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A NOTE ON POISSON DERIVATIONS

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Abstract. Free Poisson algebras are very closely connected with polynomial algebras, and the Poisson brackets are used to solve many problems in affine algebraic geometry. In this note, we study Poisson derivations on the symplectic Poisson algebra, and give a connection between the Jacobian conjecture with derivations on the symplectic Poisson algebra.

Keywords: Poisson algebra; derivation; Jacobian conjecture

MSC 2010: 13N15, 17B63

1. Introduction

Free Poisson algebras are very closely connected with polynomial algebras, free associative algebras, and free Lie algebras. Free Poisson algebras and Poisson brackets have been recently used to solve problems in affine algebraic geometry.

To solve Nagata's conjecture, Shestakov and Umirbaev in [6], [7] constructed a theory for deciding wildness of polynomial automorphisms in three variables. The approach they use is different from the traditional ones. The novelty consists in the imbedding of the polynomial ring into the free Poisson algebra (or the algebra of universal Poisson brackets) on the same set of generators and in the systematical use of Poisson brackets as an additional tool.

Free Poisson algebras and their derivations and automorphisms have attracted the interest of many mathematicians. Makar-Limanov and Umirbaev in [3], [4] proved an analog of the Bergman Centralizer theorem and an analog of the Freiheitssatz theorem in the free Poisson algebra. Makar-Limanov and Umirbaev in [2] expanded the Rentschler theorem to the free Poisson algebra in two variables and proved

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that the automorphisms of the free Poisson algebra in two variables are tame, and the locally nilpotent derivations are triangulable. Makar-Limanov and Shestakov in [1] proved that any two Poisson dependent elements in a free Poisson algebra are algebraically dependent, and applied this result to give a new proof of the tameness of automorphisms for the free Poisson algebras of rank two. Umirbaev in [8] studied the universal enveloping algebras and universal derivations of Poisson algebras, and proved that the Jacobian conjecture is equivalent to the Poisson conjecture in the Poisson algebra.

In this note, we show some connections between derivations on polynomial algebras and derivations on symplectic Poisson algebras. We also show an equivalent description of the Jacobian conjecture in the symplectic Poisson algebra $k\{x, y\}$.

2. Main results

A vector space P over a field k is said to be a Poisson algebra if B is endowed with two bilinear operations: $x \cdot y$ (a multiplication) and $\{x, y\}$ (a Poisson bracket), and satisfies the following statements:

(1) B is a commutative associative algebra under the multiplication;
(2) B is a Lie algebra under the Poisson bracket;
(3) $\{x \cdot y, z\} = x \cdot \{y, z\} + \{x, z\} \cdot y$ for $x, y, z \in B$.

Let B be a Poisson algebra and D a linear map on B. If $D(xy) = D(x)y + xD(y)$ and $D(\{x, y\}) = \{D(x), y\} + \{x, D(y)\}$ for $x, y \in B$, then D is said to be a Poisson derivation of B.

If $P = k\{x_1, \ldots, x_n\}$ is a free Poisson algebra with free generators $x_1, \ldots, x_n$ or $P = k[x_1, x_2, \ldots, x_n]$ is a polynomial algebra over k, then every derivation of P can be uniquely written as

$$D = \sum_{i=1}^{n} D(x_i) \frac{\partial}{\partial x_i}.$$

Recall that by the divergence D^* of D we mean

$$D^* = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} D(x_i).$$

There are many important classes of polynomial Poisson algebras, one of which is the symplectic algebra S_n. S_n is a polynomial algebra $k[x_1, y_1, \ldots, x_n, y_n]$ endowed with the Poisson bracket defined by

$$\{x_i, y_j\} = \delta_{ij}, \quad \{x_i, x_j\} = 0, \quad \{y_i, y_j\} = 0,$$

where δ_{ij} is the Kronecker symbol and $1 \leqslant i, j \leqslant n$.

It is natural to identify the polynomial algebra $k[x_1, x_2, \dots, x_n]$ with the subspace of the free Poisson algebra $k\{x_1, x_2, \dots, x_n\}$ generated by the elements

$$x_1^{r_1} x_2^{r_2} \dots x_n^{r_n}, \quad r_i \geqslant 0.$$

If $f, g \in k[x_1, x_2, \dots, x_n]$, then it follows immediately from the definition of the Poisson bracket that

$$\{f, g\} = \sum_{1 \leqslant i < j \leqslant n} \Big(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_i} \Big) \{x_i, x_j\}.$$

In particular, the Poisson bracket of two elements of S_n is

$$\{f, g\} = \sum_{1 \leqslant i < j \leqslant n} \Big(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial x_i} \Big).$$

Theorem 2.1. *If D is a derivation of the symplectic Poisson algebra $S_n = k[x_1, y_1, \dots, x_n, y_n]$, then $D^* = 0$ and there exists $f \in S_n$ such that $D = \mathrm{ad}_f$, that is, $D(g) = \{f, g\}$ for any $g \in S_n$.*

Proof. Since $0 = D(\{x_i, y_j\}) = \{D(x_i), y_j\} + \{x_i, D(y_j)\}$, we have

$$\frac{\partial}{\partial x_j} D(x_i) = -\frac{\partial}{\partial y_i} D(y_j).$$

Similarly, it follows from $D(\{x_i, x_j\}) = D(\{y_i, x_j\}) = 0$ that

$$\frac{\partial}{\partial y_j} D(x_i) = \frac{\partial}{\partial y_i} D(x_j), \qquad \frac{\partial}{\partial x_j} D(y_i) = \frac{\partial}{\partial x_i} D(y_j).$$

By Euler's lemma (see [5], Lemma 2.5.3) there exists $f \in S_n$ such that

$$D(x_i) = -\frac{\partial f}{\partial y_i}, \qquad D(y_j) = \frac{\partial f}{\partial x_j}.$$

Thus,

$$D = -\frac{\partial f}{\partial y_1} \frac{\partial}{\partial x_1} + \frac{\partial f}{\partial x_1} \frac{\partial}{\partial y_1} - \dots - \frac{\partial f}{\partial y_n} \frac{\partial}{\partial x_n} + \frac{\partial f}{\partial x_n} \frac{\partial}{\partial y_n} = \mathrm{ad}_f.$$

Hence $D^* = -\frac{\partial^2 f}{\partial x_1 \partial y_1} + \frac{\partial^2 f}{\partial x_1 \partial y_1} - \dots - \frac{\partial^2 f}{\partial x_n \partial y_n} + \frac{\partial^2 f}{\partial x_n \partial y_n} = 0.$ $\square$

Corollary 2.2. *Let $P = k\{x, y\}$ be the symplectic Poisson algebra S_2, then D is a derivation of P if and only if D is a divergence zero derivation of the polynomial algebra $k[x, y]$.*

P r o o f. The sufficiency immediately follows from Theorem 2.1. Conversely, if $D = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$ is a derivation of the polynomial algebra $k[x,y]$ and $D^* = 0$, then $P_x + Q_y = 0$. Thus, there exists $f \in k[x,y]$ such that for all $g \in k[x,y]$, we have $D(g) = \{f,g\}$, and D is a derivation of P. □

Proposition 2.3. *The following statements are equivalent.*

(1) *The Jacobian conjecture is true for the polynomial algebra $k[x,y]$.*

(2) *If D is a derivation of the symplectic Poisson algebra $k\{x,y\}$ with a slice, that is, there exists $s \in k\{x,y\}$ such that $D(s) = 1$, then D is locally nilpotent.*

P r o o f. (1)⇒(2) Let D be a derivation of $k\{x,y\}$ with $s \in k\{x,y\}$ such that $D(s) = 1$. Then by Theorem 2.1, there exists $f \in k\{x,y\}$ such that $D(g) = \{f,g\} = \det J(f,g)$ for any $g \in k\{x,y\}$. Since $D(s) = 1$, we have $\{f,s\} = \det J(f,s) = 1$. Since the Jacobian conjecture is true for $k[x,y]$, we have $k[f,s] = k[x,y]$. Consequently, D is actually a partial derivation of $k[x,y]$, thus D is a locally nilpotent derivation.

(2)⇒(1) Let $F = (f,g)$ be a polynomial endomorphism with $\det JF = 1$. Then $\frac{\partial}{\partial f} = g_y\frac{\partial}{\partial x} - g_x\frac{\partial}{\partial y}$ is a derivation of $k\{x,y\}$ with divergence zero and $\frac{\partial}{\partial f} f = 1$. So by hypothesis, $\frac{\partial}{\partial f}$ is locally nilpotent. Similarly, $\frac{\partial}{\partial g}$ is locally nilpotent. Then it follows from [9], Proposition 2.2.10, that F is a polynomial automorphism. □

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