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REPRESENTATIONS OF THE GENERAL LINEAR GROUP OVER SYMMETRY CLASSES OF POLYNOMIALS

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Abstract. Let V be the complex vector space of homogeneous linear polynomials in the variables $x_1, \ldots, x_m$. Suppose G is a subgroup of S_m, and χ is an irreducible character of G. Let $H_d(G,\chi)$ be the symmetry class of polynomials of degree d with respect to G and χ.

For any linear operator T acting on V, there is a (unique) induced operator $K_\chi(T) \in \operatorname{End}(H_d(G,\chi))$ acting on symmetrized decomposable polynomials by

$$K_\chi(T)(f_1 * f_2 * \ldots * f_d) = Tf_1 * Tf_2 * \ldots * Tf_d.$$

In this paper, we show that the representation $T \mapsto K_\chi(T)$ of the general linear group $GL(V)$ is equivalent to the direct sum of $\chi(1)$ copies of a representation (not necessarily irreducible) $T \mapsto B^G_\chi(T)$.

Keywords: symmetry class of polynomials; general linear group; representation; irreducible character; induced operator

MSC 2010: 20C15, 15A69, 05E05

1. Introduction

Symmetry classes of polynomials are introduced in [8], by Shahryari. In [7], Rodtes studied symmetry classes of polynomials associated with the irreducible characters of the semidihedral group. In [2], Zamani and Babaei studied symmetry classes of polynomials with respect to irreducible characters of the direct product of permutation groups. In [1], [3], [9], [10], they computed the dimensions of symmetry classes of polynomials with respect to irreducible characters of dihedral, symmetric, dicyclic and cyclic groups, respectively. Also, they discussed the existence of an o-basis for these classes. In [6], [11], the authors studied some algebraic and geometric properties

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of induced operators on symmetry classes of polynomials with respect to linear characters of an arbitrary group and the space of homogeneous polynomials, respectively. In this paper, we study representations of the general linear group over symmetry classes of polynomials. We first give a review of symmetry classes of polynomials.

Let $H_d[x_1,\ldots,x_m]$ be the complex space of homogeneous polynomials of degree d with independent commuting variables $x_1,\ldots,x_m$. Let $\Gamma_{d,m}$ be the set of all sequences $\alpha=(\alpha_1,\ldots,\alpha_d)$ such that $1\leqslant\alpha_i\leqslant m$ for any $1\leqslant i\leqslant d$. Suppose G is a subgroup of S_m. Then G acts on $\Gamma_{d,m}$ by

$$\sigma\alpha=(\sigma(\alpha_1),\ldots,\sigma(\alpha_d)),$$

where $\alpha=(\alpha_1,\ldots,\alpha_d)\in\Gamma_{d,m}$ and $\sigma\in G$. Also the symmetric group S_d acts on $\Gamma_{d,m}$ by

$$\alpha\sigma=(\alpha(\sigma(1)),\ldots,\alpha(\sigma(d))),\quad \alpha\in\Gamma_{d,m},\ \sigma\in S_d.$$

Let $G_{d,m}$ be the subset of $\Gamma_{d,m}$ consisting of all nondecreasing sequences. We define the weight of $\alpha\in G_{d,m}$ by

$$\omega(\alpha)=\prod_{i=1}^{d}x_{\alpha(i)}.$$

Then, the set $\mathfrak{B}=\{\omega(\alpha)\colon\ \alpha\in G_{d,m}\}$ is a basis of $H_d[x_1,\ldots,x_m]$. An inner product on $H_d[x_1,\ldots,x_m]$ is defined by

$$(\omega(\alpha),\omega(\beta))=\delta_{\alpha,\beta}\nu(\alpha),$$

where $\nu(\alpha)$ is the product of the factorials of the multiplicities of the distinct integers appearing in α. If $d=1$, then the inner product on $H_d[x_1,\ldots,x_m]$ is $(x_i,x_j)=\delta_{ij}$.

For any $\sigma\in S_m$, we define the linear operator $Q(\sigma)$ by

$$Q(\sigma)q(x_1,\ldots,x_m)=q^{\sigma}(x_1,\ldots,x_m)=q(x_{\sigma^{-1}(1)},\ldots,x_{\sigma^{-1}(m)}),$$

for all $q(x_1,\ldots,x_m)\in H_d[x_1,\ldots,x_m]$.

We prove that $\sigma\mapsto Q(\sigma)$ is a unitary representation of G.

Theorem 1.1. *Suppose G is a subgroup of S_m. Then $\sigma\mapsto Q(\sigma)$ is a unitary representation of G.*

P r o o f. Observe that

$$\begin{aligned}Q(\tau)Q(\sigma)q(x_1,\ldots,x_m)&=Q(\tau)q(x_{\sigma^{-1}(1)},\ldots,x_{\sigma^{-1}(m)})\\&=q(x_{\sigma^{-1}(\tau^{-1}(1))},\ldots,x_{\sigma^{-1}(\tau^{-1}(m))})\\&=q(x_{(\tau\sigma)^{-1}(1)},\ldots,x_{(\tau\sigma)^{-1}(m)})\\&=Q(\tau\sigma)q(x_1,\ldots,x_m),\end{aligned}$$

thus $Q(\tau)Q(\sigma) = Q(\tau\sigma)$ for all $\tau, \sigma \in G$. Hence $\sigma \mapsto Q(\sigma)$ is a representation of G on the vector space $H_d[x_1, \ldots, x_m]$.

For any $\alpha, \beta \in G_{d,m}$, we have

$$\begin{aligned}(Q(\sigma)\omega(\alpha), \omega(\beta)) &= (\omega(\alpha)^{\sigma}, \omega(\beta)) \\ &= (\omega(\sigma^{-1}\alpha), \omega(\beta)) \\ &= (\omega(\sigma^{-1}\alpha\tau), \omega(\beta)) \qquad \tau \in S_d,\ \sigma^{-1}\alpha\tau \in G_{d,m} \\ &= \delta_{\sigma^{-1}\alpha\tau,\beta}\nu(\beta),\end{aligned}$$

the last expression is equal to zero or $\nu(\beta)$, in this case $\alpha\tau = \sigma\beta$. On the other hand,

$$\begin{aligned}(\omega(\alpha), Q(\sigma)^{-1}\omega(\beta)) = (\omega(\alpha), \omega(\beta)^{\sigma^{-1}}) &= (\omega(\alpha), \omega(\sigma\beta)) \\ &= (\omega(\alpha), \omega(\alpha)) = \nu(\alpha) = \nu(\beta).\end{aligned}$$

Hence

$$(Q(\sigma)\omega(\alpha), \omega(\beta)) = (\omega(\alpha), Q(\sigma)^{-1}\omega(\beta)).$$

Therefore $Q(\sigma)^* = Q(\sigma)^{-1}$, $\sigma \in G$, so the result holds. □

Definition 1.1. Suppose G is a subgroup of the symmetric group S_m. Let χ be an irreducible complex character of G. The image of $H_d[x_1, \ldots, x_m]$ under the symmetrizer,

$$S_\chi = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma)Q(\sigma)$$

is called the *symmetry class of polynomials of degree d with respect to G and χ*, and it is denoted by $H_d(G, \chi)$.

For any $q \in H_d[x_1, \ldots, x_m]$,

$$q^* = S_\chi(q)$$

is called a *symmetrized polynomial* with respect to G and χ. If χ is a linear character of G, then $H_d(G, \chi)$ is the set of all $q \in H_d[x_1, \ldots, x_m]$ such that for any $\sigma \in G$, we have $q^\sigma = \chi(\sigma^{-1})q$.

Lemma 1.1. *Let G be a subgroup of S_m. Let χ be an irreducible character of G. If $\tau \in G$ is fixed but arbitrary, then $Q(\tau)S_\chi = S_\chi Q(\tau)$.*

P r o o f.

$$\begin{aligned}Q(\tau)S_\chi &= \frac{\chi(1)}{|G|}\sum_{\sigma\in G}\chi(\sigma)Q(\tau)Q(\sigma) = \frac{\chi(1)}{|G|}\sum_{\sigma\in G}\chi(\sigma)Q(\tau\sigma)\\ &= \frac{\chi(1)}{|G|}\sum_{\sigma\in G}\chi(\tau^{-1}\sigma)Q(\sigma) = \frac{\chi(1)}{|G|}\sum_{\sigma\in G}\chi(\sigma\tau^{-1})Q(\sigma)\\ &= \frac{\chi(1)}{|G|}\sum_{\sigma\in G}\chi(\sigma)Q(\sigma\tau) = \frac{\chi(1)}{|G|}\sum_{\sigma\in G}\chi(\sigma)Q(\sigma)Q(\tau) = S_\chi Q(\tau).\end{aligned}$$

□

Observe that

$$S_\chi^* = \frac{\chi(1)}{|G|}\sum_{\sigma\in G}\overline{\chi(\sigma)}Q(\sigma)^* = \frac{\chi(1)}{|G|}\sum_{\sigma\in G}\chi(\sigma^{-1})Q(\sigma^{-1}) = S_\chi.$$

By using the orthogonality relations of characters, one can easily see that $\{S_\chi\colon \chi\in\mathrm{Irr}(G)\}$ is a complete set of orthogonal idempotent operators, where $\mathrm{Irr}(G)$ is the set of irreducible complex characters of G (see [4]).

Therefore, we have the following orthogonal direct sum decomposition:

$$H_d[x_1,\dots,x_m] = \bigoplus_{\chi\in\mathrm{Irr}(G)} H_d(G,\chi).$$

Notice that $q\in H_d(G,\chi)$ if and only if $q = \chi(1)|G|^{-1}\sum\limits_{\sigma\in G}\chi(\sigma)q^\sigma$.

For any $1\leqslant i\leqslant d$, suppose

$$f_i = \sum_{j=1}^{m} a_{ij}x_j.$$

It is clear that $f_1f_2\dots f_d\in H_d[x_1,\dots,x_m]$. The polynomial $S_\chi(f_1f_2\dots f_d)$ is called a *symmetrized decomposable polynomial* and is denoted by $f_1 * f_2 * \dots * f_d$.

Example 1.1. Let $G = S_3$. Consider the irreducible character $\chi = \mathbf{Fix} - \mathbf{1}$ of G, namely,

$$\chi(1) = 2,\quad \chi((12)) = 0,\quad \chi((123)) = -1.$$

It is easy to see that the symmetry class $H_d(G,\chi)$ consists of all polynomials of three variables which satisfy the equation

$$q(x,y,z) + q(z,x,y) + q(y,z,x) = 0.$$

Let V be the complex vector space of homogeneous linear polynomials in the variables $x_1, \ldots, x_m$. For any linear operator $T \in \mathrm{End}(V)$, there is a linear operator $P(T)$ (see [11]) acting on $H_d[x_1, \ldots, x_m]$ by

$$P(T)q(x_1, \ldots, x_m) = q(Tx_1, \ldots, Tx_m).$$

It is easy to see that $P(T)S_\chi = S_\chi P(T)$. So $H_d(G, \chi)$ is an invariant subspace of $P(T)$. Denote by $K_\chi(T)$ the restriction of $P(T)$ to $H_d(G, \chi)$. Then $K_\chi(T)$ is called the *induced operator associated with G and χ*. The induced operator $K_\chi(T)$ acts on symmetrized decomposable polynomials by

$$K_\chi(T)(f_1 * f_2 * \ldots * f_d) = Tf_1 * Tf_2 * \ldots * Tf_d.$$

Some algebraic and geometric properties of the induced operator $K_\chi(T)$ have been studied in [6]. The map $T \mapsto K_\chi(T)$ defines a representation of the general linear group $GL(V)$ over the symmetry class $H_d(G, \chi)$. The main aim of this paper is to study the representation K_χ.

2. Main results

Let χ be an irreducible character of the subgroup G of S_m. Suppose $\sigma \mapsto A(\sigma) = (a_{ij}(\sigma))$ is a representation of G that affords χ. For any $1 \leqslant i \leqslant \chi(1)$, the linear operator $S_i(G, A)$ on $H_d[x_1, \ldots, x_m]$ is defined by

$$S_i(G, A) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} a_{ii}(\sigma) Q(\sigma).$$

Theorem 2.1. *Let χ be an irreducible character of the subgroup G of S_m. Suppose $\sigma \mapsto A(\sigma) = (a_{ij}(\sigma))$ is a representation of G that affords χ. Then*

(i) $S_i(G, A) S_j(G, A) = \delta_{ij} S_i(G, A)$,

(ii) $\sum_{i=1}^{\chi(1)} S_i(G, A) = S_\chi$,

(iii) *if A is a unitary representation, then $S_i(G, A)$ is Hermitian.*

P r o o f.

(i) By using Schur relations ([5], Theorem 4.21), we have

$$\begin{aligned}
S_i(G,A)S_j(G,A) &= \Big(\frac{\chi(1)}{|G|}\Big)^2\Big(\sum_{\sigma\in G} a_{ii}(\sigma)Q(\sigma)\Big)\Big(\sum_{\tau\in G} a_{jj}(\tau)Q(\tau)\Big)\\
&= \Big(\frac{\chi(1)}{|G|}\Big)^2 \sum_{\sigma,\tau\in G} a_{ii}(\sigma)a_{jj}(\tau)Q(\sigma\tau)\\
&= \frac{\chi(1)}{|G|}\sum_{\tau\in G}\Big(\frac{\chi(1)}{|G|}\sum_{\sigma\in G} a_{ii}(\sigma)a_{jj}(\sigma^{-1}\tau)\Big)Q(\tau)\\
&= \frac{\chi(1)}{|G|}\sum_{\tau\in G}\delta_{ij}a_{ii}(\tau)Q(\tau)\\
&= \delta_{ij}S_i(G,A).
\end{aligned}$$

(ii) It is trivial.

(iii) Since for any $\sigma\in G$, $A(\sigma)$ is unitary, we have $A(\sigma)A(\sigma)^* = I$. On the other hand, $A(\sigma)A(\sigma^{-1}) = I$, hence $A(\sigma)^* = A(\sigma^{-1})$. Thus $\overline{a_{ii}(\sigma)} = a_{ii}(\sigma^{-1})$. Now, by Theorem 1.1, $S_i(G,A)$ is Hermitian. □

Definition 2.1. Let χ be an irreducible character of the subgroup G of S_m. Suppose $\sigma\mapsto A(\sigma) = (a_{ij}(\sigma))$ is a representation of G that affords χ. Denote the image of $S_i(G,A)$ by $H_d^i(G,A)$. For any $q(x_1,\dots,x_m)\in H_d[x_1,\dots,x_m]$, $1\leqslant i\leqslant\chi(1)$, we have $q\in H_d^i(G,A)$ if and only if $q = (\chi(1)/|G|)\sum\limits_{\sigma\in G} a_{ii}(\sigma)q^\sigma$.

Corollary 2.1. *Let χ be an irreducible character of the subgroup G of S_m. Suppose $\sigma\mapsto A(\sigma) = (a_{ij}(\sigma))$ is a representation of G that affords χ. Then*

$$H_d(G,\chi) = \bigoplus_{i=1}^{\chi(1)} H_d^i(G,A). \tag{2.1}$$

Moreover, if A is a unitary representation of G, then the direct sum in equation (2.1) *is orthogonal.*

P r o o f. The result is an immediate consequence of Theorem 2.1 and Definition 2.1. □

Example 2.1. Consider the representation A of S_3 that affords the character χ of S_3 from Example 1.1, as follows:

$$\begin{aligned} A(1) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A((12)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ A((23)) &= \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, \quad A((123)) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \\ A((132)) &= \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad A((13)) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}. \end{aligned}$$

Then

$$S_1(S_3, A) = \frac{1}{3}\left(Q(1) - Q((23)) - Q((132)) + Q((13))\right),$$

and

$$S_2(S_3, A) = \frac{1}{3}\left(Q(1) + Q((23)) - Q((123)) - Q((13))\right).$$

Therefore,

$$\begin{aligned} H_d^1(S_3, A) &= \{q \in H_d[x, y, z] \colon\ 2q(x, y, z) + q(x, z, y) + q(y, z, x) - q(z, y, x) = 0\}, \\ H_d^2(S_3, A) &= \{q \in H_d[x, y, z] \colon\ 2q(x, y, z) - q(x, z, y) + q(z, x, y) + q(z, y, x) = 0\}. \end{aligned}$$

Since $P(T)$ commutes with $Q(\sigma)$, $\sigma \in S_m$, it commutes with $S_i(G, A)$. Therefore, $H_d^i(G, A)$ is an invariant subspace of $P(T)$ and so it is an invariant subspace of $K_\chi(T)$. Denote by $K_A^i(T)$ the restriction of $K_\chi(T)$ to $H_d^i(G, A)$.

Theorem 2.2. *Let χ be an irreducible character of the subgroup G of S_m. Suppose $\sigma \mapsto A(\sigma) = (a_{ij}(\sigma))$ is a representation of G that affords χ. If $H_d(G, \chi) \neq 0$, then K_A^i and K_A^j are equivalent representations of $GL(V)$, $1 \leqslant i, j \leqslant \chi(1)$.*

Proof. By Lemma 1.1, $Q(\sigma)$ commutes with S_χ, so $H_d(G, \chi)$ is an invariant subspace of $Q(\sigma)$, $\sigma \in G$. Denote the restriction of $Q(\sigma)$ to $H_d(G, \chi)$ by $Q_\chi(\sigma)$, $\sigma \in G$. Hence $\sigma \mapsto Q_\chi(\sigma)$ is a representation of G and

$$\frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) Q_\chi(\sigma) = I_{H_d(G, \chi)}$$

is the identity operator on $H_d(G, \chi)$, because

$$\begin{aligned} \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) Q_\chi(\sigma) S_\chi &= \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) S_\chi Q(\sigma) \\ &= S_\chi \left(\frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) Q(\sigma) \right) = S_\chi. \end{aligned}$$

Let ξ be an irreducible character of G different from χ, $\xi \neq \chi$. Then

$$Z = \frac{\xi(1)}{|G|} \sum_{\sigma \in G} \xi(\sigma) Q_\chi(\sigma)$$

is a linear operator on $H_d(G, \chi)$. Now, by using the generalized orthogonality relation of characters, we have

$$\begin{aligned} Z = Z \circ I_{H_d(G,\chi)} &= \left(\frac{\xi(1)}{|G|} \sum_{\sigma \in G} \xi(\sigma) Q_\chi(\sigma)\right)\left(\frac{\chi(1)}{|G|} \sum_{\tau \in G} \chi(\tau) Q_\chi(\tau)\right) \\ &= \frac{\xi(1)\chi(1)}{|G|^2} \sum_{\mu \in G}\left(\sum_{\sigma \in G} \xi(\sigma)\chi(\sigma^{-1}\mu)\right) Q_\chi(\mu) \\ &= \frac{\xi(1)\chi(1)}{|G|^2} \sum_{\mu \in G} \delta_{\xi,\chi}\chi(\mu) Q_\chi(\mu) = 0. \end{aligned}$$

Hence

$$0 = \operatorname{tr}(Z) = \frac{\xi(1)}{|G|} \sum_{\sigma \in G} \xi(\sigma) \operatorname{tr}(Q_\chi(\sigma)) = \xi(1)(\bar{\xi}, \eta)_G,$$

where η is the character afforded by $\sigma \mapsto Q_\chi(\sigma)$, $\sigma \in G$. Clearly, the character η contains no irreducible constituent different from $\bar{\chi}$. It follows that there exists a basis B of $H_d(G, \chi)$ such that the matrix representation of $Q_\chi(\sigma)$ with respect to B is the direct sum of $C(\sigma) = A(\sigma^{-1})^t$ with itself $N = (\bar{\chi}, \eta)_G$ times. In other words, with respect to B, the matrix representation of $Q_\chi(\sigma)$ is

$$\text{(2.2)} \qquad [Q_\chi(\sigma)]_B = \begin{pmatrix} C(\sigma) & 0 & \dots & 0 \\ 0 & C(\sigma) & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & C(\sigma) \end{pmatrix}_{N \times N} = I_N \otimes C(\sigma).$$

Now, we compute the matrix representation of $S_i(G, A)$ with its restriction to $H_d(G, \chi)$ with respect to B. By equation (2.2) and Schur relations, we have

$$\begin{aligned} \text{(2.3)} \quad [S_i(G,A)]_B &= \frac{\chi(1)}{|G|} \sum_{\sigma \in G} a_{ii}(\sigma)([Q_\chi(\sigma)]_B) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} a_{ii}(\sigma)(I_N \otimes C(\sigma)) \\ &= I_N \otimes \left(\frac{\chi(1)}{|G|} \sum_{\sigma \in G} a_{ii}(\sigma) A(\sigma^{-1})^t\right) = I_N \otimes E_{ii}, \end{aligned}$$

where E_{ii} is the $\chi(1) \times \chi(1)$ matrix whose only nonzero entry is a 1 in position (i, i).

Suppose $T \in \operatorname{End}(V)$ is a (not necessarily invertible) linear operator on V. Partition the matrix $[K_\chi(T)]_B = (K_{st})$ into N^2 blocks K_{st} of size $\chi(1) \times \chi(1)$, $1 \leqslant s, t \leqslant N$.

Since $[K_\chi(T)]_B$ commutes with $[Q_\chi(\sigma)]_B = I_N \otimes C(\sigma)$, and C is irreducible, by Schur lemma, we deduce that K_{st} is a multiple of $I_{\chi(1)}$, $1 \leqslant s, t \leqslant N$. In other words,

$$[K_\chi(T)]_B = \begin{pmatrix} b_{11} I_{\chi(1)} & \dots & b_{1N} I_{\chi(1)} \\ \vdots & \ddots & \vdots \\ b_{N1} I_{\chi(1)} & \dots & b_{NN} I_{\chi(1)} \end{pmatrix} = B(T) \otimes I_{\chi(1)} \tag{2.4}$$

for some $N \times N$ matrix $B(T)$. By [5], page 147, there exists a permutation matrix Q such that

$$Q^t (B \otimes C) Q = C \otimes B$$

for all $B \in M_N(\mathbb{C})$ and all $C \in M_{\chi(1)}(\mathbb{C})$. Similarity by Q merely permutes the elements of the ordered basis B into a new ordered basis B'. Thus, from equation (2.3),

$$[S_i(G, A)]_{B'} = E_{ii} \otimes I_N.$$

It follows that the first N elements of B' form a basis B_1 of $H_d^1(G, A)$, the second N elements form a basis B_2 of $H_d^2(G, A)$, and so on. Applying this observation to

$$[K_\chi(T)]_{B'} = I_{\chi(1)} \otimes B(T),$$

we deduce that with respect to B_i, the matrix representation of $K_A^i(T)$ is $B(T)$, $1 \leqslant i \leqslant \chi(1)$. Therefore, K_A^i and K_A^j are equivalent. $\square$

Denote the restriction of $P(T)$ to $H_d^1(G, A)$ by $B_\chi^G(T)$, $T \in \mathrm{End}(V)$.

The matrix $B(T)$ that occurs in equation (2.4) is a matrix representation of the linear operator $B_\chi^G(T)$.

It follows from Theorem 2.2 that, as long as $H_d(G, \chi) \neq 0$, the representation $T \mapsto K_\chi(T)$, $T \in GL(V)$, is equivalent to the direct sum of $B_\chi^G(T)$ with itself $\chi(1)$ times,

$$K_\chi(T) = \bigoplus_{i=1}^{\chi(1)} B_\chi^G(T).$$

Remark 2.1. If $G = S_m$, then proving the irreducibility of the representation $T \mapsto B_\chi^G(T)$ is more complicated and it is yet an open problem as far as we know.

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