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ABSTRACT KOROVKIN TYPE THEOREMS ON MODULAR SPACES BY $\mathscr{A}$-SUMMABILITY

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Abstract. Our aim is to change classical test functions of Korovkin theorem on modular spaces by using $\mathscr{A}$-summability.

Keywords: $\mathscr{A}$-summability; modular space; abstract Korovkin theory

MSC 2010: 40C05, 41A36

1. Introduction

Approximation theory is one of the most thriving areas within functional analysis. Korovkin has proved a well known approximation theorem which states the uniform convergence in $C[a,b]$, the space of continuous real functions defined on $[a,b]$, of a sequence of positive linear operators by stating the convergence only on three test functions $\{1, x, x^2\}$. Korovkin theory provides a useful technique for approaching behavior of positive linear operators within the area of approximation theory. This theory has been studied by many authors in various directions. There is a deep insight into the relation between summability theory and approximation theory. Based on this relation, we give some abstract Korovkin type theorems via modular convergence in the sense of $\mathscr{A}$-summability and strong convergence in the sense of $\mathscr{A}$-summability. These notions enable us to give generalizations of the Korovkin theorem. Our aim is to change classical test functions of Korovkin theorem on modular spaces by using $\mathscr{A}$-summability. Similar problems have been studied in [1], [2], [3], [4].

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We recall the foundations of the theory of modular function spaces and some notions which are needed. We refer the reader to [11], [17].

Let us start by considering the notion of $\mathscr{A}$-summability of a sequence introduced by Bell (see [12]). Assume that $\mathscr{A} = \{A^{(n)}\} = (a_{kj}^{(n)})$, $j,k,n \in \mathbb{N}$ is a sequence of infinite matrices. $(Ax)_k^{(n)} := \sum_j a_{kj}^{(n)} x_j$ is said to be the $\mathscr{A}$-transform of x whenever the series converges for all k and n. Then a sequence x is said to be $\mathscr{A}$-summable (or $\mathscr{A}$-convergent) to some number L provided that

$$\lim_{k\to\infty} (Ax)_k^{(n)} = L \quad \text{uniformly in } n \in \mathbb{N}.$$

Also, $\mathscr{A}$ is said to be a regular method of matrices if $\lim_{j\to\infty} x_j = L$ implies $\lim_{k\to\infty} (Ax)_k^{(n)} = L$ uniformly in $n \in \mathbb{N}$. This method has the advantage of summing some divergent sequences and has been used in approximation theory (see [21]).

Let I be a locally compact Hausdorff topological space, endowed with a uniform structure $\mathcal{U} \subset 2^{I\times I}$ which generates the topology of I. Let μ be a regular measure defined on $\mathcal{B}$ which is the σ-algebra of all Borel sets of I. Then, by $X(I)$ we denote the space of all real-valued μ-measurable functions on I equipped with the equality μ-a.e. As usual, let $C(I)$ denote the space of all continuous real valued functions on I. The space of all real-valued continuous and bounded functions on I is denoted by $C_{\rm b}(I)$ and also the subspace of $C_{\rm b}(I)$ of all functions with compact support on I is denoted by $C_{\rm c}(I)$. We say that a functional $\varrho\colon X(I) \to [0,\infty]$ is a *modular* on $X(I)$ provided that the following conditions hold:

(i) $\varrho[f] = 0$ if and only if $f = 0$ μ-almost everywhere on I,

(ii) $\varrho[-f] = \varrho[f]$ for every $f \in X(I)$,

(iii) $\varrho[\alpha f + \beta g] \leqslant \varrho[f] + \varrho[g]$ for every $f, g \in X(I)$ and for any $\alpha, \beta \geqslant 0$ with $\alpha + \beta = 1$.

A modular ϱ is said to be Q-quasi convex if there exists a constant $Q \geqslant 1$ such that the inequality

$$\varrho[\alpha f + \beta g] \leqslant Q\alpha\varrho[Qf] + Q\beta\varrho[Qg]$$

holds for every $f, g \in X(I)$, $\alpha, \beta \geqslant 0$ with $\alpha + \beta = 1$. In particular, if $Q = 1$, then ϱ is called convex.

A modular ϱ is said to be Q-quasi semiconvex if there exists a constant $Q \geqslant 1$ such that the inequality

$$\varrho[af] \leqslant Qa\varrho[Qf]$$

holds for every nonnegative function $f \in X(I)$ and $a \in (0,1]$.

It is clear that every Q-quasi convex modular is Q-quasi semiconvex. We now consider some subspaces of $X(I)$ by means of a modular ϱ as follows:

$$L^{\varrho}(I) := \Big\{ f \in X(I)\colon \lim_{\lambda\to 0^+} \varrho[\lambda f] = 0 \Big\}$$

and

$$E^{\varrho}(I) := \{f \in L^{\varrho}(I)\colon\ \varrho[\lambda f] < \infty \text{ for all } \lambda > 0\}$$

is called the modular space generated by ϱ and the space of the finite elements of $L^{\varrho}(I)$, respectively. Observe that if ϱ is Q-quasi semiconvex, then the space

$$\{f \in X(I)\colon\ \varrho[\lambda f] < \infty \text{ for some } \lambda > 0\}$$

coincides with $L^{\varrho}(I)$. The notions about modulars have been introduced in [19] and have been widely discussed in [4], [5], [7], [9]–[11], [13], [14], [16]–[18], and [20].

We need some of the following assumptions on modulars:

▷ ϱ is monotone, i.e. for $f, g \in X(I)$ if $|f| \leqslant |g|$, then $\varrho[f] \leqslant \varrho[g]$.

▷ ϱ is strongly finite, i.e. $\chi_A \in E^{\varrho}(I)$ for all $A \in \mathcal{B}$ with $\mu(A) < \infty$.

▷ ϱ is absolutely continuous, i.e. there exists $\alpha > 0$ such that for every $f \in X(I)$ with $\varrho[f] < \infty$:

  ⊳⊳ for each $\varepsilon > 0$ there exists a set $A \in \mathcal{B}$ with $\mu(A) < \infty$ and $\varrho[\alpha f \chi_{I \setminus A}] \leqslant \varepsilon$,

  ⊳⊳ for each $\varepsilon > 0$ there is $\delta > 0$ with $\varrho[\alpha f \chi_B] \leqslant \varepsilon$ for every $B \in \mathcal{B}$ with $\mu(B) < \delta$.

According to [8], recall that $\{f_j\}$ is modularly convergent to a function $f \in L^{\varrho}(I)$ if and only if

$$\lim_{j\to\infty} \varrho[\lambda_0(f_j - f)] = 0 \quad \text{for some } \lambda_0 > 0,$$

also $\{f_j\}$ is strongly convergent to a function $f \in L^{\varrho}(I)$ if and only if

$$\lim_{j\to\infty} \varrho[\lambda(f_j - f)] = 0 \quad \text{for every } \lambda > 0.$$

Moreover, we recall the following convergences in modular spaces which have also been studied in [15]. Let $\{f_j\}$ be a function sequence whose terms belong to $L^{\varrho}(I)$. Then $\{f_j\}$ is modularly convergent to a function $f \in L^{\varrho}(I)$ in the sense of $\mathscr{A}$-summability if and only if

$$\lim_k \sum_{j=1}^{\infty} a_{kj}^{(n)} \varrho[\lambda_0(f_j - f)] = 0 \quad \text{for some } \lambda_0 > 0 \text{ uniformly in } n.$$

Also, $\{f_j\}$ is strongly convergent to a function $f \in L^{\varrho}(I)$ in the sense of $\mathscr{A}$-summability if and only if

$$\lim_k \sum_{j=1}^{\infty} a_{kj}^{(n)} \varrho[\lambda(f_j - f)] = 0 \quad \text{for every } \lambda > 0 \text{ uniformly in } n.$$

If there exists a constant $M > 0$ such that for all $u \geqslant 0$

$$\varrho[2u] \leqslant M\varrho[u]$$

holds, then it is said that ϱ satisfies the Δ_2-condition. The key property of the Δ_2-condition is the following theorem.

Theorem 1. *Let $L^\varrho(I)$ be a modular space. Δ_2-condition is sufficient in order that strong convergence in the sense of $\mathscr{A}$-summability and modular convergence in the sense of $\mathscr{A}$-summability be equivalent in $L^\varrho(I)$.*

P r o o f. Obviously, strong convergence of $\{f_j\}$ to f in the sense of $\mathscr{A}$-summability is equivalent to the condition $\lim_k \sum_{j=1}^{\infty} a_{kj}^{(n)} \varrho[2^N \lambda(f_j - f)] = 0$ uniformly in n for some $\lambda > 0$ and all $N = 1, 2, \ldots$ Let $\{f_j\}$ be modularly convergent to f in the sense of $\mathscr{A}$-summability. Then there exists $\lambda > 0$ such that $\lim_k \sum_{j=1}^{\infty} a_{kj}^{(n)} \varrho[\lambda(f_j - f)] = 0$ uniformly in n. Δ_2-condition implies by induction that $\varrho[2^N \lambda(f_j - f)] \leqslant M^N \varrho[\lambda(f_j - f)]$. Therefore we get

$$\lim_k \sum_{j=1}^{\infty} a_{kj}^{(n)} \varrho[2^N \lambda(f_j - f)] = 0.$$

This completes the proof. □

2. Main results

In this section we give some Korovkin-type theorems by using different test functions from the ordinary ones $\{1, x, x^2\}$ in the sense of $\mathscr{A}$-summability.

Observe now that if a modular ϱ is monotone and finite, then we have $C(I) \subset L^\varrho(I)$ (see [11]). In a similar manner, if ϱ is monotone and strongly finite, then $C(I) \subset E^\varrho(I)$. Let ϱ be monotone and finite modular on $X(I)$. Assume that D is a set satisfying $C_{\rm b}(I) \subset D \subset X(I)$. Assume further that $T := \{T_j\}$ is a sequence of positive linear operators from D into $X(I)$. Also we say that the sequence T satisfies condition:

(∗) If there exists a subset $X_T \subset D \cap L^\varrho(I)$ with $C_{\rm b}(I) \subset X_T$ and a positive real constant R with $T_j f \in L^\varrho(I)$ for all $f \in X_T$ and $j \in \mathbb{N}$ such that

$$\limsup_k \sum_j a_{kj}^{(n)} \varrho[\tau(T_j f)] \leqslant R \varrho[\tau f]$$

for every $f \in X_T$ and $\tau > 0$.

Assume that $e_0(t) = 1$ for all $t \in I$ and let e_i, a_i be functions in $C_{\rm b}(I)$ for $i = 0, 1, \ldots, m$. Put

$$P_s(t) = \sum_{i=0}^{m} a_i(s) e_i(t), \quad s, t \in I \tag{1}$$

and suppose that $P_s(t), s, t \in I$, satisfies the following conditions:

(K1) $P_s(s) = 0$ for all $s \in I$,

(K2) for every neighbourhood $U \in \mathcal{U}$ there is a positive real number η with $P_s(t) \geqslant \eta$ whenever $s, t \in I$, $(s,t) \notin U$.

Some examples of P_s for which (K1) and (K2) are satisfied have been given in [4].

Theorem 2. *Let $\mathscr{A} = \{A^{(n)}\}$ be a sequence of infinite nonnegative real matrices and let ϱ be a strongly finite, monotone and Q-quasi semiconvex modular. Assume that e_i and a_i, $i = 0, 1, \ldots, m$ satisfy properties* (K1) *and* (K2)*. Let $\{T_j\}$, $j \in \mathbb{N}$ be a sequence of positive linear operators satisfying condition* $(*)$*. If*

$$\lim_{k\to\infty} \sum_{j=1}^{\infty} a_{kj}^{(n)} \varrho[\lambda(T_j e_i - e_i)] = 0 \quad \textit{uniformly in } n$$

for some $\lambda > 0$ and $i = 0, 1, \ldots, m$, then for every $f \in C_{\mathrm{c}}(I)$

$$\lim_{k\to\infty} \sum_{j=1}^{\infty} a_{kj}^{(n)} \varrho[\gamma(T_j f - f)] = 0 \quad \textit{uniformly in } n$$

for some $\gamma > 0$. Moreover, if

$$\lim_{k\to\infty} \sum_{j=1}^{\infty} a_{kj}^{(n)} \varrho[\lambda(T_j e_i - e_i)] = 0 \quad \textit{uniformly in } n$$

for every $\lambda > 0$ and $i = 0, 1, \ldots, m$, then for every $f \in C_{\mathrm{c}}(I)$

$$\lim_{k\to\infty} \sum_{j=1}^{\infty} a_{kj}^{(n)} \varrho[\lambda(T_j f - f)] = 0 \quad \textit{uniformly in } n$$

for every $\lambda > 0$.

P r o o f. Let $f \in C_{\mathrm{c}}(I)$. Since I is endowed with $\mathcal{U}$ uniformity, f is uniformly continuous and bounded on I. Let $\varepsilon > 0$. Without loss of generality we can choose $0 < \varepsilon \leqslant 1$. From the uniform continuity of f there exists $U \in \mathcal{U}$ such that

$$|f(s) - f(t)| \leqslant \varepsilon, \quad s, t \in I,\ (s,t) \in U.$$

For every $s, t \in I$ and in correspondence with U let $P_s(t)$ be as in (1) and $\eta > 0$ satisfy condition (K2). If $M = \sup\limits_{t\in I} |f(t)|$, for $s, t \in I$, $(s,t) \notin U$, we have

$$|f(s) - f(t)| \leqslant 2M \leqslant \frac{2M}{\eta} P_s(t).$$

For every $s, t \in I$ we obtain

$$|f(s) - f(t)| \leqslant 2M \leqslant \varepsilon + \frac{2M}{\eta} P_s(t).$$

Therefore for every $s, t \in I$ we get

$$(2) \qquad -\varepsilon - \frac{2M}{\eta} P_s(t) \leqslant f(s) - f(t) \leqslant \varepsilon + \frac{2M}{\eta} P_s(t).$$

Since T_j is a linear positive operator, using (2) for each $j \in \mathbb{N}$ and every $s \in I$ we have

$$-\varepsilon(T_j e_0)(s) - \frac{2M}{\eta}(T_j P_s(s)) \leqslant f(s)(T_j e_0)(s) - (T_j f)(s) \leqslant \varepsilon(T_j e_0)(s) + \frac{2M}{\eta}(T_j P_s)(s)$$

and hence

$$\begin{aligned} |(T_j f)(s) - f(s)| &\leqslant |(T_j f)(s) - f(s)(T_j e_0)(s)| + |f(s)(T_j e_0)(s) - f(s)| \\ &\leqslant \varepsilon(T_j e_0)(s) + \frac{2M}{\eta}(T_j P_s)(s) + M|(T_j e_0)(s) - e_0(s)|. \end{aligned}$$

Let $\gamma > 0$. Using the modular ϱ in the last inequality, for each $j \in \mathbb{N}$ we have

$$(3) \qquad \begin{aligned} \varrho[\gamma(T_j f - f)] &\leqslant \varrho[3\gamma\varepsilon(T_j e_0)] + \varrho[3\gamma M(T_j e_0 - e_0)] + \varrho\Big[6\gamma\frac{M}{\eta}(T_j P_{(\cdot)})(\cdot)\Big] \\ &= J_1 + J_2 + J_3. \end{aligned}$$

So to prove the theorem it is sufficient to show that there exists a positive real number γ such that $\lim\limits_{k\to\infty} \sum\limits_j a_{kj}^{(n)} \varrho[\gamma(T_j f - f)] = 0$ uniformly in n. From hypothesis there exists $\lambda > 0$ such that for each $i = 0, 1, \ldots, m$

$$\lim_{k\to\infty} \sum_j a_{kj}^{(n)} \varrho[\lambda(T_j e_i - e_i)] = 0 \quad \text{uniformly in } n.$$

For each $i = 0, 1, \ldots, m$ and $s \in I$ choose $N > 0$ and $\gamma > 0$ such that $|a_i(s)| \leqslant N$ and $\max\{3\gamma M, 6\gamma M\eta^{-1}(m+1)N\} \leqslant \lambda$. Consider condition (K1), for each $j \in \mathbb{N}$ and $i = 0, 1, \ldots, m$ we get

$$\begin{aligned} J_3 = \varrho\Big[6\gamma\frac{M}{\eta}(T_j P_{(\cdot)})(\cdot)\Big] &= \varrho\Big[6\gamma\frac{M}{\eta}(T_j P_{(\cdot)})(\cdot) - P_{(\cdot)}(\cdot)\Big] \\ &\leqslant \sum_{i=0}^{m} \varrho\Big[6\gamma\frac{M}{\eta}(m+1)N(T_j e_i - e_i)\Big] \leqslant \sum_{i=0}^{m} \varrho[\lambda(T_j e_i - e_i)]. \end{aligned}$$

Hence we obtain

$$\lim_{k\to\infty}\sum_j a_{kj}^{(n)} J_3 = 0 \quad \text{uniformly in } n.$$

Moreover, from choosing λ and γ it is clear that $\lim\limits_{k\to\infty}\sum\limits_j a_{kj}^{(n)} J_2 = 0$. Since ϱ is Q-quasi semiconvex and $0 < \varepsilon \leqslant 1$, we have

$$\varrho[3\gamma\varepsilon e_0] \leqslant Q\varepsilon\varrho[3\gamma Q e_0]. \tag{4}$$

If condition $(*)$ is considered in (3) and (4), we get uniformly in n

$$\begin{aligned} 0 &\leqslant \limsup_k \sum_j a_{kj}^{(n)} \varrho[\gamma(T_j f - f)] \leqslant \limsup_k \sum_j a_{kj}^{(n)} \varrho[3\gamma\varepsilon(T_j e_0)] \\ &\leqslant N\varrho[3\gamma\varepsilon e_0] \leqslant NQ\varepsilon\varrho[3\gamma Q e_0]. \end{aligned} \tag{5}$$

Since ε is arbitrary positive real number and ϱ is strongly finite using (5), we have

$$\limsup_k \sum_j a_{kj}^{(n)} \varrho[\gamma(T_j f - f)] = 0 \quad \text{uniformly in } n$$

and hence

$$\lim_k \sum_j a_{kj}^{(n)} \varrho[\gamma(T_j f - f)] = 0 \quad \text{uniformly in } n.$$

This means that $\{T_j f\}$ is modularly convergent to f in the sense of $\mathscr{A}$-summability on $L^{\varrho}(I)$. The second part can be proved similarly to the first one. $\square$

The next theorem is similar to Theorem 2.1 of [15] (see also [4]) under weaker condition by using different test functions.

Theorem 3. *Let $\mathscr{A} = \{A^{(n)}\}$ be a sequence of infinite nonnegative real matrices and let ϱ be a strongly finite, monotone, absolutely continuous and Q-quasi semiconvex modular on $X(I)$. Let T_j, $j \in \mathbb{N}$ be a sequence of positive linear operators satisfying condition $(*)$. If*

$$\lim_{k\to\infty}\sum_{j=1}^{\infty} a_{kj}^{(n)} \varrho[\lambda(T_j e_i - e_i)] = 0 \quad \textit{uniformly in } n$$

for every $\lambda > 0$ and $i = 0, 1, \ldots, m$, then for every $f \in L^{\varrho}(I) \cap D$ with $f - C_{\mathrm{b}}(I) \subset X_T$,

$$\lim_{k\to\infty}\sum_{j=1}^{\infty} a_{kj}^{(n)} \varrho[\gamma(T_j f - f)] = 0 \quad \textit{uniformly in } n$$

for some $\gamma > 0$, where X_T and D are as before.

P r o o f. Let $f \in L^{\varrho}(I) \cap D$ such that $f - C_{\mathrm{b}}(I) \subset X_T$. From Proposition 3.2 of [4] there exist $\lambda > 0$ and a sequence (f_m) in $C_{\mathrm{c}}(I)$ such that $\varrho[3\lambda f] < \infty$ and $\lim_m \varrho[3\lambda(f_m - f)] = 0$. Take arbitrary fixed $\varepsilon > 0$ and choose a positive integer $\overline{m}$ such that

$$\varrho[3\lambda(f_{\overline{m}} - f)] \leqslant \varepsilon. \tag{6}$$

For each $j \in \mathbb{N}$ we have

$$\varrho[\lambda(T_j f - f)] \leqslant \varrho[3\lambda(T_j f - T_j f_{\overline{m}})] + \varrho[3\lambda(T_j f_{\overline{m}} - f_{\overline{m}})] + \varrho[3\lambda(f_{\overline{m}} - f)]. \tag{7}$$

Using a similar technique as in the previous theorem, we obtain

$$\begin{aligned} 0 &= \lim_k \sum_j a_{kj}^{(n)} \varrho[3\lambda(T_j f_{\overline{m}} - f_{\overline{m}})] \\ &= \limsup_k \sum_j a_{kj}^{(n)} \varrho[3\lambda(T_j f_{\overline{m}} - f_{\overline{m}})] \quad \text{uniformly in } n. \end{aligned} \tag{8}$$

From condition $(*)$ there exists $R > 0$ such that

$$\lim_k \sum_j a_{kj}^{(n)} \varrho[3\lambda(T_j f - T_j f_{\overline{m}})] \leqslant R\varrho[3\lambda(f - f_{\overline{m}})] \leqslant R\varepsilon \quad \text{uniformly in } n. \tag{9}$$

From (6)–(9) and the subadditivity of the operator lim sup we have

$$0 \leqslant \limsup_k \sum_j a_{kj}^{(n)} \varrho[\lambda(T_j f - f)] \leqslant \varepsilon(R+1) \quad \text{uniformly in } n. \tag{10}$$

From (10) and the arbitrariness of ε we get for $\gamma = \lambda$

$$\limsup_k \sum_j a_{kj}^{(n)} \varrho[\lambda(T_j f - f)] = 0 \quad \text{uniformly in } n.$$

This implies $\lim_k \sum_j a_{kj}^{(n)} \varrho[\lambda(T_j f - f)] = 0$ uniformly in n. □

3. Concluding remarks and examples

In this section we give some remarks and an example to show that our theorems are generalizations of known theorems. We remark that if $A^{(n)}$ equals to identity matrix for every $n \in \mathbb{N}$, then $\mathscr{A}$-summability reduces to the ordinary convergence. In this case our Theorem 3 is similar to Theorem 3.2 of [8].

Take $I = [0,1]$ and let $\varphi\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a convex continuous function with $\varphi(0) = 0$, $\varphi(u) > 0$ for $u > 0$, $\varphi(u) \to \infty$ as $u \to \infty$. Then it is easily shown that

$$\varrho[f] = U_\varphi[f] = \int_I \varphi(|f(t)|)\,\mathrm{d}\mu(t)$$

is a convex modular on the space $X(I)$. U_φ is known as an Orlicz modular in $X(I)$. The respective modular space $L_\varphi^\varrho(I)$ is called the Orlicz space. Now let us consider the following linear positive operator on the space $L_\varphi^\varrho(I)$ which is defined as

$$B_j(f;x) := s_j \sum_{r=0}^{j} \binom{j}{r} x^r (1-x)^{j-r} (j+1) \int_{r/(j+1)}^{(r+1)/(j+1)} f(t)\,\mathrm{d}t \quad \text{for } x \in I, \tag{11}$$

where $\{s_j\}$ is a sequence of zeros and ones which is $\mathscr{A}$-summable to 1, but not ordinary convergent. Also we assume that $\mathscr{A}$ is a regular method of matrices. Observe that the operators B_j map the Orlicz space L_φ^ϱ into itself. By Lemma 5.1 of [8], for every $h \in X_B := L_\varphi^\varrho$, all $\lambda > 0$ and for a positive constant N we get

$$U_\varphi[\lambda B_j h] \leqslant s_j N U_\varphi[\lambda h].$$

Then we have

$$\lim_k \sum_j a_{kj}^{(n)} U_\varphi[\lambda B_j h] \leqslant N U_\varphi[\lambda h].$$

It is easily seen that

$$\begin{aligned}
B_j(e_0;x) &= s_j,\\
B_j(e_1;x) &= s_j\Big(\frac{jx}{j+1} + \frac{1}{2(j+1)}\Big),\\
B_j(e_2;x) &= s_j\Big(\frac{j(j-1)x^2}{(j+1)^2} + \frac{2jx}{(j+1)^2} + \frac{1}{3(j+1)^2}\Big),
\end{aligned}$$

where $e_i(t) = t^i$, $i = 0,1,2$. Therefore we can observe for any $\lambda > 0$ that

$$\lambda|B_j(e_0;x) - e_0(x)| = \lambda(1-s_j)$$

which implies

$$U_\varphi[\lambda(B_j e_0 - e_0)] = U_\varphi[\lambda(1-s_j)] = \int_0^1 \varphi[\lambda(1-s_j)]\,\mathrm{d}x$$
$$= \varphi[\lambda(1-s_j)] = (1-s_j)\varphi(\lambda)$$

because of the definition of $\{s_j\}$. Now we get for any $\lambda > 0$

$$\lim_k \sum_j a_{kj}^{(n)} U_\varphi[\lambda(B_j e_0 - e_0)] = 0 \quad \text{uniformly in } n.$$

Also since

$$\lambda|B_j(e_1;x) - e_1(x)| \leqslant \lambda\Big\{(1-s_j) + \frac{3s_j}{2(j+1)}\Big\}$$

by the definition of $\{s_j\}$ and U_φ, we may write that

$$U_\varphi[\lambda(B_j e_1 - e_1)] \leqslant U_\varphi\Big[\lambda\Big\{(1-s_j) + \frac{3s_j}{2(j+1)}\Big\}\Big] \leqslant U_\varphi[2\lambda(1-s_j)] + U_\varphi\Big[\frac{3\lambda s_j}{j+1}\Big]$$
$$= \varphi[2\lambda(1-s_j)] + \varphi\Big[\frac{3\lambda s_j}{j+1}\Big],$$

which implies for any $\lambda > 0$ that

$$U_\varphi[\lambda(B_j e_1 - e_1)] \leqslant (1-s_j)\varphi[2\lambda] + s_j\varphi\Big[\frac{3\lambda}{j+1}\Big].$$

Since φ is continuous, we have $\lim\limits_j \varphi[3\lambda/(j+1)] = \varphi\big[\lim\limits_j 3\lambda/(j+1)\big] = \varphi(0) = 0$. Therefore we have for every $\lambda > 0$

$$\lim_k \sum_j a_{kj}^{(n)} U_\varphi[\lambda(B_j e_1 - e_1)] = 0 \quad \text{uniformly in } n.$$

Finally, since

$$\lambda|B_j(e_2;x) - e_2(x)| \leqslant \lambda\Big\{(1-s_j) + s_j\frac{15j+4}{3(j+1)^2}\Big\},$$

we get

$$U_\varphi[\lambda(B_j e_2 - e_2)] \leqslant U_\varphi[2\lambda(1-s_j)] + U_\varphi\Big[\lambda s_j\frac{30j+8}{3(j+1)^2}\Big]$$
$$= \varphi[2\lambda(1-s_j)] + \varphi\Big[\lambda s_j\frac{30j+8}{3(j+1)^2}\Big],$$

which yields

$$U_\varphi[\lambda(B_j e_2 - e_2)] \leqslant (1-s_j)\varphi[2\lambda] + s_j\varphi\Big[\lambda\frac{30j+8}{3(j+1)^2}\Big]. \tag{12}$$

Considering the continuity of φ, it follows from (12) for any $\lambda > 0$ that

$$\lim_k \sum_j a_{kj}^{(n)} U_\varphi[\lambda(B_j e_2 - e_2)] = 0 \quad \text{uniformly in } n.$$

The sequence of operators $\{B_j\}$ defined by (11) satisfies all conditions of Theorem 3. So we conclude that

$$\lim_k \sum_j a_{kj}^{(n)} U_\varphi[\lambda_0(B_j(f) - f)] = 0 \quad \text{uniformly in } n$$

holds for $\lambda_0 > 0$ and every $f \in L_\varphi^\varrho(I)$. However, since $\{s_j\}$ is not convergent to zero, it is clear that $\{B_j\}$ is not modularly convergent to f.

Also remark that if we assume $I = [0,1]$, $e_0(t) = 1$, $e_1(t) = t$, $e_2(t) = t^2$, $a_0(s) = s^2$, $a_1(s) = -2s$, $a_2(s) = 1$, $s,t \in I$ in equation (1), $A^{(n)} = I$ for every $n \in \mathbb{N}$ and that ϱ is a sup-norm on $C(I)$ which is the set of all continuous functions on I in Theorem 3, the classical Korovkin theorem is obtained.

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