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**BEST PROXIMITY POINT FOR PROXIMAL BERINDE  
NONEXPANSIVE MAPPINGS ON STARSHAPED SETS**

NUTTAWUT BUNLUE AND SUTHEP SUANTAI

ABSTRACT. In this paper, we introduce the new concept of proximal mapping, namely proximal weak contractions and proximal Berinde nonexpansive mappings. We prove the existence of best proximity points for proximal weak contractions in metric spaces, and for proximal Berinde nonexpansive mappings on starshaped sets in Banach spaces. Examples supporting our main results are also given. Our main results extend and generalize some of well-known best proximity point theorems of proximal nonexpansive mappings in the literatures.

## 1. INTRODUCTION

Fixed point theory plays an important role in solving nonlinear equations arising in different areas such as difference and differential equations, discrete and continuous dynamic systems, variational analysis, physics, engineering and economics.

These problems can be modeled as fixed point equation of the form  $x = Tx$  where  $T: A \rightarrow X$  is a nonlinear mapping from a subset  $A$  of  $X$ . In the case that  $A \cap T(A) = \emptyset$ , the fixed point equation  $x = Tx$  has no solution because  $d(x, Tx) > 0$  for all  $x \in A$ . Under this circumstance, it is of interest to determine an approximate solution  $x$  such that the distance between  $x$  and  $Tx$  is minimum. For more precisely, suppose  $T: A \rightarrow B$  where  $A, B$  are subsets of a metric space  $(X, d)$ . It is noted that  $d(x, Tx) \geq D(A, B)$ , where  $D(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$ . It is natural to ask the question of finding  $x$  such that  $d(x, Tx) = D(A, B)$ , such point  $x$  is known as a best proximity point of  $T$ . It is clear that if  $T$  is a self-mapping, a best proximity point is a fixed point, that is,  $x = Tx$ .

Existence of best proximity point of nonself-mappings have been studied by many authors, see [2, 5, 6, 13, 15, 17, 18, 19, 21] and [22]. Best proximity point theorems can be applied to study equilibrium point in economics, see [10]–[12], so this topic attracts attentions of many mathematicians.

Basha [1] introduced a new concept of proximal contraction which can be reduced to a contraction in the case of self-mappings.

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**Definition 1** ([1]). Let  $(A, B)$  be a pair of nonempty subsets of a metric space  $(X, d)$ . A mapping  $T: A \rightarrow B$  is said to be a proximal contraction if there exists constant number  $\alpha \in (0, 1)$  such that

$$\left. \begin{aligned} d(u_1, Tx_1) = D(A, B) \\ d(u_2, Tx_2) = D(A, B) \end{aligned} \right\} \implies d(u_1, u_2) \leq \alpha d(x_1, x_2)$$

for all  $x_1, x_2, u_1, u_2 \in A$ .

In 2013, Gabeleh [7] introduced a new concept of proximal nonexpansive mappings and proved existence of best proximity point of such mapping when  $(A, B)$  is a pair of nonempty closed convex subsets of  $X$  and  $A$  is a compact set.

**Definition 2** ([7]). Let  $(A, B)$  be a pair of nonempty subsets of a metric space  $(X, d)$ . A mapping  $T: A \rightarrow B$  is said to be a proximal nonexpansive if

$$\left. \begin{aligned} d(u_1, Tx_1) = D(A, B) \\ d(u_2, Tx_2) = D(A, B) \end{aligned} \right\} \implies d(u_1, u_2) \leq d(x_1, x_2)$$

for all  $x_1, x_2, u_1, u_2 \in A$ .

Motivated by weak contraction of Berinde [3] and Suzuki [20], in 2014, Gabeleh [9] introduced a new classes of proximal contractions which is called Berinde weak proximal contraction.

**Definition 3** ([9]). Let  $(A, B)$  be a pair of nonempty closed subsets of a metric space  $(X, d)$ . A mapping  $T: A \rightarrow B$  is said to be a Berinde weak proximal contraction if there exist  $\alpha \in [0, 1)$  and  $\beta \in [0, \infty)$  such that for all  $x, y, u, v \in A$  with  $d(u, Tx) = D(A, B) = d(v, Ty)$ , we have

$$\frac{1}{1 + \alpha + \beta} d^*(x, Tx) \leq d(x, y) \implies d(u, v) \leq \alpha d(x, y) + \beta d^*(Tx, y),$$

where  $d^*(x, Tx) = d(x, Tx) - D(A, B)$ .

Chen [4] proved an interesting existence theorem of proximity points for proximal nonexpansive mappings under starshaped sets  $A$  and  $B$ .

**Theorem 1.1** ([4]). *Let  $(A, B)$  be a pair of nonempty, closed subsets of a Banach space  $X$  such that  $A$  is a  $p$ -starshaped set,  $B$  is a  $q$ -starshaped set, and  $\|p - q\| = D(A, B)$ . Suppose  $A$  is compact,  $(B, A)$  is a semi-sharp proximal pair. Assume that  $T: A \rightarrow B$  satisfies the following conditions:*

- (1)  $T$  is a proximal nonexpansive,
- (2)  $T(A_0) \subseteq B_0$ .

*Then there exists an elements  $x^*$  in  $A_0$  such that*

$$\|x^* - Tx^*\| = D(A, B).$$

Motivated by above results, we aim to introduce new concept of generalized proximal contraction and proximal nonexpansive mapping, called proximal weak contraction and proximal Berinde nonexpansive, respectively, and prove existence of best proximity point of such mappings under certain conditions. We also give an example supporting our main results.

## 2. PRELIMINARIES

Let  $(A, B)$  be a pair of nonempty subsets of metric space  $(X, d)$ . We denote by  $A_0$  and  $B_0$  the following sets:

$$A_0 = \{x \in A : d(x, y) = D(A, B) \text{ for some } y \in B\},$$

$$B_0 = \{y \in B : d(x, y) = D(A, B) \text{ for some } x \in A\},$$

where

$$D(A, B) = \inf\{d(x, y) : x \in A, y \in B\}.$$

A nonempty subset  $A$  of a linear space  $X$  is called a  $p$ -starshape set if there exist a point  $p$  in  $A$  such that  $\alpha p + (1 - \alpha)x \in A$ , for all  $x \in A$ ,  $\alpha \in [0, 1]$ , and  $p$  is called a center of  $A$ . It is easy to see that each convex set  $C$  is a  $p$ -starshaped set for each  $p \in C$ .

Notice that in a normed space  $(X, \|\cdot\|)$ , if both of  $A$  and  $B$  are closed and  $A_0$  is nonempty, then  $A_0$  is a closed set. Consider on starshape set, if  $A$  is a  $p$ -starshape set,  $B$  is a  $q$ -starshaped set and  $\|p - q\| = D(A, B)$ , implies that  $A_0$  is a  $p$ -starshape set and  $B_0$  is a  $q$ -starshaped set.

**Definition 4** ([14]). Let  $(A, B)$  be a pair of nonempty subsets of a metric space  $(X, d)$ . The pair  $(A, B)$  is said to be a semi-sharp proximal pair if for each  $x$  in  $A$  (respectively, in  $B$ ) there exists at most one  $x^*$  in  $B$  (respectively, in  $A$ ) such that  $d(x, x^*) = D(A, B)$ .

**Definition 5** ([8]). Let  $(A, B)$  be a pair of nonempty subsets of a metric space  $(X, d)$  with  $A \neq \emptyset$ . Then the pair  $(A, B)$  is said to have the weak  $P$ -property if for all  $x_1, x_2 \in A_0$  and  $y_1, y_2 \in B_0$ ,

$$\left. \begin{array}{l} d(x_1, y_1) = D(A, B) \\ d(x_2, y_2) = D(A, B) \end{array} \right\} \implies d(x_1, x_2) \leq d(y_1, y_2).$$

In Definition 5, if  $d(x_1, x_2) = d(y_1, y_2)$ , we said  $(A, B)$  have  $P$ -property [16]. It is clear that the weak  $P$ -property is weaker than the  $P$ -property and  $(A, B)$  has the  $P$ -property if and only if both  $(A, B)$  and  $(B, A)$  have the weak  $P$ -property. Moreover, if a pair  $(A, B)$  has the weak  $P$ -property then  $(B, A)$  must be a semi-sharp proximal pair. Obviously a semi-sharp proximal pair  $(A, B)$  is not necessarily to have the weak  $P$ -property.

## 3. MAIN RESULTS

**3.1. Proximity point for the proximal weak contraction.** We begin this section by giving definition and proving a theorem on the existence of best proximity points for proximal weak contraction in metric spaces.

**Definition 6.** Let  $(A, B)$  be a pair of nonempty subset of a metric space  $(X, d)$ . A mapping  $T: A \rightarrow B$  is said to be *proximal weak contraction* if there exist  $\alpha \in (0, 1)$  and  $L \geq 0$  such that

$$\left. \begin{array}{l} d(u_1, Tx_1) = D(A, B) \\ d(u_2, Tx_2) = D(A, B) \end{array} \right\} \implies d(u_1, u_2) \leq \alpha d(x_1, x_2) + Ld(x_2, u_1)$$

for all  $x_1, x_2, u_1, u_2 \in A$ .

For self-mapping, we see that the proximal weak contraction reduces to the weak contraction mapping introduced by Berinde in [3].

**Theorem 3.1.** *Let  $(X, d)$  be a complete metric space and  $(A, B)$  a pair of nonempty subsets of  $X$  such that  $A_0$  is nonempty and closed. Suppose that  $T: A \rightarrow B$  is a proximal weak contraction and  $T(A_0) \subseteq B_0$ . Then*

(1) *there exists  $x^* \in A$  such that  $d(x^*, Tx^*) = D(A, B)$ , and if  $1 - \alpha - L > 0$ , then  $x^*$  is unique,*

(2) *the sequence  $\{x_n\}$ , defined by  $x_0, x_1 \in A_0$  and*

$$d(x_{n+1}, Tx_n) = D(A, B), \quad \text{for all } n \in \mathbb{N},$$

*converges to  $x^*$ .*

**Proof.** Let  $x_0 \in A_0$ . Then there exist  $x_1 \in A_0$  such that

$$d(x_1, Tx_0) = D(A, B)$$

because  $Tx_0 \in T(A_0) \subseteq B_0$ . Continuing this process, we get a sequence  $\{x_n\}$  in  $A_0$  such that

$$d(x_{n+1}, Tx_n) = D(A, B), \quad \text{for all } n \in \mathbb{N}.$$

Next, we show that  $\{x_n\}$  is a Cauchy sequence and its limit is a best proximity point of  $T$ . From

$$d(x_n, Tx_{n-1}) = D(A, B) \quad \text{and} \quad d(x_{n+1}, Tx_n) = D(A, B), \quad \text{for all } n \in \mathbb{N},$$

by proximal weak contractiveness of  $T$ , we have

$$\begin{aligned} d(x_{n+1}, x_n) &\leq \alpha d(x_n, x_{n-1}) + Ld(x_n, x_n) \\ &\leq \alpha d(x_n, x_{n-1}). \end{aligned}$$

Therefore, for each  $p \in \mathbb{N}$ ,

$$\begin{aligned} d(x_{n+p}, x_n) &\leq d(x_{n+p}, x_{n+p-1}) + d(x_{n+p-1}, x_{n+p-2}) + \cdots + d(x_{n+1}, x_n) \\ &\leq \alpha^{n+p-1}d(x_1, x_0) + \alpha^{n+p-2}d(x_1, x_0) + \cdots + \alpha^n d(x_1, x_0) \\ &\leq \frac{\alpha^n}{1 - \alpha}d(x_1, x_0). \end{aligned}$$

This implies that  $\{x_n\}$  is a Cauchy sequence in  $A$ . Since  $X$  complete and  $A_0$  closed, there exists  $x^* \in A_0$  such that  $x_n \rightarrow x^*$ . By the assumption  $T(A_0) \subseteq B_0$  again, there exists  $u \in A_0$  such that

$$d(u, Tx^*) = D(A, B).$$

Since  $d(x_{n+1}, Tx_n) = D(A, B)$  for all  $n \in \mathbb{N}$ , by proximal weak contractiveness of  $T$ , we have

$$d(x_{n+1}, u) \leq \alpha d(x_n, x^*) + Ld(x^*, x_{n+1}).$$

It implies that  $d(x_{n+1}, u) \rightarrow 0$ . Therefore  $x_n \rightarrow u$  and hence  $u = x^*$ . That is

$$d(x^*, Tx^*) = D(A, B).$$

Finally, we show that if  $1 - \alpha - L > 0$  then the best proximity point of  $T$  is unique. Suppose there exists  $x^{**} \in A_0$  such that

$$d(x^{**}, Tx^{**}) = D(A, B).$$

Since  $T$  is proximal weak contraction, we get

$$d(x^*, x^{**}) \leq \alpha d(x^*, x^{**}) + Ld(x^*, x^{**}),$$

which implies

$$(1 - \alpha - L)d(x^*, x^{**}) \leq 0.$$

Hence  $x^*$  and  $x^{**}$  are same point. □

An immediate consequence of Theorem 3.1 is the following.

**Corollary 3.2** ([4]). *Let  $(X, d)$  be a complete metric space and  $(A, B)$  a pair of nonempty closed subsets of  $X$  such that  $A_0 \neq \emptyset$ . Suppose that  $T: A \rightarrow B$  satisfies the following conditions:*

- (1)  $T$  is a proximal contraction,
- (2)  $T(A_0) \subseteq B_0$ .

*Then there exists a unique  $x^* \in A$  such that  $d(x^*, Tx^*) = D(A, B)$ .*

**Example 3.3.** Let  $X = \mathbb{R}^2$  with the usual metric,

$$A_1 = \{(x, 0) : x \in [0, 1]\},$$

$$A_2 = \{(1, y) : y \in [-1, 0]\},$$

$$B_1 = \{(x, y) : x^2 + y^2 = 1, x \in [-1, 0]\},$$

$$B_2 = \{(x, 1) : x \in (0, 1]\},$$

$$A = A_1 \cup A_2,$$

$$B = B_1 \cup B_2,$$

and let  $T: A \rightarrow B$  be given by

$$T(x, y) = \begin{cases} (\frac{x^2}{2}, 1), & \text{if } (x, y) \in A_1, \\ (\frac{y}{2}, -\sqrt{1 - \frac{y^2}{4}}), & \text{if } (x, y) \in A_2. \end{cases}$$

To show that  $T$  is proximal weak contraction, let  $x_1 = (x'_1, y'_1), x_2 = (x'_2, y'_2) \in A$  and  $u_1, u_2 \in A$  be such that  $d(u_1, Tx_1) = d(u_2, Tx_2) = D(A, B)$ . We will consider the following 5 cases.

*Case 1.*  $x_1, x_2 \in A_1$ . Then  $u_1 = (\frac{x'^2_1}{2}, 0), u_2 = (\frac{x'^2_2}{2}, 0)$ . This implies

$$d(u_1, u_2) = d\left(\left(\frac{x'^2_1}{2}, 0\right), \left(\frac{x'^2_2}{2}, 0\right)\right) \leq \frac{1}{2}d(x_1, x_2) \leq \frac{1}{2}d(x_1, x_2) + Ld(x_2, u_1)$$

for any  $L \geq 0$ .

*Case 2.*  $x_1, x_2 \in A_2 - \{(1, 0)\}$ . Then  $u_1 = u_2 = (0, 0)$ . This implies

$$d(u_1, u_2) \leq \frac{1}{2}d(x_1, x_2) \leq \frac{1}{2}d(x_1, x_2) + Ld(x_2, u_1)$$

for any  $L \geq 0$ .

*Case 3.* Let  $x_1 \in A_1$ ,  $x_2 \in A_2$ . Then  $u_1 = (\frac{x_1^2}{2}, 0)$ ,  $u_2 \in \{(0, 0), (1, -1)\}$ . If  $x_1' \in [0, \frac{1}{2}]$  and  $u_2 = (0, 0)$ , we have

$$d(u_1, u_2) \leq \frac{1}{8} \leq \frac{1}{2}d(x_1, x_2) \leq \frac{1}{2}d(x_1, x_2) + Ld(x_2, u_1)$$

for any  $L \geq 0$ .

If  $x_1' \in [0, \frac{1}{2}]$  and  $u_2 = (1, -1)$ , then  $x_2 = (1, 0)$ . This implies

$$d(u_1, u_2) \leq \sqrt{2}, \quad d(x_1, x_2) \geq \frac{1}{2} \quad \text{and} \quad d(x_2, u_1) \geq \frac{7}{8}.$$

Thus

$$d(u_1, u_2) \leq \sqrt{2} \leq \frac{1}{2}d(x_1, x_2) + Ld(x_2, u_1)$$

where  $L \geq \frac{8}{7}(\sqrt{2} - \frac{1}{2})$ .

If  $x_1' \in (\frac{1}{2}, 1)$  and  $u_2 = (0, 0)$ , we get

$$d(u_1, u_2) \leq \frac{1}{2}, \quad d(x_1, x_2) \geq 0 \quad \text{and} \quad d(x_2, u_1) \geq \frac{1}{2}.$$

Thus

$$d(u_1, u_2) \leq \frac{1}{2} \leq \frac{1}{2}d(x_1, x_2) + Ld(x_2, u_1)$$

where  $L \geq 1$ .

If  $x_1' \in (\frac{1}{2}, 1)$  and  $u_2 = (1, -1)$ , then  $x_2 = (1, 0)$ . This implies

$$d(u_1, u_2) \leq \frac{\sqrt{113}}{8}, \quad d(x_1, x_2) \geq 0 \quad \text{and} \quad d(x_2, u_1) \geq \frac{1}{2}.$$

Thus

$$d(u_1, u_2) \leq \frac{\sqrt{113}}{8} \leq \frac{1}{2}d(x_1, x_2) + Ld(x_2, u_1)$$

where  $L \geq \frac{\sqrt{113}}{4}$ .

*Case 4.*  $x_1 \in A_2$ ,  $x_2 \in A_1$ . Then  $u_1 \in \{(0, 0), (1, -1)\}$ ,  $u_2 = (\frac{x_2^2}{2}, 0)$ . If  $x_2' \in [0, \frac{1}{2}]$  and  $u_1 = (0, 0)$ , then

$$d(u_1, u_2) \leq \frac{1}{8} \quad \text{and} \quad d(x_1, x_2) \geq \frac{1}{2}.$$

Thus

$$d(u_1, u_2) \leq \frac{1}{2}d(x_1, x_2) \leq \frac{1}{2}d(x_1, x_2) + Ld(x_2, u_1)$$

for any  $L \geq 0$ .

If  $x_2' \in [0, \frac{1}{2}]$  and  $u_1 = (1, -1)$ , then  $x_1 = (1, 0)$ . This implies

$$d(u_1, u_2) \leq \sqrt{2}, \quad d(x_1, x_2) \geq \frac{1}{2} \quad \text{and} \quad d(x_2, u_1) \geq \frac{\sqrt{5}}{2}.$$

Thus

$$d(u_1, u_2) \leq \sqrt{2} \leq \frac{1}{2}d(x_1, x_2) + Ld(x_2, u_1)$$

where  $L \geq \frac{2}{\sqrt{5}}(\sqrt{2} - \frac{1}{2})$ .

If  $x'_2 \in (\frac{1}{2}, 1)$  and  $u_1 = (0, 0)$ , then

$$d(u_1, u_2) \leq \frac{1}{2}, \quad d(x_1, x_2) \geq 0 \quad \text{and} \quad d(x_2, u_1) \geq \frac{1}{2}.$$

Thus

$$d(u_1, u_2) \leq \frac{1}{2} \leq \frac{1}{2}d(x_1, x_2) + Ld(x_2, u_1)$$

where  $L \geq 1$ .

If  $x'_2 \in (\frac{1}{2}, 1)$  and  $u_1 = (1, -1)$ , then  $x_2 = (1, 0)$ . This implies

$$d(u_1, u_2) \leq \frac{\sqrt{113}}{8}, \quad d(x_1, x_2) \geq 0 \quad \text{and} \quad d(x_2, u_1) \geq 1.$$

Thus

$$d(u_1, u_2) \leq \frac{\sqrt{113}}{8} \leq \frac{1}{2}d(x_1, x_2) + Ld(x_2, u_1)$$

where  $L \geq \frac{\sqrt{113}}{8}$ .

*Case 5.*  $x_1 = x_2 = (1, 0)$ . Then  $u_1, u_2 \in \{(0, 0), (1, -1)\}$ . Suppose that  $u_1 \neq u_2$ , we have

$$d(u_1, u_2) = \sqrt{2}, \quad d(x_1, x_2) = 0 \quad \text{and} \quad d(x_2, u_1) = 1.$$

Therefore

$$d(u_1, u_2) \leq \frac{1}{2}d(x_1, x_2) + Ld(x_2, u_1)$$

where  $L \geq \sqrt{2}$ .

We can conclude from all of above cases that  $T$  is proximal weak contraction with  $\alpha = \frac{1}{2}$  and  $L = \frac{\sqrt{113}}{4}$ . We also note that the point  $x = (0, 0)$  is a best proximity point of  $T$ .

We remark that  $T$  is neither a Berinde weak proximal contraction nor a proximal contraction. Let  $x = (1, 0)$ ,  $y = (1, -1)$ . Then  $u = (1, -1)$ ,  $v = (0, 0)$  are such that  $d(u, Tx) = d(v, Ty) = 1 = D(A, B)$ . We obtain

$$d(u, v) = \sqrt{2}, \quad d(x, y) = 1, \quad d^*(y, Tx) = 0 \quad \text{and} \quad d^*(x, Tx) = \sqrt{2} - 1.$$

This implies for each  $\alpha \in (0, 1)$ ,  $\beta \in (0, \infty)$ ,

$$\frac{1}{1 + \alpha + \beta}d^*(x, Tx) \leq d^*(x, Tx) \leq d(x, y),$$

but

$$d(u, v) = \sqrt{2} \geq \alpha d(x, y) = \alpha d(x, y) + \beta d^*(y, Tx).$$



**3.2. Proximity point for the proximal Berinde nonexpansive.** In this section, we first introduce a new concept of proximal nonexpansive mapping, called proximal Berinde nonexpansive mapping. This concept motivated by weak contraction of Berinde.

**Definition 7.** Let  $(A, B)$  be a pair of nonempty subsets of a metric space  $(X, d)$ . A mapping  $T: A \rightarrow B$  is said to be *proximal Berinde nonexpansive* if there exist  $L \geq 0$  such that

$$\left. \begin{aligned} d(u_1, Tx_1) = D(A, B) \\ d(u_2, Tx_2) = D(A, B) \end{aligned} \right\} \implies d(u_1, u_2) \leq d(x_1, x_2) + Ld(x_2, u_1)$$

for all  $x_1, x_2, u_1, u_2 \in A$ .

It is obvious that every proximal nonexpansive mappings is proximal Berinde nonexpansive with  $L = 0$ .

**Theorem 3.4.** *Let  $X$  be a Banach space,  $(A, B)$  be a pair of nonempty, closed subsets of  $X$  such that  $A$  is a  $p$ -starshaped set,  $B$  is a  $q$ -starshaped set, and  $\|p - q\| = D(A, B)$ . Assume  $A_0$  is compact,  $(B, A)$  is a semi-sharp proximal pair. Suppose that  $T: A \rightarrow B$  satisfies the following conditions:*

- (1)  $T$  is a proximal Berinde nonexpansive,
- (2)  $T(A_0) \subseteq B_0$ .

Then there exists  $x^*$  in  $A_0$  such that

$$\|x^* - Tx^*\| = D(A, B).$$

**Proof.** For each  $n \in \mathbb{N}$ , define  $T_n: A_0 \rightarrow B_0$  by

$$T_n x = (1 - a_n)Tx + a_n q, \quad x \in A_0,$$

where  $\{a_n\}$  is a sequence in  $(0, 1)$  such that  $\lim_{n \rightarrow \infty} a_n = 0$ . Since  $B_0$  is a  $q$ -starshaped set and  $T(A_0) \subseteq B_0$ , we get  $T_n(A_0) \subseteq B_0$ .

Next, we will show that  $T_n$  is proximal weak contraction for each  $n \in \mathbb{N}$ . Let  $x_1, x_2, u_1, u_2 \in A_0$  be such that

$$(1) \quad \|u_1 - T_n x_1\| = \|u_2 - T_n x_2\| = D(A, B).$$

Since  $Tx_1, Tx_2 \in B_0$ , there exist  $s_1, s_2 \in A_0$  such that

$$\|s_1 - Tx_1\| = \|s_2 - Tx_2\| = D(A, B).$$

By definition of  $T$ , we have

$$(2) \quad \|s_1 - s_2\| \leq \|x_1 - x_2\| + L\|x_2 - s_1\|,$$

for some  $L \geq 0$ . Now we set

$$v_1 = (1 - a_n)s_1 + a_n p \text{ and } v_2 = (1 - a_n)s_2 + a_n p.$$

Since  $A_0$  is a  $p$ -starshaped set, then  $v_1, v_2 \in A_0$ . We note that

$$\begin{aligned} D(A_0, B_0) &\leq \|v_1 - T_n x_1\| \\ &= \|(1 - a_n)v_1 + a_n p - (1 - a_n)Tx_1 - a_n q\| \\ &\leq (1 - a_n)\|s_1 - Tx_1\| + a_n\|p - q\| \end{aligned}$$

$$= D(A_0, B_0).$$

Therefore  $\|v_1 - T_n x_1\| = D(A_0, B_0)$ . Since  $(B, A)$  is a semi-sharp proximal pair and equation (1), this implies  $v_1 = u_1$ . Using the same method, we get  $v_2 = u_2$ . By proximal Berinde nonexpansiveness and (2), we have

$$\begin{aligned} \|u_1 - u_2\| &= \|v_1 - v_2\| \\ &= \|(1 - a_n)(s_1 - s_2)\| \\ &\leq (1 - a_n)\|x_1 - x_2\| + (1 - a_n)L\|x_2 - s_1\|. \end{aligned}$$

Thus for each  $n$ ,  $T_n$  is proximal weak contraction with  $k_n = 1 - \alpha_n$  and  $L'_n = (1 - a_n)L$ . By Theorem 3.1,  $T_n$  has a best proximity point  $x_n^* \in A_0$  such that

$$(3) \quad \|x_n^* - T_n x_n^*\| = D(A_0, B_0)$$

Since  $A_0$  is compact and  $\{x_n^*\}$  is a sequence in  $A_0$ , without loss of generality, we assume that, there exist  $x^* \in A_0$  such that  $x_n^* \rightarrow x^*$ .

Next, let us show that  $x^*$  is a best proximity point of  $T$ . Since  $Tx_n^* \in B_0$  for any  $n$ , there exist  $x'_n \in A_0$  such that

$$(4) \quad \|x'_n - Tx_n^*\| = D(A, B).$$

From

$$\begin{aligned} D(A_0, B_0) &\leq \|(1 - a_n)x'_n + a_n p - T_n x_n^*\| \\ &= \|(1 - a_n)x'_n + a_n p - (1 - a_n)Tx_n^* - a_n q\| \\ &\leq (1 - a_n)\|x'_n - Tx_n^*\| + a_n \|p - q\| \\ &= D(A_0, B_0), \end{aligned}$$

which implies

$$(5) \quad \|(1 - a_n)x'_n + a_n p - T_n x_n^*\| = D(A, B).$$

Since  $(B, A)$  is a semi-sharp proximal pair, we obtain by (3) and (5) that

$$x_n^* = (1 - a_n)x'_n + a_n p,$$

which implies

$$\|x_n^* - x'_n\| = a_n \|x'_n - p\| \rightarrow 0 \text{ as } n \rightarrow \infty \text{ (because } a_n \rightarrow 0).$$

That is  $\lim_{n \rightarrow \infty} x'_n = \lim_{n \rightarrow \infty} x_n^* = x^*$ . As we know  $T(x^*) \in B_0$ , so there exists  $u \in A_0$  such that

$$(6) \quad \|u - Tx^*\| = D(A, B).$$

By (4) and (6), we get

$$\|x'_n - u\| \leq \|x_n^* - x^*\| + L\|x^* - x'_n\|,$$

which implies  $\|x'_n - u\| \rightarrow 0$  as  $n \rightarrow \infty$ . Thus  $u = x^*$  and then  $x^*$  is a best proximity point of  $T$ . □

It is clear that if a pair  $(A, B)$  has the weak P-property, then  $(B, A)$  is a semi-sharp proximal pair. So we have the following result.

**Corollary 3.5.** *Let  $X$  be a Banach space,  $(A, B)$  be a pair of nonempty, closed subsets of  $X$  such that  $A$  is a  $p$ -starshaped set,  $B$  is a  $q$ -starshaped set, and  $\|p - q\| = D(A, B)$ . Assume  $A_0$  is compact,  $(A, B)$  have the weak  $P$ -property. Suppose that  $T: A \rightarrow B$  satisfies the following conditions:*

- (1)  $T$  is a proximal Berinde nonexpansive,
- (2)  $T(A_0) \subseteq B_0$ .

Then there exists  $x^*$  in  $A_0$  such that

$$\|x^* - Tx^*\| = D(A, B).$$

The following results are directly obtained by Theorem 3.4.

**Corollary 3.6** ([4]). *Let  $X$  be a Banach space,  $(A, B)$  be a pair of nonempty, closed subsets of  $X$  such that  $A_0$  is a  $p$ -starshaped set,  $B$  is a  $q$ -starshaped set, and  $\|p - q\| = D(A, B)$ . Assume  $A$  is compact,  $(B, A)$  is a semi-sharp proximal pair. Suppose that  $T: A \rightarrow B$  satisfies the following conditions:*

- (1)  $T$  is a proximal nonexpansive,
- (2)  $T(A_0) \subseteq B_0$ .

Then there exists  $x^*$  in  $A_0$  such that

$$\|x^* - Tx^*\| = D(A, B).$$

**Corollary 3.7.** *Let  $X$  be a Banach space,  $A$  be a nonempty compact subsets of  $X$  such that  $A$  is a  $p$ -starshaped set. Suppose  $T: A \rightarrow A$  is Berinde nonexpansive. Then there exists  $x^*$  in  $A$  such that  $x^* = Tx^*$ .*

**Example 3.8.** Let  $X = \mathbb{R}^2$  with  $\|(x, y)\| = |x| + |y|$ ,

$$A = \{(x, 0) : x \in [0, 2]\},$$

$$B_1 = \{(x, y) : y - x = 1, x \in [-1, 0)\},$$

$$B_2 = \{(x, 1) : x \in [0, \frac{3}{2}]\},$$

$$B = B_1 \cup B_2,$$

and let  $T: A \rightarrow B$  be defined by

$$T(x, 0) = \begin{cases} (x^2, 1), & \text{if } x \in [0, \frac{1}{2}], \\ (-\frac{x^2}{2}, -\frac{x^2}{2} + 1), & \text{if } x \in (\frac{1}{2}, \frac{3}{4}], \\ (\frac{x+3}{4}, 1), & \text{if } x \in (\frac{3}{4}, \frac{3}{2}), \\ (-\frac{x}{4} + \frac{9}{8}, 1), & \text{if } x \in [\frac{3}{2}, 2]. \end{cases}$$

We see that the following properties are satisfied:

- (1)  $A$  is a convex set,  $B$  is not convex set but it is a  $(0, 1)$ -starshaped set,
- (2)  $A_0$  is a compact set,
- (3)  $A_0 \subseteq A$ ,  $B_0 = B$  and  $T(A_0) \subseteq B_0$ ,
- (4)  $(B, A)$  is a semi-sharp proximal pair,  $(A, B)$  is not a semi-sharp proximal pair because  $\|(0, 0) - (-1, 0)\| = D(A, B) = \|(0, 0) - (0, 1)\|$  but  $(-1, 0) \neq (0, 1)$ .
- (5)  $(0, 0)$  and  $(1, 0)$  are best proximity points of  $T$ .

We can conclude that  $T$  is a proximal Berinde nonexpansive mapping with  $L = \frac{14}{3}$ . It is easy to see that  $T$  is not proximal nonexpansive mapping because when  $x_1 = (\frac{3}{2}, 0)$ ,  $x_2 = (\frac{5}{4}, 0)$ , we get  $u_1 = (\frac{3}{4}, 0)$ ,  $u_2 = (\frac{17}{16}, 0)$  and  $\|u_1 - u_2\| = \frac{5}{16} \geq \frac{1}{4} = \|x_1 - x_2\|$ . We also note that if  $x_1 = (\frac{3}{2}, 0)$ ,  $x_2 = (\frac{3}{4}, 0)$ , then  $u_1 = (\frac{3}{4}, 0)$ ,  $u_2 = (0, 0)$ . So we have  $\|x_2 - u_1\| = 0$  and  $\|u_1 - u_2\| = \frac{3}{4} = \|x_1 - x_2\|$ . Which implies that  $T$  is not proximal weak contraction.

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