^{1/p}.
\end{aligned}$$

This implies that

$$\begin{aligned}
w(B)^{-\theta}(1+rm_{\mathcal{V}}(x_B))^{\alpha}&\left(\int_B |Q_1(x)|^p w(x)\,\mathrm{d}x\right)^{1/p}\\
&\lesssim \sum_{j=0}^{\infty}(1+2^j rm_{\mathcal{V}}(x_B))^{\alpha}w(B)^{1/p-\theta}w(2^jB)^{-1/p}\|f\chi_{2^jB}\|_{L^p(w)}.
\end{aligned}$$

By the same estimate as $T(f_2)$ in (3.5), we get

$$\|Q_1\|_{M^{p,s}_{\alpha,\theta}(w)} \lesssim \|f\|_{M^{p,s}_{\alpha,\theta}(w)}. \tag{3.10}$$

Next, we will estimate Q_2. Using (3.1) again leads to

$$\begin{aligned}
Q_2(x)&\leqslant \int_{(2B)^{\mathrm{c}}} |b(z)-b_B||K(x,z)||f(z)|\,\mathrm{d}z\\
&\lesssim \sum_{j=0}^{\infty}\int_{S_j(B)}\frac{1}{(1+|x-z|m_{\mathcal{V}}(x))^m}\frac{1}{|x-z|^n}|b(z)-b_B||f(z)|\,\mathrm{d}z.
\end{aligned}$$

We provide an analogue of (3.5) to obtain

$$\begin{aligned} Q_2(x) &\lesssim \sum_{j=0}^{\infty}(1+2^j r m_{\mathcal{V}}(x_B))^{-m/(k_0+1)}(2^j r)^{-n}\int_{2^jB}|b(z)-b_{2^jB}||f(z)|\,\mathrm{d}z \\ &\quad+\sum_{j=0}^{\infty}(1+2^j r m_{\mathcal{V}}(x_B))^{-m/(k_0+1)}(2^j r)^{-n}|b_{2^jB}-b_B|\int_{2^jB}|f(z)|\,\mathrm{d}z \\ &:= H_1+H_2 \end{aligned}$$

for almost every $x\in B$.

Applying Hölder inequality, we come up to

$$\int_{2^jB}|b(z)-b_{2^jB}||f(z)|\,\mathrm{d}z \leqslant \biggl(\int_{2^jB}|b(z)-b_{2^jB}|^{p'}w^{-p'/p}(z)\,\mathrm{d}z\biggr)^{1/p'}\|f\chi_{2^jB}\|_{L^p(w)}.$$

The second part of Lemma 2.4 together with $w^{-p'/p}\in A_{p'}$ gives us

$$\int_{2^jB}|b(z)-b_{2^jB}||f(z)|\,\mathrm{d}z \lesssim \|b\|_{\mathrm{BMO}}w^{-p'/p}(2^jB)^{1/p'}\|f\chi_{2^jB}\|_{L^p(w)}.$$

Moreover, taking into account that $w\in A_p$, we get

$$\int_{2^jB}|b(z)-b_{2^jB}||f(z)|\,\mathrm{d}z \lesssim |2^jB|w(2^jB)^{-1/p}\|f\chi_{2^jB}\|_{L^p(w)}.$$

Therefore

$$H_1 \lesssim \|b\|_{\mathrm{BMO}}\sum_{j=0}^{\infty}(1+2^j r m_{\mathcal{V}}(x_B))^{-m/(k_0+1)}w(2^jB)^{-1/p}\|f\chi_{2^jB}\|_{L^p(w)}.$$

On the other hand, observe that

$$\begin{aligned} &w(B)^{-\theta}(1+r m_{\mathcal{V}}(x_B))^{\alpha}\biggl(\int_B|H_1|^p w(x)\,\mathrm{d}x\biggr)^{1/p}\,\mathrm{d}x \\ &\quad\lesssim w(B)^{1/p-\theta}(1+r m_{\mathcal{V}}(x_B))^{\alpha} \\ &\qquad\times\sum_{j=0}^{\infty}(1+2^j r m_{\mathcal{V}}(x_B))^{-m/(k_0+1)}\frac{1}{w(2^jB)^{1/p}}\|f\chi_{2^jB}\|_{L^p(w)}. \end{aligned}$$

Using estimate (3.5) for $T(f_2)$ we arrive at

$$\|H_1\|_{M^{p,s}_{\alpha,\theta}(w)} \lesssim \|f\|_{M^{p,s}_{\alpha,\theta}(w)}. \tag{3.11}$$

Combining Lemma 3.4 with the first part of Lemma 2.4 yields

$$H_2 \lesssim \|b\|_{\mathrm{BMO}} \sum_{j=0}^{\infty} j(1+2^j r m_{\mathcal{V}}(x_B))^{-m/(k_0+1)} w(2^j B)^{-1/p} \|f\chi_{2^j B}\|_{L^p(w)}.$$

Using the same argument as in (3.5) gives us

$$\begin{aligned} w(B)^{-\theta}(1+rm_{\mathcal{V}}(x_B))^{\alpha}\Big(\int_B |H_2|^p w(x)\,\mathrm{d}x\Big)^{1/p} \\ \lesssim \sum_{j=0}^{\infty} \frac{j}{2^{j\delta(1/p-\theta)}}(1+2^j rm_{\mathcal{V}}(x_B))^{\alpha} w(2^j B)^{-\theta}\|f\chi_{2^j B}\|_{L^p(w)}. \end{aligned}$$

The summability of the series $\sum\limits_{j=0}^{\infty} \frac{j}{2^{j\delta(1/p-\theta)}}$ due to $1/p-\theta>0$ yields

$$\|H_2\|_{M^{p,s}_{\alpha,\theta}(w)} \lesssim \|f\|_{M^{p,s}_{\alpha,\theta}(w)}. \tag{3.12}$$

Hence, estimate (3.11) together with (3.12) ensures that

$$\|Q_2\|_{M^{p,s}_{\alpha,\theta}(w)} \leqslant C\|f\|_{M^{p,s}_{\alpha,\theta}(w)}.$$

Combining this with estimate (3.10), we conclude

$$\|[b,T](f_2)\|_{M^{p,s}_{\alpha,\theta}(w)} \lesssim \|f\|_{M^{p,s}_{\alpha,\theta}(w)}.$$

Case 2: $p=1$. For any given $\tau>0$, by the linearity of the commutator operator $[b,T]$, we have

$$\begin{aligned} & w(B)^{-\theta}(1+rm_{\mathcal{V}}(x_B))^{\alpha} w(\{x\in B\colon\ [b,T]f(x)>t\}) \\ & \quad \leqslant w(B)^{-\theta}(1+rm_{\mathcal{V}}(x_B))^{\alpha} w(\{x\in B\colon\ [b,T]f_1(x)>t/2\}) \\ & \qquad + w(B)^{-\theta}(1+rm_{\mathcal{V}}(x_B))^{\alpha} w(\{x\in B\colon\ [b,T]f_2(x)>t/2\}) := R_1+R_2. \end{aligned}$$

We first consider the term R_1. Using Lemma 3.3, (2.3) and the fact that $t\leqslant\Phi(t)$ we obtain

$$\begin{aligned} R_1 &\lesssim w(B)^{-\theta}(1+rm_{\mathcal{V}}(x_B))^{\alpha}\int_{2B}\Phi\Big(\frac{|f(x)|}{t}\Big)w(x)\,\mathrm{d}x \\ &\lesssim \Big(\frac{w(2B)}{w(B)}\Big)^{\theta}(1+2rm_{\mathcal{V}}(x_B))^{\alpha}\frac{w(2B)}{w(2B)^{\theta}}\|\Phi\Big(\frac{|f|}{t}\Big)\|_{L\log L(w),2B} \\ &\lesssim \Big\|\Phi\Big(\frac{|f|}{t}\Big)\Big\|_{M^{1,\theta,\alpha}_{L\log L}(w)}, \end{aligned} \tag{3.13}$$

where the second inequality comes from (3.7).

We now turn to deal with the term R_2. To proceed, we use the following inequality:

$$|[b,T]f_2(x)| \leqslant |b(x)-b_B||Tf_2(x)| + |T((b_B-b)f_2)(x)|$$

and thus, one can further decompose R_2 as

$$\begin{aligned} R_2 &\lesssim w(B)^{-\theta}(1+rm_{\mathcal{V}}(x_B))^{\alpha} w(\{x\in B\colon\ |b(x)-b_B||Tf_2(x)|>t/4\}) \\ &\quad + w(B)^{-\theta}(1+rm_{\mathcal{V}}(x_B))^{\alpha} w(\{x\in B\colon\ |T((b_B-b)f_2)(x)|>t/4\}) \\ &\quad = J_1+J_2. \end{aligned}$$

It follows from Kolmogorov inequality that

$$J_1 \lesssim w(B)^{-\theta}(1+rm_{\mathcal{V}}(x_B))^{\alpha} t^{-1}\int_B |b(x)-b_B||Tf_2(x)|w(x)\,\mathrm{d}x.$$

We then invoke estimates (3.2) and (3.4) to deduce that

$$J_1 \lesssim \sum_{j=2}^{\infty} \frac{w(B)^{-\theta}(1+rm_{\mathcal{V}}(x_B))^{\alpha}}{(1+2^j rm_{\mathcal{V}}(x_B))^{m/(k_0+1)}} \frac{1}{(2^j r)^n}\int_{2^jB}\frac{|f(z)|}{t}\,\mathrm{d}z \int_B |b(x)-b_B|w(x)\,\mathrm{d}x.$$

On the other hand, we obtain the following estimate thanks to Lemma (2.5):

$$\int_B |b(x)-b_B|w(x)\,\mathrm{d}x \leqslant \|b\|_{\mathrm{BMO}}|B|\inf_{y\in B} w(y).$$

Combining the above estimate with (3.8) and the fact that $t\leqslant \Phi(t)$, we get

$$\begin{aligned} J_1 &\lesssim \sum_{j=0}^{\infty} 2^{-jn} w(B)^{-\theta}(1+2^j rm_{\mathcal{V}}(x_B))^{\alpha}\int_{2^jB}\frac{|f(z)|}{t}w(z)\,\mathrm{d}z \\ &\lesssim \sum_{j=0}^{\infty} 2^{-jn}\Big(\frac{w(2^jB)}{w(B)}\Big)^{\theta} w(2^jB)^{-\theta}(1+2^j rm_{\mathcal{V}}(x_B))^{\alpha}\int_{2^jB}\Phi\Big(\frac{|f(z)|}{t}\Big)w(z)\,\mathrm{d}z \\ &\lesssim \sum_{j=0}^{\infty} 2^{-jn(1-\theta)}(1+2^j rm_{\mathcal{V}}(x_B))^{\alpha} w(2^jB)^{1-\theta}\Big\|\Phi\Big(\frac{|f|}{t}\Big)\Big\|_{L\log L(w),2^{j+1}B}, \end{aligned}$$

where the validity of the third inequality comes from Lemma 2.1.

Thus,

$$J_1 \lesssim \Big\|\Phi\Big(\frac{|f|}{t}\Big)\Big\|_{M^{1,\theta,\alpha}_{L\log L}(w)}\sum_{j=2}^{\infty}2^{-jn(1-\theta)} \lesssim \|\Phi\Big(\frac{|f|}{t}\Big)\|_{M^{1,\theta,\alpha}_{L\log L}(w)},$$

where the summability of the series is due to $1-\theta>0$.

Taking into account Kolmogorov inequality leads us to

$$J_2 \lesssim w(B)^{-\theta}(1+rm_{\mathcal{V}}(x_B))^{\alpha} t^{-1} \int_B |T((b_B-b)f_2)(x)| w(x)\,\mathrm{d}x.$$

We estimate $T((b_B-b)f_2)(x)$ similarly as in (3.2):

$$\begin{aligned}|T((b_B-b)f_2)(x)| &\lesssim \int_{(2B)^{\mathrm{c}}} |K(x,z)||b(z)-b_B||f(z)|\,\mathrm{d}z \\ &\lesssim \sum_{j=0}^{\infty} \frac{1}{(1+2^j rm_{\mathcal{V}}(x_B))^{m/(k_0+1)}} \frac{1}{(2^j r)^n} \int_{2^j B} |b(z)-b_B||f(z)|\,\mathrm{d}z.\end{aligned}$$

It follows from (3.8) that

$$J_2 \lesssim w(B)^{1-\theta} \sum_{j=0}^{\infty} w(2^jB)^{-1}(1+2^j rm_{\mathcal{V}}(x_B))^{\alpha} \frac{w(2^jB)}{|2^jB|} \int_{2^jB} |b(z)-b_B| \frac{|f(z)|}{t}\,\mathrm{d}z.$$

We deduce from (3.8), $w \in A_1$, and $t \leqslant \Phi(t)$ that

$$\begin{aligned}J_2 &\lesssim w(B)^{1-\theta} \sum_{j=0}^{\infty} w(2^jB)^{-1}(1+2^j rm_{\mathcal{V}}(x_B))^{\alpha} \int_{2^{j+1}B} |b(z)-b_B| \Phi\Big(\frac{|f(z)|}{t}\Big) w(z)\,\mathrm{d}z \\ &\lesssim w(B)^{1-\theta} \sum_{j=0}^{\infty} w(2^jB)^{-1}(1+2^j rm_{\mathcal{V}}(x_B))^{\alpha} \int_{2^jB} |b(z)-b_{2^jB}| \Phi\Big(\frac{|f(z)|}{t}\Big) w(z)\,\mathrm{d}z \\ &\quad + w(B)^{1-\theta} \sum_{j=0}^{\infty} w(2^jB)^{-1}(1+2^j rm_{\mathcal{V}}(x_B))^{\alpha} |b_B - b_{2^jB}| \int_{2^jB} \Phi\Big(\frac{|f(z)|}{t}\Big) w(z)\,\mathrm{d}z \\ &\lesssim w(B)^{1-\theta} \sum_{j=0}^{\infty} w(2^jB)^{-1}(1+2^j rm_{\mathcal{V}}(x_B))^{\alpha} \int_{2^jB} |b(z)-b_{2^jB}| \Phi\Big(\frac{|f(z)|}{t}\Big) w(z)\,\mathrm{d}z \\ &\quad + \|b\|_{\mathrm{BMO}} w(B)^{1-\theta} \sum_{j=0}^{\infty} j w(2^jB)^{-1}(1+2^j rm_{\mathcal{V}}(x_B))^{\alpha} \int_{2^jB} \Phi\Big(\frac{|f(z)|}{t}\Big) w(z)\,\mathrm{d}z \\ &:= K_1 + K_2.\end{aligned}$$

We next use (2.2) to estimate K_1 as

$$K_1 \lesssim w(B)^{1-\theta} \sum_{j=0}^{\infty} (1+2^j rm_{\mathcal{V}}(x_B))^{\alpha} \|b-b_{2^jB}\|_{\exp L(w),2^jB} \Big\|\Phi\Big(\frac{|f|}{t}\Big)\Big\|_{L\log L(w),2^jB}.$$

We now combine Lemma 2.6 with Lemma 2.1 to obtain the following estimate

$$\begin{aligned}K_1 &\lesssim \|b\|_{\mathrm{BMO}} \Big\|\Phi\Big(\frac{|f|}{t}\Big)\Big\|_{M^{1,\theta,\alpha}_{L\log L}(w)} \sum_{j=0}^{\infty} \Big(\frac{w(B)}{w(2^jB)}\Big)^{1-\theta} \\ &\lesssim \Big\|\Phi\Big(\frac{|f|}{t}\Big)\Big\|_{M^{1,\theta,\alpha}_{L\log L}(w)} \sum_{j=0}^{\infty} 2^{-j\delta(1-\theta)}.\end{aligned}$$

By the summability of the series due to $1-\theta>0$, we obtain

$$K_1 \lesssim \Big\|\Phi\Big(\frac{|f|}{t}\Big)\Big\|_{M^{1,\theta,\alpha}_{L\log L}(w)}. \tag{3.14}$$

To estimate (or to deal with) K_2, we proceed as follows. Using the first part of Lemma 2.4 together with the facts $w \in A_1$ and $t \leqslant \Phi(t)$, we deduce that

$$\begin{aligned} K_2 &\lesssim w(B)^{1-\theta}\sum_{j=0}^{\infty} j(1+2^j r m_{\mathcal{V}}(x_B))^{\alpha} w(2^jB)^{-1}\int_{2^jB}\Phi\Big(\frac{|f(z)|}{t}\Big)w(z)\,\mathrm{d}z \\ &\lesssim \sum_{j=0}^{\infty} j(1+2^j r m_{\mathcal{V}}(x_B))^{\alpha}\Big(\frac{w(B)}{w(2^jB)}\Big)^{1-\theta} w(2^jB)^{1-\theta}\|\Phi\Big(\frac{|f|}{t}\Big)\|_{L\log L(w),2^jB} \\ &\lesssim \Big\|\Phi\Big(\frac{|f|}{t}\Big)\Big\|_{M^{1,\theta,\alpha}_{L\log L}(w)}\sum_{j=0}^{\infty}\frac{j}{2^{j\delta(1-\theta)}}. \end{aligned}$$

Thus,

$$K_2 \lesssim \Big\|\Phi\Big(\frac{|f|}{t}\Big)\Big\|_{M^{1,\theta,\alpha}_{L\log L}(w)}. \tag{3.15}$$

We then arrive at the following estimate thanks to (3.14) and (3.15):

$$J_2 \lesssim \Big\|\Phi\Big(\frac{|f|}{t}\Big)\Big\|_{M^{1,\theta,\alpha}_{L\log L}(w)}.$$

Combining this with (3.14) we obtain

$$R_2 \lesssim \Big\|\Phi\Big(\frac{|f|}{t}\Big)\Big\|_{M^{1,\theta,\alpha}_{L\log L}(w)}$$

and thus complete the proof of the theorem thanks to estimate (3.13). □

4. Fractional integral and its commutator

In 1974, Muckenhoupt and Wheeden ([22]) studied the weighted boundedness of I_β and obtained the following result.

Lemma 4.1. *Let $0<\beta<n$, $1<s<\beta/n$ and $1/q=1/s-\beta/n$. If $w\in A_{p,q}$, then*

$$\|I_\beta f\|_{L^q(w^q)} \leqslant C\|f\|_{L^p(w^p)}.$$

Suppose further that $w^q\in A_1$ with $q=n/(n-\beta)$. Then there exists a constant C such that for all $t>0$,

$$w^q(\{x\in\mathbb{R}^n\colon\ |I_\beta f(x)|>t\})^{1/q} \leqslant \frac{C}{t}\|f\|_{L^1(w)}.$$

Now, we will prove the main theorem about the weighted strong-type and weak-type estimates for fractional integral I_β.

P r o o f of Theorem 1.3. We represent f as

$$f = f\chi_{2B} + f\chi_{(2B)^{\rm c}} := f_1 + f_2.$$

Consider two cases.

Case 1: $w \in A_{p,q}$. It follows from the linearity of the fractional integral I_β that

$$\begin{aligned}
w^q(B)^{-\theta}(1 + rm_{\mathcal{V}}(x_B))^{\alpha}\Big(\int_B |I_\beta f(x)|^q w^q(x)\,\mathrm{d}x\Big)^{1/q} \\
&\leqslant w^q(B)^{-\theta}(1 + rm_{\mathcal{V}}(x_B))^{\alpha}\Big(\int_B |I_\beta f_1(x)|^q w^q(x)\,\mathrm{d}x\Big)^{1/q} \\
&\quad + w^q(B)^{-\theta}(1 + rm_{\mathcal{V}}(x_B))^{\alpha}\Big(\int_B |I_\beta f_2(x)|^q w^q(x)\,\mathrm{d}x\Big)^{1/q} \\
&:= J_1 + J_2.
\end{aligned}$$

We next estimate J_1 and J_2. Indeed, the weighted $(L^p(w^q), L^p(w^p))$-boundedness of I_β (see Lemma 4.1) gives us

$$\begin{aligned}
J_1 &\leqslant w^q(B)^{-\theta}(1 + r_B m_{\mathcal{V}}(x_B))^{\alpha}\Big(\int_{2B} |f(x)|^p w^p(x)\,\mathrm{d}x\Big)^{1/p} \\
&\leqslant \Big(\frac{w^q(2B)}{w^q(B)}\Big)^{\theta} w^q(2B)^{-\theta}(1 + 2r_B m_{\mathcal{V}}(x_B))^{\alpha}\Big(\int_{2B} |f(x)|^p w^p(x)\,\mathrm{d}x\Big)^{1/p}.
\end{aligned}$$

Therefore we obtain from the doubling property of w^q that

$$J_1 \lesssim w^q(2B)^{-\theta}(1 + 2r_B m_{\mathcal{V}}(x_B))^{\alpha}\Big(\int_{2B} |f(x)|^p w^p(x)\,\mathrm{d}x\Big)^{1/p}. \tag{4.1}$$

According to the proof of Theorem 1.3 in [30], we claim that for $x \in B(u, r)$, $y \in B(u, 2r)^{\rm c}$ and for any $m \in \mathbb{N}$, there exists $C_m > 0$ such that

$$\int_0^\infty t^{\beta/2-1}|k_t(x, y)|\,\mathrm{d}t \leqslant \frac{C_m}{(1 + |x - y|m_{\mathcal{V}}(x))^m}\frac{1}{|x - y|^{n-\beta}}. \tag{4.2}$$

When $x \in B$ and $y \in S_j(B)$, $|x - y| \sim 2^j r$. Combining this with (3.4), for all $m \in \mathbb{N}$ we have

$$\begin{aligned}
|I_\beta(f_2)(x)| &\leqslant \int_{(2B)^{\rm c}} |f(y)| \int_0^\infty t^{\beta/2-1}|k_t(x, y)|\,\mathrm{d}t\,\mathrm{d}y \\
&\lesssim \sum_{j=0}^{\infty} \frac{1}{(1 + 2^j r_B m_{\mathcal{V}}(x_B))^m}\frac{1}{(2^j r)^{n-\beta}} \int_{2^j B} |f(y)|\,\mathrm{d}y.
\end{aligned} \tag{4.3}$$

By Hölder inequality and $A_{p,q}$ condition on w, we have

$$\frac{1}{(2^j r)^{n-\beta}} \int_{2^j B} |f(y)|\,\mathrm{d}y \lesssim \frac{1}{(2^j r)^{n-\beta}} \|f\chi_{2^j B}\|_{L^p(w^p)} w^{-p'}(2^j B)^{1/p'} \tag{4.4}$$
$$\lesssim \frac{|2^j B|^{1/q+1/p'}}{(2^j r)^{n-\beta}} \|f\chi_{2^j B}\|_{L^p(w^p)} w^q(2^j B)^{-1/q}$$
$$\lesssim \|f\chi_{2^j B}\|_{L^p(w^p)} w^q(2^j B)^{-1/q}.$$

Thus,

$$|I_\beta(f_2)(x)| \lesssim \sum_{j=0}^{\infty} (1 + 2^j r m_{\mathcal{V}}(x_B))^{-m/(k_0+1)} \|f\chi_{2^j B}\|_{L^p(w^p)} w^q(2^j B)^{-1/q}. \tag{4.5}$$

And hence

$$J_2 \lesssim w^q(B)^{-\theta+1/q} \sum_{j=0}^{\infty} (1 + 2^j r m_{\mathcal{V}}(x_B))^{\alpha} \|f\chi_{2^j B}\|_{L^p(w^p)} w^q(2^j B)^{-1/q},$$

which implies that

$$J_2 \lesssim w^q(B)^{-\theta+1/q} \sum_{j=0}^{\infty} (1 + 2^j r_B m_{\mathcal{V}}(x_B))^{\alpha} \|f\chi_{2^j B}\|_{L^p(w^p)} w^q(2^j B)^{-1/q}$$
$$\lesssim \sum_{j=2}^{\infty} \Big(\frac{w^q(B)}{w^q(2^j B)}\Big)^{1/q-\theta} (1 + 2^j r_B m_{\mathcal{V}}(x_B))^{\alpha} w^q(2^j B)^{-\theta} \|f\chi_{2^j B}\|_{L^p(w^p)}.$$

Using $1/q - \theta \geqslant 0$ and Lemma 2.1 yields

$$J_2 \lesssim \sum_{j=0}^{\infty} \frac{1}{2^{j(1/q-\theta)}} (1 + 2^j r_B m_{\mathcal{V}}(x_B))^{\alpha} w^q(2^j B)^{-\theta} \|f\chi_{2^j B}\|_{L^p(w^p)}.$$

It then follows from (4.1) and the above inequality that

$$\|I_\beta f\|_{M^{q,s}_{\alpha,\theta}(w^q)} \lesssim \left(1 + \sum_{j=0}^{\infty} 2^{-j\delta(1/q-\theta)}\right) \|f\|_{M^{p,s}_{\alpha,\theta}(w^p,w^q)}.$$

Since $1/q - \theta > 0$, we get

$$\|I_\beta f\|_{M^{q,s}_{\alpha,\theta}(w^q)} \lesssim \|f\|_{M^{p,s}_{\alpha,\theta}(w^p,w^q)}.$$

Case 2: $w^{n/(n-\beta)} \in A_1$. The proof for the case $w \in A_{p,q}$ is similar and hence we may skip it. □

In 1991, Segovia and Torrea [26] proved that $[b, I_\beta]$ is also bounded from $L^p(w^p)$ to $L^q(w^q)$, $1 < p < n/\beta$, whenever $b \in \mathrm{BMO}$.

Lemma 4.2. *Let* $0 < \beta < n$, $1 < p < n/\beta$, $1/q = 1/p - \beta/n$, *and* $w \in A_{p,q}$. *Suppose that* $b \in \mathrm{BMO}$. *Then the linear commutator* $[b, I_\beta]$ *is bounded from* $L^p(w^p)$ *to* $L^q(w^q)$.

In 2007, Cruz-Uribe and Fiorenza [9] discussed the weighted endpoint inequalities for commutator of fractional integral operator. This result is showed by the following Lemma.

Lemma 4.3. *Let* $0 < \beta < n$, $p = 1$, $q = n/(n-\beta)$ *and* $w^q \in A_1$. *Suppose that* $b \in \mathrm{BMO}$. *Then for any given* $\lambda > 0$ *and any bounded domain* $\Omega \subset \mathbb{R}^n$ *there is a constant* $C > 0$ *(which does not depend on* f *and* Ω*) and* $\lambda > 0$ *such that*

$$w^q(\{x \in \mathbb{R}^n \colon |[b, I_\beta]f(x)| > \lambda\})^{1/q} \leqslant C \int_\Omega \Phi\Big(\frac{|f(x)|}{\lambda}\Big) w(x)\,\mathrm{d}x.$$

The strong-type estimate and the weak-type $L\log L$ estimate of the linear commutator $[b, I_\beta]$ in our new weighted Morrey-type space associated to $\mathcal{L}$ will be proved in Theorem 1.4.

Proof of Theorem 1.4. Let $f \in M^{p,s}_{\alpha,\theta}(w^p, w^q)$. Fix $y \in \mathbb{R}^n$, the ball B and write

$$f = f\chi_{2B} + f\chi_{(2B)^c}.$$

Consider two cases.

Case 1: $p > 1$. Using the linearity of the commutator operator $[b, I_\beta]$ we can write

$$\begin{aligned} w^q(B)^{-\theta}(1 + r_B m_{\mathcal{V}}(x_B))^\alpha \Big(\int_B |[b, I_\beta](f)(x)|^q w^q(x)\,\mathrm{d}x\Big)^{1/p} \\ \leqslant w^q(B)^{-\theta}(1 + r_B m_{\mathcal{V}}(x_B))^\alpha \Big(\int_B |[b, I_\beta](f_1)(x)|^q w^q(x)\,\mathrm{d}x\Big)^{1/p} \\ + w^q(B)^{-\theta}(1 + r_B m_{\mathcal{V}}(x_B))^\alpha \Big(\int_B |[b, I_\beta](f_2)(x)|^q w^q(x)\,\mathrm{d}x\Big)^{1/p}. \end{aligned}$$

Thus

$$\begin{aligned} \|[b, I_\beta](f)\|_{M^{q,s}_{\alpha,\theta}(w^q)} &\leqslant \|[b, I_\beta](f_1)\|_{M^{q,s}_{\alpha,\theta}(w^q)} + \|[b, I_\beta](f_2)\|_{M^{q,s}_{\alpha,\theta}(w^q)} \\ &= J_1 + J_2. \end{aligned}$$

We next estimate J_1 and J_2. It follows from the weighted $L^p(w^p)-L^q(w^q)$ boundedness of $[b,T]$ that

$$\begin{aligned}
&w^q(B)^{-\theta}(1+r_B m_{\mathcal V}(x_B))^{\alpha}\Big(\int_B |[b,I_\beta](f_1)(x)|^q w^q(x)\,\mathrm{d}x\Big)^{1/p}\\
&\quad\leqslant w^q(B)^{-\theta}(1+r_B m_{\mathcal V}(x_B))^{\alpha}\Big(\int_{2B} |f(x)|^p w^p(x)\,\mathrm{d}x\Big)^{1/p}\|b\|_{\mathrm{BMO}}\\
&\quad\lesssim \Big(\frac{w^q(2B)}{w^q(B)}\Big)^{\theta} w^q(2B)^{-\theta}(1+2r_B m_{\mathcal V}(x_B))^{\alpha}\Big(\int_{2B} |f(x)|^p w^p(x)\,\mathrm{d}x\Big)^{1/p}.
\end{aligned}$$

We obtain from the doubling property of w that

$$\begin{aligned}
(4.6)\qquad &w^q(B)^{-\theta}(1+r_B m_{\mathcal V}(x_B))^{\alpha}\Big(\int_B |[b,I_\beta](f_1)(x)|^q w^q(x)\,\mathrm{d}x\Big)^{1/p}\\
&\quad\lesssim w^q(2B)^{-\theta}(1+2r_B m_{\mathcal V}(x_B))^{\alpha}\Big(\int_{2B} |f(x)|^p w^p(x)\,\mathrm{d}x\Big)^{1/p}.
\end{aligned}$$

It is not hard to see that

$$(4.7)\qquad J_1 \lesssim \|f\|_{M^{p,s}_{\alpha,\theta}(w^p,w^q)}.$$

By (4.2), for all $m\in\mathbb N$ we have

$$\begin{aligned}
|[b,I_\beta](f_2)(x)| &\leqslant \int_{(2B)^{\rm c}} |b(x)-b(z)|\Big(\int_0^\infty t^{\beta/2-1}|k_t(x,z)|\,\mathrm{d}t\Big)|f(z)|\,\mathrm{d}z\\
&\leqslant |b(x)-b_B|\int_{(2B)^{\rm c}} \frac{1}{(1+|x-z|m_{\mathcal V}(x))^m}\,\frac{1}{|x-z|^{n-\beta}}|f(z)|\,\mathrm{d}z\\
&\quad+\int_{(2B)^{\rm c}} |b(z)-b_B|\frac{1}{(1+|x-z|m_{\mathcal V}(x))^m}\,\frac{1}{|x-z|^{n-\beta}}|f(z)|\,\mathrm{d}z\\
&\lesssim Q_1+Q_2.
\end{aligned}$$

And thus

$$\|[b,I_\beta](f_2)\|_{M^{q,s}_{\alpha,\theta}(w^q)} \leqslant \|Q_1\|_{M^{q,s}_{\alpha,\theta}(w^q)}+\|Q_2\|_{M^{q,s}_{\alpha,\theta}(w^q)}.$$

Since $x\in B$ and $z\in S_j(B)$, $|x-z|\sim 2^j r$. It follows from (4.5) that

$$Q_1(x)\lesssim |b(x)-b_B|\sum_{j=0}^{\infty}(1+2^j r m_{\mathcal V}(u))^{-m/(k_0+1)}\|f\chi_{2^jB}\|_{L^p(w^p)}w^q(2^jB)^{-1/q}.$$

Consequently,

$$w^q(B)^{-\theta}(1+r_B m_{\mathcal{V}}(x_B))^{\alpha}\left(\int_B |Q_1(x)|^q w^q(x)\,\mathrm{d}x\right)^{1/q}$$
$$\lesssim w^q(B)^{-\theta}\sum_{j=0}^{\infty}(1+2^j r_B m_{\mathcal{V}}(x_B))^{\alpha}\|f\chi_{2^jB}\|_{L^p(w^p)} w^q(2^jB)^{1/q}\|b-b_B\|_{L^q(w^q)}.$$

Applying the second part of Lemma 2.4, we get

$$w^q(B)^{-\theta}(1+r_B m_{\mathcal{V}}(x_B))^{\alpha}\left(\int_B |Q_1(x)|^q w^q(x)\,\mathrm{d}x\right)^{1/p}$$
$$\lesssim \sum_{j=0}^{\infty}\left(\frac{w^q(B)}{w^q(2^jB)}\right)^{1/q-\theta}(1+2^j r_B m_{\mathcal{V}}(x_B))^{\alpha} w^q(2^jB)^{-\theta}\|f\chi_{2^jB}\|_{L^p(w^p)}.$$

This implies that

$$w^q(B)^{-\theta}(1+r_B m_{\mathcal{V}}(x_B))^{\alpha}\left(\int_B |Q_1(x)|^q w^q(x)\,\mathrm{d}x\right)^{1/q}$$
$$\lesssim \sum_{j=0}^{\infty}\frac{1}{2^{j\delta(1/q-\theta)}}(1+2^j r_B m_{\mathcal{V}}(x_B))^{\alpha} w^q(2^jB)^{-\theta}\|f\chi_{2^jB}\|_{L^p(w^p)}$$
$$\lesssim \|f\|_{M^{p,s}_{\alpha,\theta}(w^p,w^q)}\sum_{j=0}^{\infty}2^{-j\delta(1/q-\theta)}.$$

Thus, we obtain

$$\|Q_1\|_{M^{q,s}_{\alpha,\theta}(w^q)} \lesssim \|f\|_{M^{p,s}_{\alpha,\theta}(w^p,w^q)}. \tag{4.8}$$

Next, we will estimate Q_2. Using (3.1) again leads to

$$Q_2 \lesssim \sum_{j=0}^{\infty}\int_{S_j(B)}\frac{1}{(1+|x-z|m_{\mathcal{V}}(x))^m}\frac{1}{|x-z|^{n-\beta}}|b(z)-b_B||f(z)|\,\mathrm{d}z.$$

We provide an analogue of (3.5) to obtain

$$Q_2 \lesssim \sum_{j=0}^{\infty}(1+2^j r_B m_{\mathcal{V}}(x_B))^{-m/(k_0+1)}(2^j r)^{\beta-n}\int_{2^jB}|b(z)-b_{2^jB}||f(z)|\,\mathrm{d}z$$
$$+\sum_{j=0}^{\infty}(1+2^j r_B m_{\mathcal{V}}(x_B))^{-m/(k_0+1)}(2^j r)^{\beta-n}|b_{2^jB}-b_B|\int_{2^jB}|f(z)|\,\mathrm{d}z$$
$$:= H_1+H_2$$

for almost every $x\in B$.

Taking Hölder inequality into account, we get

$$\int_{2^jB} |b(z)-b_{2^jB}||f(z)|\,\mathrm{d}z \leqslant \left(\int_{2^jB} |b(z)-b_{2^jB}|^{p'} w^{-p'}(z)\,\mathrm{d}z\right)^{1/p'} \|f\chi_{2^jB}\|_{L^p(w^p)}.$$

The second part of Lemma 2.4 together with $w^{-p'} \in A_{p'}$ implies that

$$\begin{aligned}\left(\int_{2^jB} |b(z)-b_{2^jB}|^{p'} w^{-p'}(z)\,\mathrm{d}z\right)^{1/p'} &\lesssim \|b\|_{\mathrm{BMO}} w^{-p'}(2^jB)^{1/p'}\\ &\lesssim |2^jB|^{1/q+1/p'} w^q(2^jB)^{-1/q}.\end{aligned}$$

Since $w \in A_{p,q}$, we have

$$\int_{2^jB} |b(z)-b_{2^jB}||f(z)|\,\mathrm{d}z \lesssim (2^jr)^{n-\beta} w^q(2^jB)^{-1/q}\|f\chi_{2^jB}\|_{L^p(w^p)}.$$

Therefore

$$H_1 \lesssim \sum_{j=0}^{\infty}(1+2^jr_Bm_{\mathcal{V}}(x_B))^{-m/(k_0+1)} w^q(2^jB)^{-1/q}\|f\chi_{2^jB}\|_{L^p(w^p)}.$$

Using the same argument as in (4.5) we can deduce

$$\|H_1\|_{M^{q,s}_{\alpha,\theta}(w^q)} \lesssim \|f\|_{M^{p,s}_{\alpha,\theta}(w^p,w^q)}. \tag{4.9}$$

Combining Lemma 3.4 with the first part of Lemma 2.4 yields

$$\begin{aligned}H_2 &\lesssim \|b\|_{\mathrm{BMO}} \sum_{j=0}^{\infty} j(1+2^jr_Bm_{\mathcal{V}}(x_B))^{-m/(k_0+1)}(2^jr)^{\beta-n}\int_{2^jB}|f(z)|\,\mathrm{d}z\\ &\lesssim \sum_{j=0}^{\infty} j(1+2^jr_Bm_{\mathcal{V}}(x_B))^{-m/(k_0+1)}\|f\chi_{2^jB}\|_{L^p(w^p)} w^q(2^jB)^{-1/q},\end{aligned}$$

where (4.4) is used in the last inequality.

Thus, we obtain

$$\begin{aligned}&w^q(B)^{-\theta}(1+rm_{\mathcal{V}}(u))^{\alpha}\left(\int_B |H_2|^q w^q(x)\,\mathrm{d}x\right)^{1/q}\\ &\quad\lesssim w^q(B)^{1/q-\theta}\sum_{j=0}^{\infty} j(1+2^jr_Bm_{\mathcal{V}}(x_B))^{\alpha}\|f\chi_{2^jB}\|_{L^p(w^p)} w^q(2^jB)^{-1/q}.\end{aligned}$$

Using the same argument as in (3.12) yields

$$\|H_2\|_{M^{q,s}_{\alpha,\theta}(w^q)} \lesssim \|f\|_{M^{p,s}_{\alpha,\theta}(w^p,w^q)}. \tag{4.10}$$

It then follows from (4.9) and (4.10) that

$$\|Q_2\|_{M^{q,s}_{\alpha,\theta}(w^q)} \lesssim \|f\|_{M^{p,s}_{\alpha,\theta}(w^p,w^q)}.$$

Combining this with estimate (4.8), we conclude

$$J_2 \lesssim \|f\|_{M^{p,s}_{\alpha,\theta}(w^p,w^q)}. \tag{4.11}$$

By (4.7) and (4.11) we get

$$\|[b,I_\beta](f)\|_{M^{q,s}_{\alpha,\theta}(w^q)} \lesssim \|f\|_{M^{p,s}_{\alpha,\theta}(w^p,w^q)}.$$

Case 2: $p=1$. For any given $\tau>0$, by the linearity of the commutator operator $[b,T]$, we write

$$\begin{aligned}
&w^q(B)^{-\theta}(1+rm_{\mathcal V}(x_B))^\alpha w^q(\{x\in B\colon\ [b,I_\beta]f(x)>t\})^{1/q}\\
&\quad\leqslant w^q(B)^{-\theta}(1+rm_{\mathcal V}(x_B))^\alpha w^q(\{x\in B\colon\ [b,I_\beta]f_1(x)>t/2\})^{1/q}\\
&\qquad+w^q(B)^{-\theta}(1+rm_{\mathcal V}(x_B))^\alpha w^q(\{x\in B\colon\ [b,I_\beta]f_2(x)>t/2\})^{1/q}\\
&\quad:=R_1+R_2.
\end{aligned}$$

We first consider the term R_1. Using Lemma 4.3, (2.3) and the fact that $t\leqslant\Phi(t)$, we get

$$\begin{aligned}
R_1&\lesssim w^q(B)^{-\theta}(1+rm_{\mathcal V}(x_B))^\alpha\int_{2B}\Phi\Big(\frac{|f(x)|}{t}\Big)w(x)\,\mathrm{d}x\\
&\lesssim\Big(\frac{w^q(2B)}{w^q(B)}\Big)^\theta(1+2rm_{\mathcal V}(x_B))^\alpha\frac{w(2B)}{w^q(2B)^\theta}\Big\|\Phi\Big(\frac{|f|}{t}\Big)\Big\|_{L\log L(w),2B}\\
&\lesssim\|\Phi\Big(\frac{|f|}{t}\Big)\|_{M^{1,\theta,\alpha}_{L\log L}(w,w^q)},
\end{aligned} \tag{4.12}$$

where the second inequality comes from (3.7).

We now turn to deal with the term R_2. To proceed, we use the following inequality:

$$|[b,I_\beta]f_2(x)|\leqslant|b(x)-b_B||I_\beta f_2(x)|+|I_\beta((b_B-b)f_2)(x)|$$

which helps to further decompose R_2 as

$$\begin{aligned}
R_2&\lesssim w^q(B)^{-\theta}(1+rm_{\mathcal V}(x_B))^\alpha w^q(\{x\in B\colon\ |b(x)-b_B||I_\beta f_2(x)|>t/4\})\\
&\quad+w^q(B)^{-\theta}(1+rm_{\mathcal V}(x_B))^\alpha w^q(\{x\in B\colon\ |I_\beta((b_B-b)f_2)(x)|>t/4\})\\
&:=J_1+J_2.
\end{aligned}$$

It follows from Chebyshev inequality that

$$J_1 \lesssim w^q(B)^{-\theta}(1+rm_{\mathcal{V}}(x_B))^{\alpha}t^{-1}\Big(\int_B |b(x)-b_B|^q|I_\beta f_2(x)|^q w^q(x)\,\mathrm{d}x\Big)^{1/q}.$$

We then invoke estimates (4.12) and (3.4) to deduce that

$$J_1 \lesssim \sum_{j=0}^{\infty} \frac{w^q(B)^{-\theta}(1+rm_{\mathcal{V}}(x_B))^{\alpha}}{(1+2^j rm_{\mathcal{V}}(x_B))^{m/(k_0+1)}} \frac{1}{(2^j r)^{n-\beta}} \int_{2^jB} \frac{|f(z)|}{t}\,\mathrm{d}z \\ \times \Big(\int_B |b(x)-b_B|^q w^q(x)\,\mathrm{d}x\Big)^{1/q}.$$

Thanks to the second part of Lemma (2.4), we obtain

$$\Big(\int_B |b(x)-b_B|^q w^q(x)\,\mathrm{d}x\Big)^{1/q} \leqslant \|b\|_{\mathrm{BMO}} w^q(B)^{1/q}.$$

Combining the above estimate with (3.8) and the fact that $t \leqslant \Phi(t)$, we get

$$J_1 \lesssim \sum_{j=0}^{\infty}(1+2^j rm_{\mathcal{V}}(x_B))^{\alpha} \frac{w^q(B)^{1/q-\theta}}{|2^jB|^{1-\beta/n}} \int_{2^jB} \Phi\Big(\frac{|f(z)|}{t}\Big)\,\mathrm{d}z.$$

Moreover, by applying Hölder inequality and the reverse Hölder inequality in succession, we can show that $w^q \in A_1$ if only if $w \in A_1 \cap RH_q$ (see [18]). And hence

$$J_1 \lesssim \sum_{j=0}^{\infty} \frac{w^q(B)^{1/q-\theta}|2^jB|^{\beta/n}}{w(2^jB)} \frac{w(2^jB)}{|2^jB|}(1+2^j rm_{\mathcal{V}}(x_B))^{\alpha} \int_{2^jB} \Phi\Big(\frac{|f(z)|}{t}\Big)\,\mathrm{d}z \\ \lesssim \sum_{j=0}^{\infty} w^q(B)^{1/q-\theta} \frac{|2^jB|^{\beta/n}}{w(2^jB)}(1+2^j rm_{\mathcal{V}}(x_B))^{\alpha} \int_{2^jB} \Phi\Big(\frac{|f(z)|}{t}\Big) w(z)\,\mathrm{d}z,$$

where $w \in A_1$ and Lemma 2.1 are used in the second inequality.

In addition, note that $w \in RH_q$. For any $j \in \mathbb{Z}^+$ we have

$$w^q(2^jB)^{1/q} = \Big(\int_{2^jB} w^q(x)\,\mathrm{d}x\Big)^{1/q} \lesssim |2^jB|^{1/q-1} w(2^jB)$$

or equivalently,

$$\frac{|2^jB|^{\beta/n}}{w(2^jB)} \lesssim \frac{1}{w^q(2^jB)^{1/q}}. \tag{4.13}$$

Consequently,

$$\begin{aligned}J_1 &\lesssim \sum_{j=0}^{\infty} \frac{w^q(B)^{1/q-\theta}}{w^q(2^jB)^{1/q}}(1+2^jrm_{\mathcal V}(x_B))^{\alpha} w(2^jB)\Big\|\Phi\Big(\frac{|f|}{t}\Big)\Big\|_{L\log L(w),2^jB}\\ &\lesssim \sum_{j=0}^{\infty}\Big(\frac{w^q(B)}{w^q(2^jB)}\Big)^{1/q-\theta}\frac{w(2^jB)}{w^q(2^jB)^{\theta}}(1+2^jrm_{\mathcal V}(x_B))^{\alpha}\Big\|\Phi\Big(\frac{|f|}{t}\Big)\Big\|_{L\log L(w),2^jB}\\ &\lesssim \Big\|\Phi\Big(\frac{|f|}{t}\Big)\Big\|_{M^{1,\theta,\alpha}_{L\log L}(w,w^q)}\sum_{j=0}^{\infty}\frac{1}{2^{j\delta(1/q-\theta)}}.\end{aligned}$$

Thus

$$J_1 \lesssim \Big\|\Phi\Big(\frac{|f|}{t}\Big)\Big\|_{M^{1,\theta,\alpha}_{L\log L}(w,w^q)}, \tag{4.14}$$

where the summability of the series is due to $1/q-\theta>0$.

Using Chebyshev inequality gives us

$$J_2 \lesssim w^q(B)^{-\theta}(1+rm_{\mathcal V}(x_B))^{\alpha}t^{-1}\Big(\int_B |I_\beta((b_B-b)f_2)(x)|^q w^q(x)\,\mathrm{d}x\Big)^{1/q}.$$

By (4.2), we get

$$\begin{aligned}&|I_\beta((b_B-b)f_2)(x)|\\ &\quad\lesssim \sum_{j=0}^{\infty}\frac{1}{(1+2^jrm_{\mathcal V}(x_B))^{m/(k_0+1)}}\frac{1}{|2^jB|^{1-\beta/n}}\int_{2^jB}|b(z)-b_B||f(z)|\,\mathrm{d}z.\end{aligned}$$

In view of (3.8) and $w\in A_1$, it follows from A_1 condition and the fact that $t\leqslant \Phi(t)$ that

$$\begin{aligned}J_2 &\lesssim w^q(B)^{1/q-\theta}\sum_{j=0}^{\infty}\frac{|2^jB|^{\beta/n}}{w(2^jB)}(1+2^jrm_{\mathcal V}(x_B))^{\alpha}\frac{w(2^jB)}{|2^jB|}\int_{2^jB}|b(z)-b_B|\frac{|f(z)|}{t}\,\mathrm{d}z\\ &\lesssim w^q(B)^{1/q-\theta}\sum_{j=0}^{\infty}\frac{|2^jB|^{\beta/n}}{w(2^jB)}(1+2^jrm_{\mathcal V}(x_B))^{\alpha}\int_{2^jB}|b(z)-b_B|\Phi\Big(\frac{|f(z)|}{t}\Big)w(z)\,\mathrm{d}z.\end{aligned}$$

Thus,

$$\begin{aligned}J_2 &\lesssim \sum_{j=0}^{\infty}\frac{w^q(B)^{1/q-\theta}|2^jB|^{\beta/n}}{w(2^jB)}(1+2^jrm_{\mathcal V}(x_B))^{\alpha}\int_{2^jB}|b(z)-b_{2^jB}|\Phi\Big(\frac{|f(z)|}{t}\Big)w(z)\,\mathrm{d}z\\ &\quad+\sum_{j=0}^{\infty}\frac{w^q(B)^{1/q-\theta}|2^jB|^{\beta/n}}{w(2^jB)}(1+2^jrm_{\mathcal V}(x_B))^{\alpha}|b_B-b_{2^jB}|\int_{2^jB}\Phi\Big(\frac{|f(z)|}{t}\Big)w(z)\,\mathrm{d}z\\ &:= K_1+K_2.\end{aligned}$$

Using (2.2) and (3.8) to estimate K_1 we obtain

$$K_1 \lesssim w^q(B)^{1/q-\theta} \\ \times \sum_{j=0}^{\infty} |2^j B|^{\beta/n}(1+2^j r m_{\mathcal{V}}(x_B))^{\alpha} \|b-b_{2^j B}\|_{\exp L(w),2^j B} \Big\|\Phi\Big(\frac{|f|}{t}\Big)\Big\|_{L\log L(w),2^j B}.$$

We now combine Lemma 2.6 with Lemma 2.1 to arrive at the following estimate:

$$K_1 \lesssim \|b\|_{\mathrm{BMO}} \Big\|\Phi\Big(\frac{|f|}{t}\Big)\Big\|_{M^{1,\theta,\alpha}_{L\log L}(w,w^q)} \sum_{j=0}^{\infty} w^q(B)^{1/q} \frac{|2^j B|^{\beta/n}}{w(2^j B)} \Big(\frac{w^q(2^j B)}{w^q(B)}\Big)^{\theta}.$$

By virtue of (4.13) one can get

$$\begin{aligned} K_1 &\lesssim \Big\|\Phi\Big(\frac{|f|}{t}\Big)\Big\|_{M^{1,\theta,\alpha}_{L\log L}(w,w^q)} \sum_{j=0}^{\infty} w^q(B)^{1/q} w^q(2^j B)^{-1/q} \Big(\frac{w^q(2^j B)}{w^q(B)}\Big)^{\theta} \\ &\lesssim \Big\|\Phi\Big(\frac{|f|}{t}\Big)\Big\|_{M^{1,\theta,\alpha}_{L\log L}(w,w^q)} \sum_{j=0}^{\infty} \Big(\frac{w^q(B)}{w^q(2^j B)}\Big)^{1/q-\theta}. \end{aligned}$$

Taking (2.1) into account we have

$$K_1 \lesssim \Big\|\Phi\Big(\frac{|f|}{t}\Big)\Big\|_{M^{1,\theta,\alpha}_{L\log L}(w,w^q)} \sum_{j=0}^{\infty} 2^{-j\delta(1-\theta)}.$$

It follows from the summability of the series (due to $1-\theta>0$) that

$$K_1 \lesssim \Big\|\Phi\Big(\frac{|f|}{t}\Big)\Big\|_{M^{1,\theta,\alpha}_{L\log L}(w,w^q)}. \tag{4.15}$$

For the last term K_2 we proceed as follows. Using the first part of Lemma 2.4 we deduce that

$$\begin{aligned} K_2 &\lesssim \sum_{j=0}^{\infty} \frac{w^q(B)^{1/q-\theta}|2^j B|^{\beta/n}}{w(2^j B)} (1+2^j r m_{\mathcal{V}}(x_B))^{\alpha} |b_B - b_{2^j B}| \int_{2^j B} \Phi\Big(\frac{|f(z)|}{t}\Big) w(z)\,\mathrm{d}z \\ &\lesssim \sum_{j=0}^{\infty} j(1+2^j r m_{\mathcal{V}}(x_B))^{\alpha} w^q(B)^{1/q-\theta} |2^j B|^{\beta/n} \Big\|\Phi\Big(\frac{|f|}{t}\Big)\Big\|_{L\log L(w),2^j B} \\ &\lesssim \Big\|\Phi\Big(\frac{|f|}{t}\Big)\Big\|_{M^{1,\theta,\alpha}_{L\log L}(w,w^q)} \sum_{j=0}^{\infty} j w^q(2^j B)^{\theta} w^q(B)^{1/q-\theta} \frac{|2^j B|^{\beta/n}}{w(2^j B)}. \end{aligned}$$

Thanks to (4.13) we get

$$K_2 \lesssim \Big\|\Phi\Big(\frac{|f|}{t}\Big)\Big\|_{M^{1,\theta,\alpha}_{L\log L}(w,w^q)} \sum_{j=0}^{\infty} j\Big(\frac{w^q(B)}{w^q(2^jB)}\Big)^{1/q-\theta}.$$

Thus

$$K_2 \lesssim \Big\|\Phi\Big(\frac{|f|}{t}\Big)\Big\|_{M^{1,\theta,\alpha}_{L\log L}(w,w^q)} \sum_{j=0}^{\infty} \frac{j}{2^{j\delta(1-\theta)}} \lesssim \Big\|\Phi\Big(\frac{|f|}{t}\Big)\Big\|_{M^{1,\theta,\alpha}_{L\log L}(w,w^q)}.$$

We then deduce from (3.14) and (3.15) that:

$$J_2 \lesssim \Big\|\Phi\Big(\frac{|f|}{t}\Big)\Big\|_{M^{1,\theta,\alpha}_{L\log L}(w,w^q)}.$$

Combining this with (3.14) yields

$$R_2 \lesssim \Big\|\Phi\Big(\frac{|f|}{t}\Big)\Big\|_{M^{1,\theta,\alpha}_{L\log L}(w,w^q)}$$

and thus completes the proof of the theorem thanks to (3.13). □

5. The Calderón-Zygmund inequalities for Schrödinger type equations

In this section we give some applications of our main results to Schrödinger equation. Let Ω be an open set in $\mathbb{R}^n$. We define the space $M^{p,s}_{\alpha,\theta}(\Omega,w,v)$ as the space of all measurable functions f satisfying $\|f\|_{M^{p,s}_{\alpha,\theta}(\Omega,w,v)} < \infty$, where

$$\|f\|_{M^{p,s}_{\alpha,\theta}(\Omega,w,v)} = \sup_{r>0}\Big[\int_{\mathbb{R}^n} ((1+rm_{\mathcal{V}}(x))^{\alpha} v(B(x,r))^{-\theta}\|f\chi_{B(x,r)\cap\Omega}\|_{L^p(w)})^s \,\mathrm{d}x\Big]^{1/s} < \infty.$$

Corollary 5.1. *Suppose that $\mathcal{V} \in RH_n$. Let $1 < p < s \leqslant \infty$, $\alpha \in (-\infty,\infty)$, $w \in A_p$ and $\theta \in [0,1/p)$. If u is a solution of $-\Delta u + \mathcal{V}u = \operatorname{div} f$, then*

$$\|\nabla u\|_{M^{p,s}_{\alpha,\theta}(\Omega,w)} \leqslant C\|f\|_{M^{p,s}_{\alpha,\theta}(\Omega,w)}.$$

Proof. Since $\nabla u = \nabla\mathcal{L}^{-1/2}(\mathcal{L}^{-1/2}\nabla)f$, the result follows similarly as in the proof of Theorem 1.1. □

Corollary 5.2. *Suppose that* $\mathcal{V} \in RH_{n/2}$. *Let* $\beta \in (0, n)$, $1 < p < n/\beta$, $\alpha \in (-\infty, \infty)$, $1 < s \leqslant \infty$. *For all* $1/q = 1/p - \beta/n$, $\theta \in [0, 1/q)$, $w \in A_{p,q}$, *if* $g \in M^{p,s}_{\alpha,\theta}(\Omega, w^p, w^q)$, *then there exists a function* $u \in M^{q,s}_{\alpha,\theta}(\Omega, w^q)$, *such that*

$$-\Delta u + \mathcal{V} u = g \text{ a.e. } x \in \Omega.$$

Futhermore,

$$\|u\|_{M^{q,s}_{\alpha,\theta}(\Omega,w^q)} \leqslant C\|g\|_{M^{p,s}_{\alpha,\theta}(\Omega,w^p,w^q)}.$$

P r o o f. It follows from the proof of Theorem 1.2 that

$$\|u\|_{M^{q,s}_{\alpha,\theta}(\Omega,w^q)} = \|\mathcal{V}^{-1}g\|_{M^{q,s}_{\alpha,\theta}(\Omega,w^q)} = \|\mathcal{I}^2 g\|_{M^{q,s}_{\alpha,\theta}(\Omega,w^q)} \leqslant C\|g\|_{M^{p,s}_{\alpha,\theta}(\Omega,w^p,w^q)}.$$

□

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