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THE SYMMETRY REDUCTION OF VARIATIONAL INTEGRALS, COMPLEMENT

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Abstract. Some open problems appearing in the primary article on the symmetry reduction are solved. A new and quite simple coordinate-free definition of Poincaré-Cartan forms and the substance of divergence symmetries (quasisymmetries) are clarified. The unbeliavable uniqueness and therefore the global existence of Poincaré-Cartan forms without any uncertain multipliers for the Lagrange variational problems are worth extra mentioning.

Keywords: Lagrange variational problem; Poincaré-Cartan form; symmetry reduction

MSC 2010: 49S05, 49N99, 70H03

1. Introduction

We will systematically refer to the primary article, especially to the open problems in *Perspectives* (see [7]). Two problems admit short solution stated below. They essentially improve the achievements of article [7]. It should be noted on this occasion that the improvements cannot be carried over for a much more involved case of the multidimensional Lagrange problem (in preparation). For the reader's convenience, let us briefly overview the core of our approach since it differs from the common literature.

We start with the notation. Our underlying spaces are infinite-dimensional manifolds $\mathbf{M}$ modelled on $\mathbb{R}^\infty$. Each function depends on a finite number of coordinates and the functions constitute the ring denoted by $\mathcal{F}(\mathbf{M})$ ($= \mathcal{F}$, abbreviation). The

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$\mathcal{F}$-module $\Phi(\mathbf{M})$ $(=\Phi)$ of differential 1-forms

$$\varphi = \sum f^i \,\mathrm{d}g^i \quad (\text{finite sum with } f^i, g^i \in \mathcal{F})$$

makes a good sense. The vector fields Z are $\mathcal{F}$-linear functions on Φ, where

$$\varphi(Z) = Z\rfloor\varphi = \sum f^i Z g^i, \quad Zf = Z\rfloor\,\mathrm{d}f = \mathrm{d}f(Z)$$

are familiar formulae. The Lie derivative $\mathcal{L}_Z\varphi = Z\rfloor\,\mathrm{d}\varphi + \mathrm{d}\varphi(Z)$ will be frequently needed. A novelty is as follows. Let $\varphi^1, \varphi^2, \ldots$ be a basis of module Φ. Then we denote

$$Z = \sum z^i \frac{\partial}{\partial\varphi^i} \quad (\text{infinite sum with } z^i = \varphi^i(Z))$$

and recall the common abbreviation $\partial/\partial f = \partial/\partial\,\mathrm{d}f$. In fact, we deal with the local theory on open subsets of $\mathbf{M}$ which are not explicitly specified.

The *Lagrange variational problem* consists of two components. First, the differential constraints for admissible curves of the problem are defined in coordinate-free terms as a certain Pfaffian system on $\mathbf{M}$ and we speak of a *diffiety* Ω. In more detail, we deal with Pfaffian equations $\omega = 0$, $\omega \in \Omega$, where $\Omega \subset \Phi$ is a submodule of codimension one. Any function $x \in \mathcal{F}$ with $\mathrm{d}x \notin \Omega$ may be taken for the *independent variable* as a technical tool. It follows that Ω is generated by *contact forms* $\mathrm{d}f - Df\,\mathrm{d}x$, $f \in \mathcal{F}$, where D is the *total derivative* vector field determined by the properties

$$Dx = 1, \quad \omega(D) = 0, \quad \omega \in \Omega.$$

If Ω is a *controllable* diffiety in the sense that $Df = 0$ if and only if $f = \text{const.}$, then there exists the *standard basis*

$$\pi^j_r, \quad j = 1, \ldots, \mu(\Omega),\ r = 0, 1, \ldots$$

of module Ω with the excellent property $\mathcal{L}_D\pi^j_r = \pi^j_{r+1}$. Second, we are interested in the values

$$\int_a^b \mathbf{n}^*\varphi \quad (\mathbf{n}\colon\ (a \leqslant t \leqslant b) \to \mathbf{M}),$$

where the curves $\mathbf{n}$ lying in $\mathbf{M}$ satisfy the differential constraints, that is, mappings $\mathbf{n}$ are solutions of Ω in the common sense $\mathbf{n}^*\omega = 0$, $\omega \in \Omega$. However,

$$\int_a^b \mathbf{n}^*\varphi = \int_a^b \mathbf{n}^*\widetilde{\varphi}, \quad \widetilde{\varphi} = \varphi + \widetilde{\omega},\ \widetilde{\omega} \in \Omega$$

for any $\widetilde{\omega}$. So the latter expression with *arbitrary* $\widetilde{\omega}$ may be called a *variational integral*. For our convenience, it is abbreviated as $\int\varphi$ and therefore $\int\varphi = \int(\varphi + \omega)$ for any fixed $\omega \in \Omega$.

The *calculus of variations* eventually appears as follows. A vector field V is called a *variation* of Ω if $\mathcal{L}_V\Omega \subset \Omega$. A vector field A is called an (admissible) *variation of solution* $\mathbf{n}$ if $\mathbf{n}^*\mathcal{L}_A\Omega = 0$. A solution $\mathbf{n}$ of Ω is called an *extremal* (of variational integral $\int \varphi$ with the constraint Ω) if

$$A\rfloor\,\mathrm{d}\widetilde{\varphi} = 0, \quad \widetilde{\varphi} = \varphi + \widetilde{\omega},\ \widetilde{\omega} = \widetilde{\omega}[\mathbf{n}] \in \Omega$$

for all variations A of $\mathbf{n}$ and an appropriate correction $\widetilde{\omega}$. The most important achievement is: in the controllable case, there exists a *universal* correction $\widetilde{\omega}$ denoted $\breve{\omega}$ which gives all extremals. It can be explicitly calculated by a mere linear algebra. This is the famous *Poincaré-Cartan* ($\mathcal{PC}$) *form* $\breve{\varphi} = \varphi + \breve{\omega}$ of the Lagrange problem and we recall the primary Definition 2.5 (see [7]) which is as follows.

The form $\breve{\varphi} = \varphi + \breve{\omega}$, $\breve{\omega} \in \Omega$ is called $\mathcal{PC}$ *form* if

$$A\rfloor\,\mathrm{d}\breve{\varphi} \cong Z\rfloor\,\mathrm{d}\breve{\varphi}\ (\mathrm{mod}\ \Omega),$$

where Z is an arbitrary vector field and $A = A[Z]$ is an appropriate variation universal for all solutions $\mathbf{n}$ such that (2.6) from [7]

$$\varphi^j(A) = \sum f^j_{kr} D^r\varphi^k(Z), \quad j = 1, 2, \ldots$$

Here $\varphi^1, \varphi^2, \ldots$ is a basis of module Φ and the coefficients f^j_{kr} are universal for all Z and $A = A[Z]$.

In the reduction theory, we speak of *variations* V *of integral* $\int \varphi$ (with the constraint Ω) if $\mathcal{L}_V\Omega \subset \Omega$ and $\mathcal{L}_V\varphi \in \Omega$. Variations V which moreover (locally) generate a one-parameter are called *infinitesimal symmetries* of the variational problem, see Definition 2.6 and Theorem 2.2 (both from [7]).

The preparation is over and we leave more details to the *Concluding comments.* Let us turn to the open problems in *Perspectives* from [7].

2. On the $\mathcal{PC}$ forms

We introduce shorter definition of Poincaré-Cartan ($\mathcal{PC}$) form.

Definition 2.1. For a special choice $\breve{\omega} \in \Omega$, the form $\breve{\varphi} = \varphi + \breve{\omega}$ is called $\mathcal{PC}$ *form related to the integral* $\int \varphi$ if

$$V\rfloor\,\mathrm{d}\breve{\varphi} \cong Z\rfloor\,\mathrm{d}\breve{\varphi}\ (\mathrm{mod}\ \Omega), \tag{2.1}$$

where $Z \in \mathcal{T}(\mathbf{M})$ is an arbitrary vector field and $V = V[Z]$ the appropriate variation.

The equivalence of both definitions of the $\mathcal{PC}$ form rests on the following observation. If A is admissible variation of all solutions $\mathbf{n}$, then $A = V$ is in reality a variation of Ω (easy). Therefore $A = A[Z]$ in Definition 2.5 from [7] can be replaced with $V = V[Z]$. (This fact was also noted in [7].) Then Definition 2.1 appears by omitting the requirement (2.6) in [7] and the supply of $\mathcal{PC}$ forms is therefore not reduced. However, there exists only one $\mathcal{PC}$ form in the sense of Definition 2.1 and so the equivalence is obvious.

Theorem 2.1. *For every variational integral $\int \varphi$ there exists a unique $\mathcal{PC}$ form $\breve{\varphi}$ in the sense of Definition* 2.1.

P r o o f. The difference $\omega \in \Omega$ of two $\mathcal{PC}$ forms in the sense of Definition 2.1 satisfies

$$V \rfloor \mathrm{d}\omega \cong Z \rfloor \mathrm{d}\omega \pmod{\Omega}. \tag{2.2}$$

In terms of (any) standard basis (4.5), see [7], we have

$$\omega = \sum a_r^j \pi_r^j, \quad \mathrm{d}\omega \cong \mathrm{d}x \wedge \sum b_r^j \pi_r^j \pmod{\Omega \wedge \Omega},$$

where $b_r^j = Da_r^j + a_{r-1}^j$ and

$$D = \frac{\partial}{\partial x} + \sum 0 \frac{\partial}{\partial \pi_r^j}, \quad Dx = 1,\ \pi_r^j(D) = 0$$

is the total derivative. This follows from the congruences

$$\mathrm{d}\pi_r^j \cong \mathrm{d}x \wedge \pi_{r+1}^j \pmod{\Omega \wedge \Omega}, \quad \mathrm{d}a_r^j \cong Da_r^j \,\mathrm{d}x \pmod{\Omega}$$

frequently appearing in [7]. On the other hand, we recall formula (4.6) from [7]

$$V = v\frac{\partial}{\partial x} + \sum D^r p^j \frac{\partial}{\partial \pi_r^j}, \quad v = Vx,\ p^j = \pi_0^j(V)$$

for the variations V. Then (2.2) implies the congruence

$$\sum b_r^j D^r p^j \,\mathrm{d}x \cong V \rfloor \mathrm{d}\omega \cong Z \rfloor \mathrm{d}\omega \cong \sum b_r^j z_r^j \,\mathrm{d}x, \quad z_r^j = \pi_r^j(Z)$$

and therefore the identity

$$Df = \sum b_r^j D^r p^j = \sum b_r^j z_r^j = g, \quad f = \sum a_r^j D^r p^j.$$

This identity is impossible for arbitrary nonvanishing function g, see the lemma below. It follows that $b_r^j = 0$ identically. However

$$b_0^j = Da_0^j,\ b_1^j = Da_1^j + a_0^j, \ldots,\ b_R^j = Da_R^j + a_{R-1}^j,\ b_{R+1}^j = a_R^j \text{ (certain } R)$$

and therefore $a_r^j = 0$, hence $\omega = 0$. □

Lemma 2.1. *The identity of the kind $g = Df$ holds true if and only if the $\mathcal{PC}$ form related to the variational integral $\int g\,\mathrm{d}x$ is closed.*

P r o o f. The identity reads

$$g\,\mathrm{d}x \cong \mathrm{d}f = Df\,\mathrm{d}x + \sum \frac{\partial f}{\partial \pi_r^j}\pi_r^j \pmod{\Omega}.$$

The $\mathcal{PC}$ form $\breve{\varphi}$ related to the integral $\int g\,\mathrm{d}x$ is the same as the $\mathcal{PC}$ form related to the integral $\int \mathrm{d}f$, therefore $\breve{\varphi} = \mathrm{d}f$ and $\mathrm{d}\breve{\varphi} = 0$. The converse is true as well. □

For our convenience, we recall (a little simplified) Theorem 2.1 from [7] which describes the role of the $\mathcal{PC}$ form in the calculus of variations.

Theorem 2.2. *A solution $\mathbf{n}$ of Ω is an extremal of variational integral $\int \varphi$ with the constraint Ω if and only if $\mathbf{n}^* Z \rfloor \mathrm{d}\breve{\varphi} = 0$ for all vector fields Z.*

In the reduction theory, the following obvious consequence of Theorem 2.1 is essential. It completely devaluates Theorems 2.2, 2.3 (see [7]) which become needless with comfortable impact for the practice: any $\mathcal{PC}$ form $\breve{\varphi}$ can be employed in (5.1) from [7]. See also the more general Theorem 3.3 below.

Theorem 2.3. *Every infinitesimal symmetry V of variational integral $\int \varphi$ preserves the relevant $\mathcal{PC}$ form $\breve{\varphi}$.*

Theorems 2.1 and 2.3 appear as a by-product of the symmetry reduction, however, they are the most important and even unbelievable achievements already for the case of the jet spaces with trivial constraint Ω. Indeed, there are always many possible standard filtrations Ω_* and construction in Theorem 4.3 (see [7]) moreover heavily depends on the choice of the initial forms of Ω_*. However, the final $\mathcal{PC}$ form is unique. In the global theory omitted here, the local uniqueness ensures the existence of the global $\mathcal{PC}$ form related to every controllable Lagrange problem, that is, the choice of quite different good filtrations (2.1) from [7] on overlapping coordinate systems does not affect the global $\mathcal{PC}$ form!

3. On the Noether theorem

Complete Definition 2.6 of [7] goes as follows.

Definition 3.1. An infinitesimal symmetry V of diffiety Ω is called a *divergence symmetry* of variational integral $\int \varphi$ if $\mathcal{L}_V\varphi - \mathrm{d}f \in \Omega$ for appropriate function $f \in \mathcal{F}(\mathbf{M})$. We occasionally denote $V = V[f]$ for clarity.

Theorem 3.1 (Noether). *The conservation law*

$$\mathbf{n}^*(\breve{\varphi}(V) - f) = c \in \mathbb{R}, \quad V = V[f] \tag{3.1}$$

holds true for every extremal $\mathbf{n}$.

P r o o f. This is a consequence of the inclusion

$$\mathcal{L}_V \breve{\varphi} - \mathrm{d}f = V \rfloor \mathrm{d}\breve{\varphi} + \mathrm{d}(\breve{\varphi}(V) - f) \in \Omega,$$

where $\mathbf{n}^* V \rfloor \mathrm{d}\breve{\varphi} = 0$. □

In reality, the divergence symmetries are merely slight generalizations of the common symmetries.

Theorem 3.2. *A vector field $V = V[f]$ is a divergence symmetry of integral $\int \varphi$ if and only if this V is a symmetry of any integral*

$$\int (\varphi - \mathrm{d}F), \quad VF = f. \tag{3.2}$$

If $\breve{\varphi}$ is the $\mathcal{PC}$ form related to integral $\int \varphi$, then $\breve{\varphi}[F] = \breve{\varphi} - \mathrm{d}F$ is the $\mathcal{PC}$ form corresponding to integral (3.2).

P r o o f. Clearly

$$\mathcal{L}_V \varphi - \mathrm{d}f = \mathcal{L}_V \varphi - \mathrm{d}VF = \mathcal{L}_V(\varphi - \mathrm{d}F) \in \Omega.$$

Substituting moreover the $\mathcal{PC}$ form $\breve{\varphi}$ for φ, we obtain

$$\mathcal{L}_V \breve{\varphi}[F] \in \Omega, \quad \breve{\varphi}[F] = \breve{\varphi} - \mathrm{d}F \tag{3.3}$$

and one can see that $\breve{\varphi}[F]$ is the $\mathcal{PC}$ form related to the integral $\int(\varphi - \mathrm{d}F)$. □

In particular, it follows that the conservation law (3.1) reads

$$\mathbf{n}^* \breve{\varphi}[F](V) = c \in \mathbb{R}, \quad V = V[f]. \tag{3.4}$$

Altogether we conclude that the reduction theory (see [7]) can be applied to the divergence symmetries as well: the $\mathcal{PC}$ form $\breve{\varphi}$ should be replaced with $\breve{\varphi}[F]$. For instance, Theorem 5.1 from [7] turns into a more general result:

Theorem 3.3. *Let $\Omega \subset \Phi(\mathbf{M})$ be a controllable diffiety and $V = V[f] \in \mathcal{T}(\mathbf{M})$ a divergence symmetry of a variational integral $\int \varphi$ with the constraint Ω. Every* *extremal is lying in a certain subspace* $\mathbf{M}[f,c] \subset \mathbf{M}$ *determined by the equation* $\breve{\varphi}(V) - f = c$*, where* $c \in \mathbb{R}$ *and there exists the orbit space* $\mathbf{M}[f,c]/V \subset \mathbf{M}/\mathbf{V}$ *of the total orbit space* $\mathbf{M}/\mathbf{V}$*, the orbit diffiety* $\Omega[f,c]/V \subset \Phi(\mathbf{M}[f,c]/V)$ *naturally induced by* Ω *and the Routh variational integral*

$$\int \breve{\varphi}[f,c], \quad \breve{\varphi}[f,c] = \breve{\varphi} - \mathrm{d}F - c\,\mathrm{d}w,\ f = VF,\ Vw = 1 \tag{3.5}$$

defined on the space $\mathbf{M}[f,c]/V$*. Projections on the orbit space of the original extremals are extremals of the Routh variational integral. If* $\breve{\varphi}[f,c]$ *is even a* $\mathcal{PC}$ *form, the projection of extremals is surjective.*

We have already employed the uniqueness of $\mathcal{PC}$ forms. One can observe that the primary Theorem 5.1 from [7] is involved for the particular case $f = \text{const}$. We omit the obvious result corresponding to Theorem 5.3 from [7]. The global theory can be comfortably investigated as well: since the $\mathcal{PC}$ forms are unique, the global reduction is ensured by any global solution F of the ordinary differential equation $VF = f + \text{const}$.

Theorem 3.4. *The divergence symmetries* $V = V[f]$ *for various* f *can be defined by the identity* $\mathcal{L}_V\,\mathrm{d}\breve{\varphi} = 0$.

P r o o f. The divergence condition $\mathcal{L}_V\varphi - \mathrm{d}f \in \Omega$ is equivalent to any of the inclusions

$$\mathcal{L}_V(\varphi - \mathrm{d}F) \in \Omega, \quad \mathcal{L}_V(\breve{\varphi} - \mathrm{d}F) \in \Omega, \quad \mathcal{L}_V\breve{\varphi}[F] \in \Omega.$$

However, Theorem 2.3 applied to the $\mathcal{PC}$ form $\breve{\varphi}[F]$ implies even

$$\mathcal{L}_V\breve{\varphi}[F] = 0, \quad \mathcal{L}_V(\breve{\varphi} - \mathrm{d}F) = 0, \quad \mathcal{L}_V\,\mathrm{d}\breve{\varphi} = 0.$$

This sequence of reasonings can be reversed. □

It follows that the divergence symmetries V can be defined by $\mathcal{L}_V\,\mathrm{d}\breve{\varphi} = 0$. In more detail

$$0 = \mathcal{L}_V\,\mathrm{d}\breve{\varphi} \cong \sum Ve^j\pi_0^j \wedge \mathrm{d}x + \sum e^j\mathcal{L}_V\pi_0^j \wedge \mathrm{d}x \ (\mathrm{mod}\ \Omega \wedge \Omega) \tag{3.6}$$

by using (4.9) from [7], where $\mathcal{L}_V\pi_0^j = \sum a_{j'}^j\pi_0^{j'}$ by virtue of Lemma 5.1 (see [7]). Then (3.6) can be interpreted by saying that the divergence symmetries $V = V[f]$ with various f are identical with such infinitesimal symmetries V of Ω which preserve the Euler-Lagrange system $e^j = 0$, $j = 1, \ldots, \mu(\Omega)$ but this is informal statement and we do not investigate subtle details.

4. Concluding comments

Since our approach differs from the actual literature, we believe that some brief comments would be useful. On this occasion, let us refer to the excellent and clear survey (see [4]) of rather special reduction problems in terms of the common jet theory and, on the contrary, to the involved prologue (see [3]) into the problem of $\mathcal{PC}$ forms in general field theories, where the exterior differential systems are used for the differential constraints.

4.1. A note on extremals. Actually, the extremals in current literature are defined by the stationarity of variational integral for one-parameter solutions $\mathbf{n}(t)$ satisfying moreover certain boundary conditions. However, for multidimensional Lagrange problems, that is, in the constrained field theories (see [3]), the existence of such true solutions satisfying moreover appropriate boundary conditions is in general doubtful. On the contrary, our approach (with Definitions 2.1–2.6 from [7]) can be almost literally applied as well. Alas, the standard basis becomes rather involved (see [6]) and the explicit formulae for all variations V do not exist. Consequently, the simplified Definition 1.1 fails and the final $\mathcal{PC}$ forms need not be unique.

4.2. A note on divergence symmetries. We may refer to the theory (see [4]), where only the variational integrals and infinitesimal symmetries

$$\int L\,\mathrm{d}t, \quad V = v\frac{\partial}{\partial t} + \sum v_i \frac{\partial}{\partial q_i} \tag{4.1}$$

with $L = L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n)$ and $v = 0$, $v_i = v_i(q, \ldots, q_n)$ independent of time variable t are discussed. These results in beautiful geometrical theory on the first-order jet spaces, however, even the classical Maupertuis theorem, cannot be involved without adaptations (see [1]). In our approach the most general case (4.1) with variable t is easily contained as well. For instance, the divergence symmetries $V = V[f]$ of the integral $\int L\,\mathrm{d}t$ are identified with the point symmetries of the variational integral

$$\int \Big(L - \Big(\frac{\partial F}{\partial t} + \sum \frac{\partial F}{\partial q_i}\dot{q}_i\Big)\Big)\,\mathrm{d}t, \quad F = F(t, q_1, \ldots, q_n),\ VF = f.$$

Altogether taken, our divergence symmetries applied within the jet spaces involve the quasisymmetries (see [4]) as a very particular subcase.

4.3. A note on several symmetries. For instance, let V and $\overline{V}$ be infinitesimal symmetries of integral $\int \varphi$. Then

$$\mathcal{L}_V \breve{\varphi} = \mathcal{L}_{\overline{V}} \breve{\varphi} = 0, \quad \mathbf{n}^*\breve{\varphi}(V) = c, \quad \mathbf{n}^*\breve{\varphi}(\overline{V}) = \bar{c}, \quad c, \bar{c} \in \mathbb{R}$$

hold true for every extremal $\mathbf{n}$. It follows that the extremals lie in subspaces $\mathbf{N}[c,\bar{c}] \subset \mathbf{M}$ defined by $\breve{\varphi}(V) = c$, $\breve{\varphi}(\overline{V}) = \bar{c}$. Alas, we have

$$V\breve{\varphi}(\overline{V}) = (\mathcal{L}_V\breve{\varphi})(\overline{V}) + \breve{\varphi}([V,\overline{V}]) = \breve{\varphi}([V,\overline{V}])$$

and the vector field V need not be in general tangent to the subspace $\mathbf{N}[c,\bar{c}]$ which therefore does not consist of V-orbits. The "gyroscopic augmentation" was inverted in order to delete this trouble, see the references in [4]. Possible implementation of this idea into our theory would be desirable.

4.4. Still two open problems. Finally, we recall that the pseudogroup symmetries depending on the choice of arbitrary functions, the *calibrations* in physics, cause many difficulties since they cannot be reasonably developed in terms of the infinitesimal symmetries (see [2], [5]). We also raise the following problem. Let us have a conservation law in an extremal principle of a field theory with differential constraints. What is the impact on the extremality if the conservation law is regarded as an additional constraint? The group symmetry reduction is a particular case of the latter problem.

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