Benjamin Cahen

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To the memory of my father, Alfred Cahen

Abstract. Let G be the semidirect product $V \rtimes K$ where K is a connected semisimple non-compact Lie group acting linearly on a finite-dimensional real vector space V. Let π be a unitary irreducible representation of G which is associated by the Kirillov-Kostant method of orbits with a coadjoint orbit of G whose little group is a maximal compact subgroup of K. We construct an invariant symbolic calculus for π, under some technical hypothesis. We give some examples including the Poincaré group.

Keywords: semidirect products; invariant symbolic calculus; coadjoint orbit; unitary representation; Berezin quantization; Weyl quantization; Poincaré group

Classification: 81S10, 22E46, 22E45, 22D30, 81R05

1. Introduction

In the context of covariant quantization, an important tool is the notion of invariant symbolic calculus, see [1], [3]. Various invariant symbolic calculi were introduced and intensively studied, in particular

(1) the Berezin symbolic calculus, see [6], [7];
(2) the Weyl calculus for symmetric domains, see [29], [2];
(3) the Stratonovich-Weyl correspondence, see [27], [30], [16], [18], [8], [15].

The following definition is adapted from [3] and [18].

Definition 1.1 ([18]). Let G be a Lie group and π a unitary representation of G on a Hilbert space $\mathcal{H}$. Let M be a homogeneous G-space and μ a (suitably normalized) G-invariant measure on M. Then an invariant symbolic calculus for the triple (G, π, M) is a linear map $\mathcal{S}$ from a vector space of operators on $\mathcal{H}$ to a vector space of (generalized) functions on M satisfying the following properties:

(1) $\mathcal{S}$ is one-to-one;
(2) reality: the function $\mathcal{S}(A^*)$ is the complex conjugate of $\mathcal{S}(A)$;
(3) invariance: we have $\mathcal{S}(\pi(g)A\pi(g)^{-1})(x) = \mathcal{S}(A)(g^{-1}x)$.

If, moreover, $\mathcal{S}$ is unitary in the sense that we have

$$\int_M \mathcal{S}(A)(x)\mathcal{S}(B)(x)\mathrm{d}\mu(x) = \mathrm{Tr}(AB)$$

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for each Hilbert-Schmidt operators A and B in the domain of $\mathcal{S}$, then $\mathcal{S}$ is called a Stratonovich-Weyl correspondence, see [18].

Note that, in Definition 1.1, M is generally taken to be a coadjoint orbit of G which is associated with π by the Kirillov-Kostant method of orbits, see [21], [22]. A simple illustration is given by the case when G is the $(2n+1)$-dimensional Heisenberg group. Each non-degenerate coadjoint orbit M of G is diffeomorphic to $\mathbb{R}^{2n}$ and is associated with a unitary irreducible representation π of G on $L^2(\mathbb{R}^n)$. In this case, the classical Weyl correspondence provides an invariant symbolic calculus for the triple (G, π, M) (which is also a Stratonovich-Weyl correspondence) [17], [18].

More sophisticated examples, involving some generalized Weyl correspondences, can be found in [3] and [29]. On the other hand, the Berezin calculus on integral coadjoint orbits is, in general, an invariant symbolic calculus, see [25], [4], and [12].

Most of the results on invariant symbolic calculi concern semisimple Lie groups. For semidirect products, the more remarkable result is the construction of a Stratonovich-Weyl correspondence for the unitary irreducible representations of the Poincaré group $\mathbb{R}^4 \rtimes SO_0(3,1)$ corresponding to the massive particules with spin, see [16]. Moreover, in paper in preparation, we extended this construction to unitary irreducible representations of $G = \mathbb{R}^{n+1} \rtimes SO_0(n,1)$ whose associated coadjoint orbits have little group $SO(n)$.

Here we consider the case when $G := V \rtimes K$ where K is a non-compact semi-simple Lie group acting linearly on a finite-dimensional real vector space V and π is a unitary irreducible representation of G associated with a coadjoint orbit $\mathcal{O}$ of G whose little group K_0 is a maximal compact subgroup of K.

This is the direct generalization of the massive coadjoint orbits (and representations) of the Poincaré group, see [23], Chapter IV, Section 3, [26], Chapter 8.

In the present paper, we aim to combine some ideas from [16], [10], [5], in order to get an invariant symbolic calculus for $(G, \pi, \mathcal{O})$.

Let us briefly describe the method we use here. Let $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{k}_0$ be the Lie algebras of G, K and K_0. Consider the Cartan decomposition $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{p}$. Then we can realize π on a Hilbert space $\mathcal{H}$ of square-integrable functions on $\mathfrak{p}$. Moreover, $\mathcal{O}$ is diffeomorphic to $\mathfrak{p}^2 \times o$ where o is a coadjoint orbit of K_0, see [10]. Thus we fix $\xi_0 \in \mathcal{O}$ and we choose a suitable operator $\Omega(\xi_0)$ on $\mathcal{H}$ which, in particular, commutes to $\pi(g)$ for each g in the stabilizer of ξ_0 in G. As pointed out in [5], this choice is crucial for the success of the method. Hence we define a *quantizer* $\Omega\colon \mathcal{O} \to \mathrm{End}(\mathcal{H})$ by

$$\Omega(g \cdot \xi_0) := \pi(g)\Omega(\xi_0)\pi(g)^{-1}$$

for each $g \in G$ and a symbolic calculus $\mathcal{S}$ by the formula

$$\mathcal{S}(A)(\xi) = \mathrm{Tr}(A\Omega(\xi))$$

for A operator on $\mathcal{H}$ and $\xi \in \mathcal{O}$.

Then $\mathcal{S}$ is clearly invariant and, at this step, the main difficulty is to prove that $\mathcal{S}$ is injective (on a suitable space of operators on $\mathcal{H}$), since the explicit computations of [16] cannot be performed in our general situation.

This paper is organized as follows. Section 2 contains some generalities on semidirect products. In Section 3, we introduce the unitary irreducible representations of G and the corresponding coadjoint orbits. In Section 4, we recall the construction of the Berezin calculus for a unitary irreducible representation of K_0. The quantizer Ω is introduced in Section 5. In Section 6, the invariant symbolic calculus $\mathcal{S}$ for π is defined and we prove that it is injective. In Section 7, we discuss the problem of extending $\mathcal{S}$ to operators which are not Hilbert-Schmidt. Finally, in Section 8, we consider two examples: the (generalized) Poincaré group and the group $su(n,1) \rtimes SU(n,1)$.

2. Preliminaries

The material of this section and of the next section is essentially taken from [24], see also [10].

We consider a connected, non-compact, semisimple real Lie group K with finite center. Let $\mathfrak{k}$ be the Lie algebra of K. For $k \in K$ and $f \in \mathfrak{k}^*$, we denote by $k \cdot f$ the coadjoint action of k on f.

We assume that K acts linearly on a finite-dimensional real vector space V, and for k in K and v in V, we denote by $k \cdot v$ the action of k on v. We also denote by $(k,p) \to k \cdot p$ the contragredient action of K on V^*. Let $(A,v) \to A \cdot v$ and $(A,p) \to A \cdot p$ be the corresponding representations of $\mathfrak{k}$ on V and V^*. For each v in V and p in V^* we define $v \wedge p \in \mathfrak{k}^*$ by $\langle v \wedge p, A\rangle = \langle p, A \cdot v\rangle = -\langle A \cdot p, v\rangle$ for $A \in \mathfrak{k}$. Note that we have

$$k \cdot (v \wedge p) = k \cdot v \wedge k \cdot p$$

for each $k \in K$, $v \in V$ and $p \in V^*$.

We can form the semidirect product $G = V \rtimes K$. The multiplication of G is

$$(v,k)(v',k') = (v + k \cdot v', kk')$$

for each v, v' in V and k, k' in K. The Lie algebra $\mathfrak{g}$ of G is the vector space $V \times \mathfrak{k}$ equipped with the Lie bracket

$$[(a,A),(a',A')] = (A \cdot a' - A' \cdot a, [A,A'])$$

for each a, a' in V and A, A' in $\mathfrak{k}$.

Then $\mathfrak{g}^*$ can be identified with $V^* \times \mathfrak{k}^*$. The coadjoint action of G on $\mathfrak{g}^*$ is thus given by

$$(v,k) \cdot (p,f) = (k \cdot p, k \cdot f + v \wedge k \cdot p)$$

for each $(v,k) \in G$ and $(p,f) \in \mathfrak{g}^*$. We can also identify K-equivariantly $\mathfrak{k}$ to its dual $\mathfrak{k}^*$ by using the Killing form of $\mathfrak{k}$. Hence $\mathfrak{g}^*$ can be identified with $V^* \times \mathfrak{k}$.

Let us consider the orbit $\mathcal{O}(\xi_0)$ of the element $\xi_0 = (p_0, f_0)$ of $\mathfrak{g}^* \simeq V^* \times \mathfrak{k}$ under the coadjoint action of G on $\mathfrak{g}^*$. Henceforth we assume that the little group $K_0 = \{k \in K : k \cdot p_0 = p_0\}$ is a maximal compact subgroup of K. Then K_0 is a connected reductive subgroup of K, see [19], and denoting by $\mathfrak{k}_0$ the Lie algebra of K_0, we have the Cartan decomposition $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{p}$ where $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}_0$ in $\mathfrak{k}$. Moreover, we can verify that $\mathfrak{p} = \{v \wedge p_0 : v \in V\}$, see [10] and [24], Lemma 1. From this, we see that, without loss of generality, we can assume that $\xi_0 = (p_0, \varphi_0)$ with $\varphi_0 \in \mathfrak{k}_0$. We denote by $o(\varphi_0) \subset \mathfrak{k}_0$ the orbit of $\varphi_0 \in \mathfrak{k}_0 \simeq \mathfrak{k}_0^*$ under the (co)adjoint action of K_0.

Let $Z(p_0)$ be the orbit of p_0 under the action of K on V^*. By [19], Chapter VI, Theorem 1.1, the map $e : T \to \exp T \cdot p_0$ is a diffeomorphism from $\mathfrak{p}$ onto $Z(p_0)$. For $p \in Z(p_0)$ we denote by $M(p)$ the unique element of $\exp \mathfrak{p}$ such that $M(p) \cdot p_0 = p$. Consequently, if $p = e(T)$ then $M(p) = \exp T$.

Let V_0 be a complement of $\{v \in V : v \wedge p_0 = 0\}$ in V. Then we have $\dim V_0 = \dim \mathfrak{p}$.

In Section 6, for some technical reasons we shall need to assume that the map

$$\gamma : T \to (e(T) - e(-T))|_{V_0}$$

is a diffeomorphism from $\mathfrak{p}$ onto V_0^*. This assumption is satisfied, for example, in the case of the massive coadjoint orbits of the Poincaré group, see Section 8. However, it seems to be difficult to characterize precisely the orbits for which this condition is fulfilled.

Let n be the dimension of $\mathfrak{p}$. We know that the restriction to $\mathfrak{p}$ of the Killing form $\langle \cdot, \cdot \rangle$ of $\mathfrak{k}$ is positive definite, see [19]. We fix an orthonormal basis $(E_1, E_2, \ldots, E_n)$ for $\mathfrak{p}$ and we denote by $(t_1, t_2, \ldots, t_n)$ the coordinates of $T \in \mathfrak{p}$ in this basis.

Let $\mathrm{d}T = \mathrm{d}t_1 \mathrm{d}t_2 \ldots \mathrm{d}t_n$ be the Lebesgue measure on $\mathfrak{p}$. Then, the K-invariant measure $\mathrm{d}\mu$ on $Z(p_0)$ is given by $\mathrm{d}\mu = e^*(\delta(T)\mathrm{d}T)$ where $\delta(T) := \mathrm{Det}\left(\frac{\sinh \operatorname{ad} T}{\operatorname{ad} T}|_{\mathfrak{p}}\right)$, see [19].

Let us denote by $\mathrm{d}v$ a Lebesgue measure on V_0. Also, let ν be an invariant measure on $o(\varphi_0)$. We fix a section (defined on a dense open subset of $o(\varphi_0)$) $\varphi \to h_\varphi$ for the action of K_0 on $o(\varphi_0)$. Such a section always exists, see [13]. The following proposition can be proved easily.

Proposition 2.1. *Let Ψ be the map from $Z(p_0) \times V_0 \times o(\varphi_0)$ to $\mathfrak{g}^*$ defined by*

$$\Psi(q, v, \varphi) = (q, M(q) \cdot (\varphi + v \wedge p_0)).$$

Then we have

(1) *Ψ is a diffeomorphism from $Z(p_0) \times V_0 \times o(\varphi_0)$ onto $\mathcal{O}(\xi_0)$;*
(2) *the image by Ψ of the measure $\mathrm{d}\mu(p)\mathrm{d}v\mathrm{d}\nu(\varphi)$ on $Z(p_0) \times V_0 \times o(\varphi_0)$ is an invariant measure μ_0 on $\mathcal{O}(\xi_0)$;*
(3) *the map $\xi = \Psi(q, v, \varphi) \to g_\xi := (M(q) \cdot v, M(q) \cdot h_\varphi)$ is a section for the action of G on $\mathcal{O}(\xi_0)$, that is, we have $g_\xi \cdot \xi_0 = \xi$ for each $\xi \in \mathcal{O}(\xi_0)$.*

3. Representations

The material of this section is essentially taken from [10].

Henceforth we assume that $o(\varphi_0)$ is associated with the unitary irreducible representation (ϱ, E) of K_0 as in [33], Section 4. Let us describe this correspondence. Let H be a maximal torus of K_0 with Lie algebra $\mathfrak{h}$. We fix an ordering on the root system $\Delta(\mathfrak{g}^c, \mathfrak{h}^c)$. Now, let $\lambda \in (i\mathfrak{h})^*$ be the highest weight of (ϱ, E). Then we define $\varphi_0 \in \mathfrak{k}_0^*$ by $\varphi_0(X) = -i\lambda(X)$ for $X \in \mathfrak{h}$ and $\varphi_0(X) = 0$ for X in the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{k}_0$ with respect to the Killing form of $\mathfrak{k}_0$. The orbit of φ_0 under the coadjoint action of K_0 is said to be associated with the representation (ϱ, E).

It is well-known that $\mathcal{O}(\xi_0)$ is integral since $o(\varphi_0)$ is assumed to be integral, see [24]. Then $\mathcal{O}(\xi_0)$ is associated with the unitarily induced representation

$$\pi = \operatorname{Ind}_{V \rtimes K_0}^{G}(e^{i\langle p_0, \cdot\rangle} \otimes \varrho).$$

By a result of G. Mackey, π is irreducible since ϱ is [28].

The representation π is usually realized on the Hilbert space $L^2(Z(p_0), E)$ which is the completion of the space of compactly supported smooth functions $\psi\colon Z(p_0) \to E$ with respect to the norm defined by

$$\|\psi\|^2 = \int_{Z(p_0)} \langle \psi(p), \psi(p)\rangle_E \, \mathrm{d}\mu(p).$$

Specifically, for each $(v, k) \in G$ the action of the operator $\pi(v, k)$ is given by

$$(\pi(v, k)\psi)(p) = e^{i\langle p, v\rangle} \varrho(M(p)^{-1} k M(k^{-1} \cdot p))\psi(k^{-1} \cdot p).$$

However, it is convenient to realize π on the Hilbert space $\mathcal{H} := L^2(\mathfrak{p}, E)$ defined as the completion of the space $C_0^\infty(\mathfrak{p}, E)$ of compactly supported smooth functions $\phi\colon \mathfrak{p} \to E$ with respect to the norm given by

$$\|\phi\|^2 = \int_{\mathfrak{p}} \langle \phi(T), \phi(T)\rangle_E \, \mathrm{d}T.$$

Then we introduce the unitary operator $\phi \to \psi$ from $\mathcal{H}$ to $L^2(Z(p_0), E)$ defined by $\psi(e(T)) = \delta(T)^{1/2}\phi(T)$. Let us denote by $k \cdot T$ the action of K on $\mathfrak{p}$ which corresponds to the action of K on $Z(p_0)$, that is, we have $e(k \cdot T) = k \cdot e(T)$ for $k \in K$ and $T \in \mathfrak{p}$. Thus we obtain

$$(\pi(v, k)\phi)(T) = \Big(\frac{\delta(T)}{\delta(k^{-1} \cdot T)}\Big)^{1/2} e^{i\langle e(T), v\rangle} \varrho(M(e(T))^{-1} k M(k^{-1} e(T)))\phi(k^{-1} \cdot T)$$

for each $(v, k) \in G$.

We recall now the explicit expression for the differential $\mathrm{d}\pi$ of π given in [10]. We need some additional notation. First, we can differentiate the action of K on

$\mathfrak{p}$ and define for $A \in \mathfrak{k}$ and $T \in \mathfrak{p}$,

$$A \cdot T := \frac{\mathrm{d}}{\mathrm{d}t}(\exp tA) \cdot T\Big|_{t=0}.$$

Furthermore, for $p \in Z(p_0)$ and $A \in \mathfrak{k}$ we set

$$L(p, A) = \frac{\mathrm{d}}{\mathrm{d}t}(M(p)^{-1} \exp(tA) M(\exp(-tA) \cdot p))|_{t=0}.$$

Let $p_{\mathfrak{k}_0}$ and $p_{\mathfrak{p}}$ be the projections of $\mathfrak{k}$ onto $\mathfrak{k}_0$ and $\mathfrak{p}$ associated with the direct decomposition $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{p}$.

Lemma 3.1 ([10]). (1) *For $A \in \mathfrak{k}$ and $T \in \mathfrak{p}$ we have*

$$A \cdot T = -\operatorname{ad} T\, p_{\mathfrak{k}_0}(A) + \frac{\operatorname{ad} T}{\tanh \operatorname{ad} T}\, p_{\mathfrak{p}}\,(A).$$

(2) *For $p = e(T) \in Z(p_0)$ and $A \in \mathfrak{k}$ we have*

$$L(p, A) = p_{\mathfrak{k}_0}(A) - \tanh\Big(\frac{1}{2} \operatorname{ad} T\Big) p_{\mathfrak{p}}(A).$$

(3) *For $A \in \mathfrak{k}$ and $T \in \mathfrak{p}$ we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta(\exp(tA) \cdot T)\Big|_{t=0} = \delta(T) \operatorname{Tr}_{\mathfrak{p}}(\gamma(\operatorname{ad} T)\ \operatorname{ad} p_{\mathfrak{p}}(A))$$

where the function γ is defined by $\gamma(z) = (z \cosh z - \sinh z)/(z \sinh z)$ if $z \neq 0$ and by $\gamma(0) = 0$.

From this lemma, we deduce the following expression of $\mathrm{d}\pi$.

Proposition 3.2 ([10]). *For each $(w, A) \in \mathfrak{g}$ and $\phi \in C_0(\mathfrak{p}, E)$, we have*

$$\begin{aligned}(\mathrm{d}\pi(w, A)\phi)(T) = {} & i\langle e(T), w\rangle \phi(T) + \mathrm{d}\varrho\Big(p_{\mathfrak{k}_0}(A) - \tanh\Big(\frac{1}{2} \operatorname{ad} T\Big) p_{\mathfrak{p}}(A)\Big) \phi(T) \\ & + \mathrm{d}\phi(T)\Big(\operatorname{ad} T p_{\mathfrak{k}_0}(A) - \frac{\operatorname{ad} T}{\tanh \operatorname{ad} T} p_{\mathfrak{p}}(A)\Big) \\ & + \frac{1}{2} \operatorname{Tr}_{\mathfrak{p}}\,(\gamma(T) \operatorname{ad} p_{\mathfrak{p}}(A))\, \phi(T).\end{aligned}$$

4. Berezin calculus on $o(\varphi_0)$

Here we recall the Berezin correspondence associated with ϱ, see for instance [6], [7], [4], [33] and [11].

Without loss of generality, we can assume that E is a space of holomorphic sections of a complex line bundle over $o(\varphi_0)$, see [32]. Let $\varphi \in o(\varphi_0)$. For each $\widehat{\varphi} \neq 0$ in the fiber over φ, there exists a unique section $e_{\widehat{\varphi}} \in E$ (a coherent state) such that $a(\varphi) = \langle a, e_{\widehat{\varphi}}\rangle_E \widehat{\varphi}$ for each $a \in V$. The Berezin calculus on $o(\varphi_0)$

associates with each operator B on V the complex-valued function $s(B)$ on $o(\varphi_0)$ defined by

$$s(B)(\varphi)=\frac{\langle Be_{\widehat{\varphi}},e_{\widehat{\varphi}}\rangle_E}{\langle e_{\widehat{\varphi}},e_{\widehat{\varphi}}\rangle_E}$$

which is called the symbol of B. We denote by $Sy(o(\varphi_0))$ the space of all such symbols.

In the following proposition, we recall some properties of s, see [25], [4] and [11].

Proposition 4.1. (1) *The map $B\to s(B)$ from* $\mathrm{End}(E)$ *onto $Sy(o(\varphi_0))$ is a linear isomorphism.*

(2) *For each operator B on E, we have $s(B^*)=\overline{s(B)}$.*

(3) *For each $\varphi\in o(\varphi_0)$, $h\in K_0$ and $B\in\mathrm{End}(E)$, we have*

$$s(B)(h\cdot\varphi)=s(\varrho(h)B\varrho(h)^{-1})(\varphi).$$

(4) *For each $U\in\mathfrak{k}_0$ and $\varphi\in o(\varphi_0)$, we have $s(d\varrho(U))(\varphi)=i\langle\varphi,U\rangle$.*

For each $\varphi\in o(\varphi_0)$, we denote by $P(\varphi)$ the orthogonal projection operator of E on the line generated by $e_{\widehat{\varphi}}$.

Proposition 4.2 ([6]). *For each operator B on E and each $\varphi\in o(\varphi_0)$, we have*

$$s(B)(\varphi)=\mathrm{Tr}(BP(\varphi)).$$

In the terminology of [18] and [5], the map $\varphi\to P(\varphi)$ is called the *quantizer* associated with s and the properties of s are reflected by similar properties of this quantizer. In particular, the invariance property of s corresponds to the fact that for each $h\in K_0$ and $\varphi\in o(\varphi_0)$, we have

$$P(h\cdot\varphi)=\varrho(h)P(\varphi)\varrho(h)^{-1}. \tag{4.1}$$

5. The quantizer for π

In this section, we introduce a quantizer Ω which will give an invariant symbol calculus for π in the next section. In order to motivate our choice for $\Omega(\xi_0)$, we will make a little digression about the classical Weyl correspondence based on [17], [18], and [14].

The Weyl correspondence $\mathcal{W}_0$ on $\mathbb{R}^{2n}$ is defined as follows. For each $f\in L^2(\mathbb{R}^{2n})$, let $\mathcal{W}_0(f)$ be the operator on $L^2(\mathbb{R}^n)$ given by

$$(\mathcal{W}_0(f)\phi)(x)=(2\pi)^{-n}\int_{\mathbb{R}^{2n}}e^{i\langle y,z\rangle}f\Big(x+\frac{1}{2}y,z\Big)\phi(x+y)\,\mathrm{d}y\,\mathrm{d}z.$$

Now, let G_0 be the Heisenberg group of dimension $2n{+}1$. We write the elements of G_0 as $[a,b,c]$ with $a,b\in\mathbb{R}^n$ and $c\in\mathbb{R}$. The multiplication of G_0 is given by

$$[a,b,c]\cdot[a',b',c']=\Big[a+a',b+b',c+c'+\frac{1}{2}(\langle a,b'\rangle-\langle a',b\rangle)\Big].$$

Then G_0 acts on $\mathbb{R}^{2n}$ by

$$g\cdot(p,q)=(p+a,q+b),\qquad g=[a,b,c]\in G_0.$$

Also, let σ be the unitary representation of G_0 on $L^2(\mathbb{R}^n)$ defined by

$$(\sigma(g)\phi)(x)=\exp\Big(i\Big(c-\langle b,x\rangle+\frac{1}{2}\langle a,b\rangle\Big)\Big)\phi(x-a)$$

for $g=[a,b,c]\in G_0$.

Consider the parity operator R_0 on $\mathbb{R}^n$ defined by $R_0\phi(x)=2^n\phi(-x)$ and note that R_0 commute with $\sigma(g)$ for each $g=[0,0,c]$ in the center of G_0. Then, for each $g=[a,b,c]\in G_0$, we can define $R(g\cdot(0,0))=\sigma(g)R_0\sigma(g)^{-1}$. More precisely, for each $(a,b)\in\mathbb{R}^{2n}$, $\phi\in L^2(\mathbb{R}^n)$ and $x\in\mathbb{R}^n$, we have

$$(R(a,b)\phi)(x)=2^n\exp(2i\langle b,a-x\rangle)\phi(2a-x).$$

Proposition 5.1 ([18])**.** (1) *For each $g\in G_0$ and each Hilbert-Schmidt operator A on $L^2(\mathbb{R}^n)$, we have*

$$\mathcal{W}_0^{-1}(\sigma(g)A\sigma(g)^{-1})(x,y)=\mathcal{W}_0^{-1}(A)(g^{-1}\cdot(x,y)).$$

(2) *For each trace class operator A on $L^2(\mathbb{R}^n)$ and each $(x,y)\in\mathbb{R}^{2n}$, we have*

$$\mathrm{Tr}(AR(x,y))=\mathcal{W}_0^{-1}(A)(x,y).$$

This implies that $\mathcal{W}_0^{-1}$ is a Stratonovich-Weyl correspondence for $(G_0,\sigma,\mathbb{R}^{2n})$ with quantizer R.

Now, we return to the construction of the quantizer for π. We begin with the following lemma which is easy but useful.

Lemma 5.2. *Let $\Omega(\xi_0)$ be an operator on $\mathcal{H}$. For each $\xi\in\mathcal{O}(\xi_0)$, let*

$$\Omega(\xi):=\pi(g_\xi)\Omega(\xi_0)\pi(g_\xi)^{-1}. \tag{5.1}$$

Then we have that

$$\Omega(g\cdot\xi):=\pi(g)\Omega(\xi)\pi(g)^{-1}$$

for each $g\in G$ and $\xi\in\mathcal{O}(\xi_0)$, if and only if $\Omega(\xi_0)$ commute with the operator $\pi(g)$ for each g in the stabilizer $G(\xi_0)$ of ξ_0 in G.

In the following lemma, we collect some easy facts.

Lemma 5.3. (1) *For each $k\in K_0$ and $p\in Z(p_0)$, we have $M(k\cdot p)=kM(p)k^{-1}$.*
(2) *For each $k\in K_0$ and $T\in\mathfrak{p}$, we have $k\cdot T=\mathrm{Ad}(k)T$.*
(3) *For each $T\in\mathfrak{p}$, we have $M(e(-T))=M(e(T))^{-1}$.*

Now we define $\Omega(\xi_0)$ from R_0 and P (see Section 4) as follows. We take

$$(\Omega(\xi_0)\phi)(T):=2^nP(\varphi_0)\phi(-T)$$

for each $\phi \in \mathcal{H}$ and $T \in \mathfrak{p}$. Then we can easily verify that $G(\xi_0)$ consists of all elements (v,k) such that $v \wedge p_0 = 0$ and k lies in the stabilizer of φ_0 in K_0. Moreover, by using Lemma 5.3, we see that for each $g \in G(\xi_0)$, $\pi(g)$ commute with $\Omega(\xi_0)$. Hence Lemma 5.2 can be applied.

For each $k \in K$ and $T \in \mathfrak{p}$ we set

$$r(k,T) := \varrho(M(e(T))^{-1}kM(k^{-1}\cdot e(T))).$$

Proposition 5.4. (1) *Let $(v,k) \in G$ and $\xi = (v,k)\cdot\xi_0$. Then for each $\psi \in \mathcal{H}$ and $T \in \mathfrak{p}$ we have*

$$\begin{aligned}(\Omega(\xi)\phi)(T) = 2^n\Big(\frac{\delta(T)}{\delta(k\cdot(-k^{-1}\cdot T))}\Big)^{1/2} e^{i\langle e(T)-e(k\cdot(-k^{-1}\cdot T)),v\rangle}\\ \times r(k,T)P(\varphi_0)r(k^{-1},-k^{-1}\cdot T)\phi(k\cdot(-k^{-1}\cdot T)).\end{aligned}$$

(2) *Let $(p,v,\varphi) \in Z(p_0)\times V_0 \times o(\varphi_0)$ and set $\xi := \Psi(p,v,\varphi)$ and $h := M(p)$. Then for each $\psi \in \mathcal{H}$ and $T \in \mathfrak{p}$ we have*

$$\begin{aligned}(\Omega(\xi)\phi)(T) = 2^n\Big(\frac{\delta(T)}{\delta(h\cdot(-h^{-1}\cdot T))}\Big)^{1/2} e^{i\langle e(h^{-1}\cdot T)-e(-h^{-1}\cdot T),v\rangle}\\ \times r(h,T)P(\varphi)r(h^{-1},-h^{-1}\cdot T)\,\phi(h\cdot(-h^{-1}\cdot T)).\end{aligned}$$

Proof: (1) follows from a simple but tedious calculation based on equation (5.1). Moreover, taking Proposition 2.1 and equation (4.1) into account, (2) follows from (1) and equation (5.1). □

6. Invariant symbolic calculus for π

We aim to prove that the quantizer Ω introduced in the preceding section gives an invariant symbolic calculus for π. To simplify writing of equations, for each $(p,v,\varphi) \in Z(p_0)\times V_0\times o(\varphi_0)$ and $\xi = \Psi(p,v,\varphi)$ we set

$$\begin{aligned}\beta(\xi,T) := 2^n\Big(\frac{\delta(T)}{\delta(M(p)\cdot(-M(p)^{-1}\cdot T))}\Big)^{1/2} e^{i\langle e(M(p)^{-1}\cdot T)-e(-M(p)^{-1}\cdot T),v\rangle}\\ \times r(M(p),T)P(\varphi)r(M(p)^{-1},-M(p)^{-1}\cdot T).\end{aligned}$$

For $k \in K$, we also denote by s_k the map from $\mathfrak{p}$ to $\mathfrak{p}$ defined by $s_k(T) = k\cdot(-k^{-1}\cdot T)$. Note that we have $s_k \circ s_k = id_{\mathfrak{p}}$ for each $k \in K$. Then for each $\phi \in \mathcal{H}$ and $T \in \mathfrak{p}$ we can write

$$(\Omega(\xi)\phi)(T) = \beta(\xi,T)\phi(s_{M(p)}(T)). \tag{6.1}$$

Now, for each trace class operator A on $\mathcal{H}$ we define

$$\mathcal{S}(A)(\xi) := \operatorname{Tr}(A\Omega(\xi))$$

and we aim to prove that $\mathcal{S}$ is an invariant symbolic calculus.

For each trace class operator A on $\mathcal{H}$, let us denote by $k_A\colon \mathfrak{p}^2 \to \mathrm{End}(V)$ the kernel of A, that is, for each $\phi \in \mathcal{H}$ and $T \in \mathfrak{p}$ we have

$$(A\phi)(T) = \int_{\mathfrak{p}} k_A(T,S)\phi(S)\,\mathrm{d}S.$$

The following lemma is well-known, see for instance [21], page 342.

Lemma 6.1. (1) *Let A and B be two Hilbert-Schmidt operators on $\mathcal{H}$. Then the kernel of AB is given by*

$$k_{AB}(T,S) = \int_{\mathfrak{p}} k_A(T,Z)k_B(Z,S)\,\mathrm{d}Z.$$

(2) *Let A be a trace-class operator on $\mathcal{H}$. Then the function $T \to k_A(T,T)$ is integrable on $\mathfrak{p}$ and we have*

$$\mathrm{Tr}(A) = \int_{\mathfrak{p}} k_A(T,T)\,\mathrm{d}T.$$

Proposition 6.2. *Let A be a trace class operator on $\mathcal{H}$. Let $\xi = \Psi(p,v,\varphi)$ where $(p,v,\varphi) \in Z(p_0) \times V_0 \times o(\varphi_0)$. Then we have*

$$\begin{aligned}\mathcal{S}(A)(\xi) = 2^n \int_{\mathfrak{p}} e^{i\langle e(T)-e(-T),v\rangle}\,&\mathrm{Tr}(k_A(M(p)\cdot(-T), M(p)\cdot T)\\ &\times r(M(p), M(p)\cdot T)P(\varphi)r(M(p)^{-1}, -T))\\ &\times \frac{\delta(T)}{\delta(M(p)\cdot T)^{1/2}\delta(M(p)\cdot(-T))^{1/2}}\,\mathrm{d}T.\end{aligned}$$

Proof: Let A be a trace class operator on $\mathcal{H}$ and $\xi = \Psi(p,v,\varphi)$. Then for each $\phi \in \mathcal{H}$ and $T \in \mathfrak{p}$ we have

$$\begin{aligned}(A\Omega(\xi)\phi)(T) &= \int_{\mathfrak{p}} k_A(T,S)(\Omega(\xi)\phi)(S)\,\mathrm{d}S\\ &= \int_{\mathfrak{p}} k_A(T,S)\beta(\xi,S)\phi(s_{M(p)}(S))\,\mathrm{d}S\\ &= \int_{\mathfrak{p}} k_A(T,s_{M(p)}(S))\beta(\xi,s_{M(p)}(S))\phi(S)\frac{\delta(S)}{\delta(s_{M(p)}(S))}\,\mathrm{d}S\end{aligned}$$

by the change of variables $S \to s_{M(p)}(S)$ and K-invariance of $\delta(S)\mathrm{d}S$.

This shows that $A\Omega(\xi)$ has kernel

$$k_{A\Omega(\xi)}(T,S) = k_A(T,s_{M(p)}(S))\beta(\xi,s_{M(p)}(S))\frac{\delta(S)}{\delta(s_{M(p)}(S))}.$$

Now, applying Lemma 6.1, we get

$$\begin{aligned}\mathcal{S}(A)(\xi) &= \int_{\mathfrak{p}} \operatorname{Tr}(k_A(T, s_{M(p)}(T))\beta(\xi, s_{M(p)}(T)))\frac{\delta(T)}{\delta(s_{M(p)}(T))}\mathrm{d}T\\ &= \int_{\mathfrak{p}} \operatorname{Tr}(k_A(s_{M(p)}(T), T)\beta(\xi, T))\,\mathrm{d}T.\end{aligned}$$

The desired result is then obtained by replacing $\beta(\xi, T)$ by its expression and by performing the change of variables $T \to M(p)\cdot T$. □

Let $p_1\colon K \to \exp\mathfrak{p}$ and $p_2\colon K \to K_0$ be the projections associated with the Cartan decomposition $K = (\exp\mathfrak{p})K_0$. We need the following lemma.

Lemma 6.3. *The map $j\colon (t,s) \to (p_1(st), p_1(st^{-1}))$ is a C^1-diffeomorphism of $(\exp\mathfrak{p})^2$. The inverse diffeomorphism j^{-1} is given by*

$$j^{-1}(u,v) = \big(p_2(vp_1(v^{-1}u)^{1/2})p_1(v^{-1}u)^{1/2}p_2(vp_1(v^{-1}u)^{1/2})^{-1}, p_1(vp_1(v^{-1}u)^{1/2})\big).$$

Proof: Let $(t,s) \in (\exp\mathfrak{p})^2$ and $u = p_1(st)$, $v = p_1(st^{-1})$. Then we can write $st = uh$, and $st^{-1} = vh'$ with $h, h' \in K_0$. From this, we deduce that $t^2 = h'^{-1}v^{-1}uh$ and, writing also $v^{-1}u = wa$ with $w \in \exp\mathfrak{p}$ and $a \in K_0$, we get $t^2 = (h'^{-1}wh')(h'^{-1}ah)$ which in turn implies that $t^2 = h'^{-1}wh'$ and $h'^{-1}ah = e$ (the identity element of K) or, equivalently, $t = h'^{-1}w^{1/2}h'$ and $h' = ah$.

Thus, substituting t and h' in the equality $st^{-1} = vh'$, we find that $s = vw^{1/2}ah$. This gives on the one hand that $s = p_1(vw^{1/2})$ and on the other hand that $h = a^{-1}p_2(vw^{1/2})^{-1}$ and

$$t = h'^{-1}w^{1/2}h' = h^{-1}a^{-1}w^{1/2}ah = p_2(vw^{1/2})w^{1/2}p_2(vw^{1/2})^{-1}.$$

This shows that $j\colon (\exp\mathfrak{p})^2 \to (\exp\mathfrak{p})^2$ is a bijection and gives the explicit expression of j^{-1}. Moreover, since the multiplication map $\exp\mathfrak{p}\times K_0 \to K$ is a C^1-diffeomorphism, we see that p_1 and p_2 are C^1-functions and consequently, that j and j^{-1} are also C^1-functions, hence the result. □

Now, we can remark that the expression of $\mathcal{S}(A)(\xi)$ given in Proposition 6.2 can be interpreted as the Fourier transform evaluated at v of some function of T. Then, in order to prove that $\mathcal{S}$ is injective by using Fourier inversion, we are led to introduce the assumption that the map $\gamma\colon T \to (e(T) - e(-T))|_{V_0}$ is a diffeomorphism from $\mathfrak{p}$ onto V_0^*, as announced in Section 2.

Proposition 6.4. *For each trace-class operator A on $\mathcal{H}$, we have $\mathcal{S}(A) = 0$ if and only if $A = 0$.*

Proof: Let A be a trace-class operator A on $\mathcal{H}$ such that $\mathcal{S}(A)(\xi) = 0$ for each $\xi \in \mathcal{O}(\xi_0)$. By performing the change of variables $q = \gamma(T)$ in the integral expression for $\mathcal{S}(A)(\xi)$ of Proposition 6.2 and by applying Fourier inversion we get

$$\operatorname{Tr}\big(k_A(M(p)\cdot(-T), M(p)\cdot T)r(M(p), M(p)\cdot T)P(\varphi)r(M(p)^{-1}, -T)\big) = 0$$

for each $p \in Z(p_0)$, $T \in \mathfrak{p}$ and $\varphi \in o(\varphi_0)$. Consequently, the Berezin symbol of the operator

$$r(M(p)^{-1}, -T)k_A(M(p)\cdot(-T), M(p)\cdot T)r(M(p), M(p)\cdot T)$$

is zero hence the operator is zero and we obtain

$$k_A(M(p)\cdot(-T), M(p)\cdot T) = 0$$

for each $p \in Z(p_0)$ and each $T \in \mathfrak{p}$.

Applying Lemma 6.3, we can conclude that $k_A = 0$ hence $A = 0$. □

Proposition 6.5. *The map $\mathcal{S}$ defined on trace-class operators on $\mathcal{H}$ is an invariant symbolic calculus.*

PROOF: $\mathcal{S}$ is invariant by construction and also injective. We have just to verify that for each trace-class operator A on $\mathcal{H}$ we have $\mathcal{S}(A^*) = \overline{\mathcal{S}(A)}$. But, for each trace-class operator A on $\mathcal{H}$ we have

$$\mathcal{S}(A^*)(\xi) = \mathrm{Tr}(A^*\Omega(\xi)) = \mathrm{Tr}(\Omega(\xi)A)^* = \overline{\mathrm{Tr}(\Omega(\xi)A} = \overline{\mathcal{S}(A)(\xi)}.$$

The result hence follows. □

7. Extension of the invariant symbolic calculus

Here we aim to extend $\mathcal{S}$ to operators on $\mathcal{H}$ which are not of trace-class. Our method is based on the Berezin-Weyl calculus on $\mathfrak{p}^2 \times o(\varphi_0)$ obtained by combining the Berezin calculus s with the usual Weyl correspondence $\mathcal{W}_0$ on $\mathfrak{p}^2 \simeq \mathbb{R}^{2n}$. Let us recall the definition of the Berezin-Weyl calculus, see [10].

We say that a smooth function $f\colon (T,S,\varphi) \to f(T,S,\varphi)$ is a symbol on $\mathfrak{p}^2 \times o(\varphi_0)$ if for each $(T,S) \in \mathfrak{p}^2$ the function $\varphi \to f(T,S,\varphi)$ is the symbol in the Berezin calculus on $o(\varphi_0)$ of an operator $\hat{f}(T,S)$ on E. Moreover, a symbol f on $\mathfrak{p}^2 \times o(\varphi_0)$ is called an S-symbol if the function $\hat{f}$ belongs to the Schwartz space of rapidly decreasing smooth functions on $\mathfrak{p}^2$ with values in $\mathrm{End}(E)$.

For any S-symbol f on $\mathfrak{p}^2 \times o(\varphi_0)$ we define the operator $\mathcal{W}(f)$ on $\mathcal{H}$ by the equation

$$(\mathcal{W}(f)\phi)(T) = (2\pi)^{-n} \int_{\mathfrak{p}^2} e^{i\langle S,Z\rangle} \hat{f}\Big(T + \frac{1}{2}S, Z\Big)\phi(T+S)\,\mathrm{d}S\,\mathrm{d}Z$$

for each $\phi \in C_0^\infty(\mathfrak{p}, E)$.

The Berezin-Weyl calculus can be extended to much larger classes of symbols [20], in particular to polynomial symbols. We say that a symbol f on $\mathfrak{p}^2 \times o(\varphi_0)$ is a P-symbol if the function $\hat{f}(T,S)$ is polynomial in S. Let f be the P-symbol defined by $f(T,S,\varphi) = u(T)S^\alpha$ where $u \in C^\infty(\mathfrak{p}, E)$ and with the usual notation $S^\alpha := s^{\alpha_1}s^{\alpha_2}\dots s^{\alpha_n}$ for each multi-index $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n)$. Then, by [31], we have

$$(\mathcal{W}(f)\phi)(T)=\Big(i\frac{\partial}{\partial S}\Big)^\alpha\Big(u\Big(T+\frac{1}{2}S\Big)\phi(T+S)\Big)\Big|_{S=0}.$$

In particular, if $f(T,S,\varphi)=u(T)$ then $(\mathcal{W}(f)\phi)(T)=u(T)\phi(T)$ and if $f(T,S,\varphi)=u(T)s_k$ then

$$(\mathcal{W}(f)\phi)(T)=i\Big(\frac{1}{2}\partial_k u(T)\phi(T)+u(T)\partial_k\phi(T)\Big)$$

where ∂_k denotes the partial derivative with respect to the variable t_k.

From this, we can deduce the following result.

Proposition 7.1 ([10])**.** *For each $X=(w,A)\in\mathfrak{g}$, the Berezin-Weyl symbol of the operator $-i\mathrm{d}\pi(X)$ is the P-symbol f_X on $\mathfrak{p}^2\times o(\varphi_0)$ given by*

$$f_X(T,S,\varphi)=\langle e(T),w\rangle+\langle\varphi,L(e(T),A)\rangle+\langle A\cdot T,S\rangle.$$

Let us introduce some additional notation. We have $\mathcal{H}=L^2(\mathfrak{p})\otimes E$. For each $\phi_0\in L^2(\mathfrak{p})$ and each $v\in E$ we denote by $\phi_0\otimes v$ the function $x\to\phi_0(x)v$. Moreover, if A_0 is an operator on $L^2(\mathfrak{p})$ and A_1 is an operator on E then we denote by $A_0\otimes A_1$ the operator on $\mathcal{H}$ defined by $(A_0\otimes A_1)(\phi_0\otimes v)=A_0\phi_0\otimes A_1 v$.

Also, if f_0 is a complex valued function on $\mathbb{R}^{2n}$ and f_1 a complex valued function on $o(\varphi_0)$, we denote by $f_0\otimes f_1$ the function on $\mathfrak{p}^2\times o(\varphi_0)$ defined by

$$(f_0\otimes f_1)(T,S,\varphi)=f_0(T,S)f_1(\varphi)$$

for $T,S\in\mathfrak{p}$ and $\varphi\in o(\varphi_0)$.

Note that if f is a function on $\mathfrak{p}^2\times o(\varphi_0)$ of the form $f=f_0\otimes f_1$ with $f_1\in Sy(o(\varphi_0))$ (see Section 4) then it is clear that we have

$$\mathcal{W}(f)=\mathcal{W}_0(f_0)\otimes s^{-1}(f_1).$$

As in Section 5, we denote by R_0 the parity operator on $L^2(\mathfrak{p})$ defined by $(R_0\phi_0)(T)=2^n\phi_0(-T)$.

Now, let A_0 be a trace class operator on $L^2(\mathfrak{p})$ and A_1 an operator on E. Let $A=A_0\otimes A_1$. The we have

$$\begin{aligned}\mathcal{S}(A)(\xi_0)&=\mathrm{Tr}(A\Omega(\xi_0))=\mathrm{Tr}(A_0R_0\otimes A_1P(\varphi_0))=\mathrm{Tr}(A_0R_0)\,\mathrm{Tr}(A_1P(\varphi_0))\\&=\mathcal{W}_0^{-1}(A_0)(0,0)s(A_1)(\varphi_0)=\mathcal{W}^{-1}(A)(0,0,\varphi_0).\end{aligned}$$

In other words, $\mathcal{S}$ and $\mathcal{W}^{-1}$ coincide at base points. This naturally suggests to extend $\mathcal{S}$ to differential operators on $\mathcal{H}$ by using the fact that $\mathcal{W}^{-1}$ can be extended to differential operators and the invariance property. More precisely, we set

$$\mathcal{S}(A)(\xi):=\mathcal{W}^{-1}(\pi(g_\xi)A\pi(g_\xi)^{-1})(0,0,\varphi_0)$$

for each operator A on $\mathcal{H}$ such that $\mathcal{W}^{-1}(\pi(g)A\pi(g)^{-1})$ is well-defined for each $g\in G$. In the rest of this section, we give some simple examples of operators A such that $\mathcal{S}(A)$ can be defined by this way.

Proposition 7.2. *For each $X_1,X_2,\ldots,X_p\in\mathfrak{g}$, $\mathcal{S}(\mathrm{d}\pi(X_1X_2\cdots X_p))$ is well-defined.*

Proof: Let $X_1,X_2,\ldots,X_p\in\mathfrak{g}$. Let $\xi\in\mathcal{O}(\xi_0)$. Let $Y_k:=\mathrm{Ad}(g_\xi)^{-1}X_k$ for $k=1,2,\ldots,p$. Then we have

$$\pi(g_\xi)^{-1}\mathrm{d}\pi(X_1X_2\cdots X_p)\pi(g_\xi)=\mathrm{d}\pi(Y_1Y_2\cdots Y_p).$$

By induction from Proposition 3.2, we see that $\mathrm{d}\pi(Y_1Y_2\cdots Y_p)$ is a sum of operators of the form $A_0\otimes A_1$ where A_0 is a differential operator on $\mathfrak{p}$ with polynomial coefficients and A_1 an operator on E. Then $\mathcal{S}(\mathrm{d}\pi(Y_1Y_2\cdots Y_p))(\xi_0)$ is well-defined hence $\mathcal{S}(\mathrm{d}\pi(X_1X_2\cdots X_p))$ is. □

Let us remark that it is not clear whether this extension of $\mathcal{S}$ is still injective, even on the class of operators on $\mathcal{H}$ considered in the preceding proposition. However, we can compute $\mathcal{S}(\mathrm{d}\pi(X))$ for $X\in\mathfrak{g}$.

Proposition 7.3. *For each $X\in\mathfrak{g}$ and $\xi\in\mathcal{O}(\xi_0)$ we have $\mathcal{S}(\mathrm{d}\pi(X))(\xi)=i\langle\xi,X\rangle$.*

Proof: Let $X=(w,U)\in\mathfrak{g}$. By Proposition 7.1, we have

$$\begin{aligned}\mathcal{S}(\mathrm{d}\pi(X))(\xi_0)&=\mathcal{W}^{-1}(\mathrm{d}\pi(X))(0,0,\varphi_0)\\&=i\langle p_0,w\rangle+i\langle\varphi_0,p_{\mathfrak{k}_0}(U)\rangle\\&=i\langle\xi_0,X\rangle.\end{aligned}$$

Hence, for each $\xi\in\mathcal{O}(\xi_0)$ we have

$$\begin{aligned}\mathcal{S}(\mathrm{d}\pi(X))(\xi)&=\mathcal{S}(\mathrm{d}\pi(X))(g_\xi\cdot\xi_0)=\mathcal{S}(\mathrm{d}\pi(\mathrm{Ad}(g_\xi^{-1})X)(\xi_0)\\&=i\langle\xi_0,\mathrm{Ad}(g_\xi^{-1})X\rangle=i\langle\xi,X\rangle.\end{aligned}$$

□

8. Examples

8.1 The Poincaré group. Here we take $V=\mathbb{R}^{n+1}$ and $K=SO_0(n,1)$, the identity component of $SO(n,1)$. Then G is the (generalized) Poincaré group. In this case, the Berezin-Weyl calculus $\mathcal{W}$ was investigated in [9] and [10]. The usual Poincaré group corresponds to the case $n=3$.

We recall that $SO(n,1)$ is the group of all real $(n+1)\times(n+1)$ matrices of determinant 1 leaving invariant the bilinear form on V defined by

$$\langle p,p'\rangle=-\left(\sum_{k=1}^{n}p_i\,p'_i\right)+p_{n+1}\,p'_{n+1}.$$

We can identify V^* to V by using this bilinear form.

Denoting by $(e_1, e_2, \dots, e_{n+1})$ the standard basis of $\mathbb{R}^{n+1}$, we take $p_0 = me_{n+1}$ where $m > 0$. Then K_0 is the subgroup of K consisting of all matrices of the form $\left(\begin{smallmatrix} k_0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ for $k_0 \in SO(n, \mathbb{R})$ and the orbit $Z(p_0)$ is the sheet of the hyperboloid $\langle p, p \rangle = m^2$ defined by $p_{n+1} > 0$. On the other hand, $\mathfrak{p}$ consists of all matrices of the form $\left(\begin{smallmatrix} 0 & b^t \\ b & 0 \end{smallmatrix}\right)$ for $b \in \mathbb{R}^n$.

Also, we can take V_0 to be the space generated by $e_1, e_2, \dots, e_n$ since $\{v \in V : v \wedge p_0 = 0\}$ is here the line generated by e_{n+1}.

We can verify that for each $T = \left(\begin{smallmatrix} 0 & b^t \\ b & 0 \end{smallmatrix}\right) \in \mathfrak{p}$ we have

$$e(T) = m\Big(\frac{\sinh|b|}{|b|} b_1, \dots, \frac{\sinh|b|}{|b|} b_n, \cosh|b|\Big).$$

Then, identifying V_0^* with V_0 by using the restriction of $\langle \cdot, \cdot \rangle$ to V_0, we can write

$$\gamma(T) = e(T) - e(-T)|_{V_0} = 2m \frac{\sinh|b|}{|b|} b.$$

Hence γ is clearly a diffeomorphism $\mathfrak{p} \to V_0^*$ whose inverse is given by $p \to T = \left(\begin{smallmatrix} 0 & b^t \\ b & 0 \end{smallmatrix}\right)$ with $b_k = \frac{\sinh^{-1}|p|}{|p|} p_k$ for $k = 1, 2, \dots, n$. Consequently, we see that the hypothesis of Section 2 is satisfied and then our method applies in this case.

In fact, we can also obtain a Stratonovich-Weyl correspondence for π by modifying suitably Ω. However, this needs precise computations of some Jacobians which are difficult to perform in the general situation considered in the present paper.

8.2 The group $su(n,1) \rtimes SU(n,1)$. Let $K = SU(n,1)$ and $V = \mathfrak{k} = su(n,1)$, the action of K on V being the adjoint action. Then we can identify V^* to V by using the bilinear form on V defined by $\langle X, Y \rangle = \frac{1}{n+1} \operatorname{Tr}(XY)$.

We take $p_0 = im \left(\begin{smallmatrix} -n & 0 \\ 0 & I_n \end{smallmatrix}\right)$ with $m \neq 0$. Then $K_0 = S(U(n) \times U(1))$ and $\mathfrak{p}$ consists of all matrices of the form $\left(\begin{smallmatrix} 0 & b^* \\ b & 0 \end{smallmatrix}\right)$ with $b \in \mathbb{C}^n$.

A simple calculation shows that for each $T \in \mathfrak{p}$ we have

$$e(T) = im \begin{pmatrix} -nI_n - (n+1)\dfrac{\sinh^2|b|}{|b|^2} b^* b & (n+1)\dfrac{\cosh|b|\sinh|b|}{|b|} b^* \\ -(n+1)\dfrac{\cosh|b|\sinh|b|}{|b|} b & (n+1)\cosh^2|b| - n \end{pmatrix}.$$

On the other hand, for each $v \in V$ and $p \in V^* \cong V$ one has $v \wedge p = [v, p]$ and then $v \wedge p_0 = 0$ if and only if $v \in \mathfrak{k}_0$. Thus we can take $V_0 = \mathfrak{p}$ and we have

$$\gamma(T) = im \begin{pmatrix} 0 & 2(n+1)\dfrac{\cosh|b|\sinh|b|}{|b|} b^* \\ -2(n+1)\dfrac{\cosh|b|\sinh|b|}{|b|} b & 0 \end{pmatrix}.$$

Hence γ is a diffeomorphism of $\mathfrak{p}$, the hypothesis of Section 2 is fulfilled here and our construction of the invariant symbolic calculus also works in this case.

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B. Cahen:
Département de mathématiques, Université de Lorraine, Site de Metz, UFR-MIM, Bâtiment A, 3 rue Augustin Fresnel, BP 45112, 57073 METZ Cedex 03, France

E-mail: benjamin.cahen@univ-lorraine.fr

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