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A CHARACTERIZATION OF REFLEXIVE SPACES OF OPERATORS

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Abstract. We show that for a linear space of operators $\mathcal{M} \subseteq \mathcal{B}(\mathscr{H}_1, \mathscr{H}_2)$ the following assertions are equivalent. (i) $\mathcal{M}$ is reflexive in the sense of Loginov-Shulman. (ii) There exists an order-preserving map $\Psi = (\psi_1, \psi_2)$ on a bilattice $\mathrm{Bil}(\mathcal{M})$ of subspaces determined by $\mathcal{M}$ with $P \leqslant \psi_1(P, Q)$ and $Q \leqslant \psi_2(P, Q)$ for any pair $(P, Q) \in \mathrm{Bil}(\mathcal{M})$, and such that an operator $T \in \mathcal{B}(\mathscr{H}_1, \mathscr{H}_2)$ lies in $\mathcal{M}$ if and only if $\psi_2(P, Q)T\psi_1(P, Q) = 0$ for all $(P, Q) \in \mathrm{Bil}(\mathcal{M})$. This extends the Erdos-Power type characterization of weakly closed bimodules over a nest algebra to reflexive spaces.

Keywords: reflexive space of operators; order-preserving map

MSC 2010: 47A15

1. Introduction and preliminaries

In [3], Erdos and Power characterized the weakly closed bimodules of a nest algebra in terms of order homomorphisms on the lattice of invariant subspaces of the algebra. Deguang showed in [1] that, given any reflexive subalgebra σ-weakly generated by its rank one operators, the σ-weakly closed bimodules over the algebra could analogously be characterized in terms of order homomorphisms on the lattice of invariant subspaces of the algebra. Li and Li, see [8], Proposition 2.6, have extended the mentioned results to the realm of Banach spaces. It is worth noticing that the bimodules considered in the Erdos-Power theorems are implicitly reflexive subspaces in the sense of Loginov-Shulman (cf. [9]). The aim of the present paper is to extend this type of characterization to all such reflexive subspaces. The main result, Theorem 9, shows that for every reflexive space $\mathcal{M}$ of operators between two

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complex Hilbert spaces, there exists an order homomorphism on a bilattice of subspaces determined by $\mathcal{M}$ which describes this subspace in the sense of Erdos-Power, see [3], Theorem 1.5. We would like to point out that although some of our results could potentially be deduced from those in [2], our proofs however use an approach based on the considerably more recent notion of bilattices introduced by Shulman and Turowska in [10] and investigated of late by Kliś-Garlicka in [6], [7].

The proof of Theorem 9 requires some auxiliary results appearing in Section 2. In the rest of the present section, apart from the notation, we shall also establish the facts about bilattices needed in the sequel.

Let $\mathscr{H}$ be a complex Hilbert space, let $\mathcal{B}(\mathscr{H})$ be the Banach algebra of all bounded linear operators on $\mathscr{H}$, and let $\mathcal{P}(\mathscr{H})$ be the set of all orthogonal projections on $\mathscr{H}$. It is well known that $\mathcal{P}(\mathscr{H})$ is a lattice when endowed with the partial order relation defined for all $P_1, P_2 \in \mathscr{H}$ by $P_1 \leqslant P_2 \iff P_1\mathscr{H} \subseteq P_2\mathscr{H}$. The join $P_1 \vee P_2$ is the orthogonal projection onto $\overline{P_1\mathscr{H} + P_2\mathscr{H}}$ and the meet $P_1 \wedge P_2$ is the orthogonal projection onto $P_1\mathscr{H} \cap P_2\mathscr{H}$. In fact, $\mathcal{P}(\mathscr{H})$ is a complete lattice whose top and bottom elements are, respectively, the identity operator I and the zero operator 0.

Recall that the lattice $\operatorname{Lat}(\mathcal{U})$ of invariant subspaces of a subset $\mathcal{U}$ of $\mathcal{B}(\mathscr{H})$ is given by

$$\operatorname{Lat}(\mathcal{U}) = \{P \in \mathcal{P}(\mathscr{H});\ P^{\perp}TP = 0 \text{ for all } T \in \mathcal{U}\},$$

where $P^{\perp} = I - P$. It is clear that $\operatorname{Lat}(\mathcal{U})$ is a sublattice of $\mathcal{P}(\mathscr{H})$ which is strongly closed and therefore complete, i.e. for every subset $\mathcal{F} \subseteq \operatorname{Lat}(\mathcal{U})$, the supremum $\bigvee \mathcal{F}$ and the infimum $\bigwedge \mathcal{F}$ lie in $\operatorname{Lat}(\mathcal{U})$ (see [5]).

If $\mathcal{U} \subseteq \mathcal{B}(\mathscr{H})$ is a nonempty subset, then let $\mathcal{U}^* = \{T^*;\ T \in \mathcal{U}\}$. We say that $\mathcal{U}$ is *selfadjoint* if $\mathcal{U}^* = \mathcal{U}$. It is obvious that $P \in \operatorname{Lat}(\mathcal{U})$ if and only if $P^{\perp} \in \operatorname{Lat}(\mathcal{U}^*)$, i.e. $\operatorname{Lat}(\mathcal{U}^*) = \operatorname{Lat}(\mathcal{U})^{\perp}$.

Let $\mathscr{H}_1, \mathscr{H}_2$ be complex Hilbert spaces. We endow the Cartesian product $\mathcal{P}(\mathscr{H}_1) \times \mathcal{P}(\mathscr{H}_2)$ with the partial order $\preceq$ defined for all (P_1, Q_1), $(P_2, Q_2) \in \mathcal{P}(\mathscr{H}_1) \times \mathcal{P}(\mathscr{H}_2)$, by

$$(P_1, Q_1) \preceq (P_2, Q_2) \quad \text{if and only if } P_1 \leqslant P_2 \text{ and } Q_1 \geqslant Q_2. \tag{1.1}$$

Hence, the operations of join and meet are given, respectively, by

$$\begin{aligned}(P_1, Q_1) \vee (P_2, Q_2) &= (P_1 \vee P_2, Q_1 \wedge Q_2) \quad \text{and}\\ (P_1, Q_1) \wedge (P_2, Q_2) &= (P_1 \wedge P_2, Q_1 \vee Q_2).\end{aligned} \tag{1.2}$$

It follows that $\mathcal{P}(\mathscr{H}_1) \times \mathcal{P}(\mathscr{H}_2)$ together with $\preceq$ is a lattice as it contains all the binary joins and meets. From now on we write $\mathcal{P}(\mathscr{H}_1) \times_{\preceq} \mathcal{P}(\mathscr{H}_2)$ whenever we consider the Cartesian product to be endowed with the partial order (1.1), i.e. with the lattice structure (1.2). The corresponding notation will also be used for Cartesian products of subsets of $\mathcal{P}(\mathscr{H}_1) \times \mathcal{P}(\mathscr{H}_2)$. Unless otherwise stated, it is assumed that the partial order under consideration will always be $\preceq$.

Following [10], we call a subset $\mathcal{L}$ of $\mathcal{P}(\mathscr{H}_1) \times_{\preceq} \mathcal{P}(\mathscr{H}_2)$ a *bilattice* if it is closed under the lattice operations (1.2) and contains the pairs $(0,0)$, $(0,I)$, and $(I,0)$. Examples of bilattices are $\mathcal{P}(\mathscr{H}_1) \times_{\preceq} \mathcal{P}(\mathscr{H}_2)$, of course, and

$$\mathrm{BIL}(\mathcal{U}) = \{(P,Q) \in \mathcal{P}(\mathscr{H}_1) \times_{\preceq} \mathcal{P}(\mathscr{H}_2);\ QTP = 0 \text{ for any } T \in \mathcal{U}\}, \tag{1.3}$$

where $\mathcal{U} \subseteq \mathcal{B}(\mathscr{H}_1, \mathscr{H}_2)$ is an arbitrary nonempty set.

Recall that for a nonempty family $\mathcal{F} \subseteq \mathcal{P}(\mathscr{H})$,

$$\mathrm{Alg}(\mathcal{F}) = \{T \in \mathcal{B}(\mathscr{H});\ P^{\perp}TP = 0 \text{ for all } P \in \mathcal{F}\}$$

is a weakly closed subalgebra of $\mathcal{B}(\mathscr{H})$ that contains the identity operator. A subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathscr{H})$ is said to be *reflexive* if $\mathrm{Alg}\,\mathrm{Lat}(\mathcal{A}) = \mathcal{A}$. The notion of reflexive algebras has been generalized in several different directions; see [4] for a general view of reflexivity. The concept of reflexivity is naturally extended to spaces of operators as follows.

For a nonempty family $\mathcal{F} \subseteq \mathcal{P}(\mathscr{H}_1) \times_{\preceq} \mathcal{P}(\mathscr{H}_2)$ let

$$\mathrm{Op}(\mathcal{F}) = \{T \in \mathcal{B}(\mathscr{H}_1, \mathscr{H}_2);\ QTP = 0 \text{ for all } (P,Q) \in \mathcal{F}\}.$$

It is easily seen that $\mathrm{Op}(\mathcal{F})$ is a weakly closed linear subspace of $\mathcal{B}(\mathscr{H}_1, \mathscr{H}_2)$. A subspace $\mathcal{M} \subseteq \mathcal{B}(\mathscr{H}_1, \mathscr{H}_2)$ is said to be *reflexive* if $\mathrm{Op}\,\mathrm{BIL}(\mathcal{M}) = \mathcal{M}$. This definition is equivalent to that of Loginov and Shulman in [9], where a subspace $\mathcal{M}$ is said to be reflexive if $\mathcal{M}$ coincides with its *reflexive cover*

$$\mathrm{Ref}(\mathcal{M}) = \{S \in \mathcal{B}(\mathscr{H}_1, \mathscr{H}_2);\ Sx \in \overline{\mathcal{M}x} \text{ for all } x \in \mathscr{H}_1\}.$$

In fact, $\mathrm{Op}\,\mathrm{BIL}(\mathcal{M}) = \mathrm{Ref}(\mathcal{M})$ (cf. [10], page 298).

Remark 1. Notice that if $\mathcal{A} \subseteq \mathcal{B}(\mathscr{H})$ is an algebra containing the identity operator, then $\mathrm{Ref}(\mathcal{A}) = \mathrm{Alg}\,\mathrm{Lat}(\mathcal{A})$.

For a linear subspace $\mathcal{M} \subseteq \mathcal{B}(\mathscr{H}_1, \mathscr{H}_2)$ let

$$\mathcal{A}_{\mathcal{M}} = \{A \in \mathcal{B}(\mathscr{H}_1);\ TA \in \mathcal{M} \text{ for all } T \in \mathcal{M}\} \tag{2.1}$$

and

$$\mathcal{B}_{\mathcal{M}} = \{B \in \mathcal{B}(\mathscr{H}_2);\ BT \in \mathcal{M} \text{ for all } T \in \mathcal{M}\}. \tag{2.2}$$

It is easily seen that $\mathcal{A}_{\mathcal{M}}$ and $\mathcal{B}_{\mathcal{M}}$ are algebras containing the identity operator and that $\mathcal{M}$ is $\mathcal{B}_{\mathcal{M}}$-$\mathcal{A}_{\mathcal{M}}$-bimodule. It is clear that these are the largest subalgebras of $\mathcal{B}(\mathscr{H}_1)$ and $\mathcal{B}(\mathscr{H}_2)$, respectively, such that $\mathcal{M}$ is a bimodule over them. If $\mathcal{M}$ is closed (weakly closed), then $\mathcal{B}_{\mathcal{M}}$ and $\mathcal{A}_{\mathcal{M}}$ are closed (weakly closed). Next we show that $\mathcal{A}_{\mathcal{M}}$ and $\mathcal{B}_{\mathcal{M}}$ are reflexive whenever $\mathcal{M}$ is a reflexive space.

Proposition 1. *If $\mathcal{M} \subseteq \mathcal{B}(\mathscr{H}_1, \mathscr{H}_2)$ is a reflexive space, then $\mathcal{A}_{\mathcal{M}}$ and $\mathcal{B}_{\mathcal{M}}$ are reflexive algebras.*

Proof. It will only be shown that $\mathcal{A}_{\mathcal{M}}$ is reflexive since the reflexivity of $\mathcal{B}_{\mathcal{M}}$ can be proved similarly. In view of Remark 1, it suffices to show that $\mathrm{Ref}(\mathcal{A}_{\mathcal{M}}) = \mathcal{A}_{\mathcal{M}}$. In other words, fixing $S \in \mathrm{Ref}(\mathcal{A}_{\mathcal{M}})$, we need to show that for all $T \in \mathcal{M}$, the operator TS lies in $\mathcal{M}$. Since this is trivially satisfied for $T = 0$, henceforth we shall assume that $T \neq 0$.

Let $x \in \mathscr{H}_1$ and $\varepsilon > 0$ be arbitrary. Since $S \in \mathrm{Ref}(\mathcal{A}_{\mathcal{M}})$, there exists $A_{x,\varepsilon} \in \mathcal{A}_{\mathcal{M}}$ such that $\|Sx - A_{x,\varepsilon}x\| < \varepsilon/\|T\|$. Hence $\|TSx - TA_{x,\varepsilon}x\| \leqslant \|T\|\|Sx - A_{x,\varepsilon}x\| < \varepsilon$. The operator $TA_{x,\varepsilon}$ lies in $\mathcal{M}$ therefore we can conclude that $TS \in \mathrm{Ref}(\mathcal{M}) = \mathcal{M}$, as required. $\square$

Corollary 1. *Let $\mathcal{M}$ be a linear subspace of $\mathcal{B}(\mathscr{H}_1, \mathscr{H}_2)$. Then $\mathrm{Ref}\big(\mathcal{A}_{\mathcal{M}}\big) \subseteq \mathcal{A}_{\mathrm{Ref}(\mathcal{M})}$ and $\mathrm{Ref}\big(\mathcal{B}_{\mathcal{M}}\big) \subseteq \mathcal{B}_{\mathrm{Ref}(\mathcal{M})}$.*

Proof. Let $A \in \mathcal{A}_{\mathcal{M}}$. If $T \in \mathrm{Ref}(\mathcal{M})$, then for any $x \in \mathscr{H}_1$ and any $\varepsilon > 0$ there exists $S_{x,\varepsilon} \in \mathcal{M}$ such that $\|TAx - S_{x,\varepsilon}Ax\| < \varepsilon$. Since $S_{x,\varepsilon}A \in \mathcal{M}$, we conclude that $TA \in \mathrm{Ref}(\mathcal{M})$. By Proposition 2, the algebra $\mathcal{A}_{\mathrm{Ref}(\mathcal{M})}$ is reflexive, from which follows that $\mathrm{Ref}\big(\mathcal{A}_{\mathcal{M}}\big) \subseteq \mathrm{Ref}\big(\mathcal{A}_{\mathrm{Ref}(\mathcal{M})}\big) = \mathcal{A}_{\mathrm{Ref}(\mathcal{M})}$.

The proof of the second inclusion is similar. $\square$

Let $\mathrm{tr}(\cdot)$ be the trace functional and let $\mathcal{C}_1(\mathscr{H}) \subseteq \mathcal{B}(\mathscr{H})$ be the ideal of trace-class operators. The dual of $\mathcal{C}_1(\mathscr{H})$ can be identified with $\mathcal{B}(\mathscr{H})$ by means of the pairing $\langle C, A\rangle = \mathrm{tr}(CA^*)$ with $C \in \mathcal{C}_1(\mathscr{H})$, $A \in \mathcal{B}(\mathscr{H})$. The preannihilator of a subset

$\mathcal{U} \subseteq \mathcal{B}(\mathscr{H})$ is $\mathcal{U}_\perp = \{C \in \mathcal{C}_1(\mathscr{H});\ \operatorname{tr}(CA^*) = 0 \text{ for all } A \in \mathcal{U}\}$ and the annihilator of $\mathcal{V} \subseteq \mathcal{C}_1(\mathscr{H})$ is $\mathcal{V}^\perp = \{A \in \mathcal{B}(\mathscr{H});\ \operatorname{tr}(CA^*) = 0 \text{ for all } C \in \mathcal{V}\}$. It is obvious that $\mathcal{U}_\perp$ and $\mathcal{V}^\perp$ are linear spaces and that a linear subspace $\mathcal{M} \subseteq \mathcal{B}(\mathscr{H})$ is σ-weakly closed if and only if $\mathcal{M} = (\mathcal{M}_\perp)^\perp$.

If $\mathcal{U}$, $\mathcal{V}$ are two nonempty sets of operators, then we denote by $\mathcal{U}\mathcal{V}$ the set of all products TS, where $T \in \mathcal{U}$ and $S \in \mathcal{V}$.

Proposition 2. *Let $\mathcal{M}$ be a linear subspace of $\mathcal{B}(\mathscr{H})$. Then the following assertions hold.*

(i) $(\mathcal{A}_\mathcal{M})^* = \mathcal{B}_{\mathcal{M}^*}$.

(ii) *If $\mathcal{M}$ is σ-weakly closed, then $\mathcal{A}_\mathcal{M} = (\mathcal{M}^*\mathcal{M}_\perp)^\perp$ and $\mathcal{B}_\mathcal{M} = (\mathcal{M}\mathcal{M}_\perp)^\perp$.*

(iii) *If $\mathcal{M}$ is selfadjoint and σ-weakly closed, then $\mathcal{A}_\mathcal{M} = \mathcal{B}_\mathcal{M}$ is a C^*-algebra.*

(iv) *If $\mathcal{M}$ is selfadjoint, σ-weakly closed and reflexive, then $\mathcal{A}_\mathcal{M} = \mathcal{B}_\mathcal{M}$ is a von Neumann algebra.*

P r o o f. (i) An operator $A \in \mathcal{B}(\mathscr{H})$ lies in $(\mathcal{A}_\mathcal{M})^*$ if and only if $TA^* \in \mathcal{M}$ for every $T \in \mathcal{M}$. However this is equivalent to $AT^* \in \mathcal{M}^*$ for any $T^* \in \mathcal{M}^*$, which by the definition means that $A \in \mathcal{B}_{\mathcal{M}^*}$.

(ii) Let $A \in \mathcal{A}_\mathcal{M}$. For arbitrary $T \in \mathcal{M}$ and $C \in \mathcal{M}_\perp$ we have $\operatorname{tr}(T^*CA^*) = \operatorname{tr}(C(TA)^*) = 0$, since $TA \in \mathcal{M}$. This proves that $A \in (\mathcal{M}^*\mathcal{M}_\perp)^\perp$. On the other hand, if $A \in (\mathcal{M}^*\mathcal{M}_\perp)^\perp$ and $T \in \mathcal{M}$, then $\operatorname{tr}(C(TA)^*) = \operatorname{tr}(T^*CA^*) = 0$ for any $C \in \mathcal{M}_\perp$. Hence $TA \in (\mathcal{M}_\perp)^\perp = \mathcal{M}$. The second equality is similarly proved.

(iii) It follows from (ii) that $\mathcal{A}_\mathcal{M} = \mathcal{B}_\mathcal{M}$. By (i), the algebra $\mathcal{A}_\mathcal{M}$ is selfadjoint and closed since $\mathcal{M}$ itself is closed. Hence $\mathcal{A}_\mathcal{M}$ is a C^*-algebra.

(iv) By (iii), $\mathcal{A}_\mathcal{M} = \mathcal{B}_\mathcal{M}$ is a C^*-algebra. However, since $\mathcal{M}$ is reflexive, it is weakly closed and therefore $\mathcal{A}_\mathcal{M}$ is also weakly closed. □

3. A characterization of reflexivity

Let $\mathcal{M} \subseteq \mathcal{B}(\mathscr{H}_1, \mathscr{H}_2)$ be a linear subspace and let $\mathcal{A}_\mathcal{M}$ and $\mathcal{B}_\mathcal{M}$ be the algebras defined in (2.1)–(2.2). The associated bilattice $\operatorname{BIL}(\mathcal{M})$ (see (1.3)) is very large. For our purposes it suffices to consider a smaller bilattice to be defined below. Firstly, we state the following lemma which is just a formalization of a remark in [10], page 298. We include a short proof.

Lemma 1. *Let $\mathcal{M}$ be a linear subspace of $\mathcal{B}(\mathscr{H}_1, \mathscr{H}_2)$. For any pair $(P, Q) \in \operatorname{BIL}(\mathcal{M})$ there exists a pair $(P', Q') \in \operatorname{BIL}(\mathcal{M})$ such that $P' \in \operatorname{Lat}(\mathcal{A}_\mathcal{M})$, $Q' \in \operatorname{Lat}(\mathcal{B}_\mathcal{M})^\perp$, $P \leqslant P'$, and $Q \leqslant Q'$.*

P r o o f. Let P' be the orthogonal projection onto $\overline{\mathcal{A}_{\mathcal{M}}P\mathscr{H}_1}$ and let Q' be the orthogonal projection onto $\overline{\mathcal{B}^*_{\mathcal{M}}Q\mathscr{H}_2}$. It is obvious that $\overline{\mathcal{A}_{\mathcal{M}}P\mathscr{H}_1}$ is invariant for any $A \in \mathcal{A}_{\mathcal{M}}$ and that $\overline{\mathcal{B}^*_{\mathcal{M}}Q\mathscr{H}_2}$ is invariant for any $B^* \in \mathcal{B}^*_{\mathcal{M}}$. Hence $P' \in \mathrm{Lat}(\mathcal{A}_{\mathcal{M}})$ and $Q' \in \mathrm{Lat}(\mathcal{B}^*_{\mathcal{M}}) = \mathrm{Lat}(\mathcal{B}_{\mathcal{M}})^{\perp}$. Observe that $P\mathscr{H}_1 \subseteq \overline{\mathcal{A}_{\mathcal{M}}P\mathscr{H}_1}$ and $Q\mathscr{H}_2 \subseteq \overline{\mathcal{B}^*_{\mathcal{M}}Q\mathscr{H}_2}$, since both algebras contain the identity operator. Consequently, $P \leqslant P'$ and $Q \leqslant Q'$.

To prove that (P', Q') lies in $\mathrm{BIL}(\mathcal{M})$, we have to see that for any $T \in \mathcal{M}$, the equality $Q'TP' = 0$ holds, i.e. $TP'\mathscr{H}_1 \perp Q'\mathscr{H}_2$. Let $x \in \mathscr{H}_1$ be arbitrary. For any $\varepsilon > 0$ there exist $A_\varepsilon \in \mathcal{A}_{\mathcal{M}}$ and $x_\varepsilon \in \mathscr{H}_1$ such that $\|P'x - A_\varepsilon P x_\varepsilon\| < \varepsilon$, and therefore $\|TP'x - TA_\varepsilon P x_\varepsilon\| < \varepsilon\|T\|$. For arbitrary $B^* \in \mathcal{B}^*_{\mathcal{M}}$ and $y \in \mathscr{H}_2$ we have $\langle TA_\varepsilon P x_\varepsilon, B^*Qy\rangle = \langle QBTA_\varepsilon P x_\varepsilon, y\rangle = 0$, since $BTA_\varepsilon \in \mathcal{M}$. Hence

$$\begin{aligned} |\langle TP'x, B^*Qy\rangle| &= |\langle TP'x - TA_\varepsilon P x_\varepsilon, B^*Qy\rangle| \\ &\leqslant \|TP'x - TA_\varepsilon P x_\varepsilon\|\|B^*Qy\| < \varepsilon\|T\|\|B^*Qy\|, \end{aligned}$$

yielding $TP'x \perp \mathcal{B}^*Q\mathscr{H}_2$, from which it follows that $TP'\mathscr{H}_1 \perp Q'\mathscr{H}_2$. $\square$

Let

$$\mathrm{Bil}(\mathcal{M}) = \mathrm{BIL}(\mathcal{M}) \cap \bigl(\mathrm{Lat}(\mathcal{A}_{\mathcal{M}}) \times_{\preceq} \mathrm{Lat}(\mathcal{B}_{\mathcal{M}})^{\perp}\bigr).$$

It is clear that $\mathrm{Bil}(\mathcal{M})$ is a bilattice.

Proposition 3. *Let $\mathcal{M}$ be a linear subspace of $\mathcal{B}(\mathscr{H}_1, \mathscr{H}_2)$. Then*

$$\mathrm{Op}\,\mathrm{BIL}(\mathcal{M}) = \mathrm{Op}\,\mathrm{Bil}(\mathcal{M}).$$

P r o o f. Since $\mathrm{Bil}(\mathcal{M})$ is a subset of $\mathrm{BIL}(\mathcal{M})$, it follows that $\mathrm{Op}\,\mathrm{BIL}(\mathcal{M}) \subseteq \mathrm{Op}\,\mathrm{Bil}(\mathcal{M})$. To show that the reverse inclusion also holds, we begin by fixing an operator $T \in \mathrm{Op}\,\mathrm{Bil}(\mathcal{M})$ and a pair of projections $(P, Q) \in \mathrm{BIL}(\mathcal{M})$. By Lemma 5, there exists a pair $(P', Q') \in \mathrm{Bil}(\mathcal{M})$ such that $P \leqslant P'$ and $Q \leqslant Q'$. Hence $P'P = P$ and $QQ' = Q$. It follows that $QTP = QQ'TP'P = 0$ and therefore T lies in $\mathrm{Op}\,\mathrm{BIL}(\mathcal{M})$, as required. $\square$

Let $\mathcal{M} \subseteq \mathcal{B}(\mathscr{H}_1, \mathscr{H}_2)$ be a linear space. Define $\varphi\colon \mathrm{Lat}(\mathcal{A}_{\mathcal{M}}) \to \mathrm{Lat}(\mathcal{B}_{\mathcal{M}})^{\perp}$ by

$$\varphi(P) = \bigvee\{Q \in \mathrm{Lat}(\mathcal{B}_{\mathcal{M}})^{\perp};\ (P, Q) \in \mathrm{Bil}(\mathcal{M})\}, \tag{3.1}$$

and similarly define $\theta\colon \mathrm{Lat}(\mathcal{B}_{\mathcal{M}})^{\perp} \to \mathrm{Lat}(\mathcal{A}_{\mathcal{M}})$ by

$$\theta(Q) = \bigvee\{P \in \mathrm{Lat}(\mathcal{A}_{\mathcal{M}});\ (P, Q) \in \mathrm{Bil}(\mathcal{M})\}. \tag{3.2}$$

Observe that none of the sets appearing in (3.1)–(3.2) is empty as $(P,0)$, $(0,Q) \in \operatorname{Bil}(\mathcal{M})$ for any $P \in \operatorname{Lat}(\mathcal{A}_{\mathcal{M}})$, $Q \in \operatorname{Lat}(\mathcal{B}_{\mathcal{M}})^{\perp}$. The next proposition lists some properties of the maps φ and θ.

Proposition 4. *Let $\mathcal{M}$ be a linear subspace of $\mathcal{B}(\mathscr{H}_1, \mathscr{H}_2)$ and let φ, θ be the maps defined in* (3.1)–(3.2)*. Then the following assertions hold.*

(i) *φ and θ are order-reversing maps.*
(ii) *$(P, \varphi(P)), (\theta(Q), Q) \in \operatorname{Bil}(\mathcal{M})$ for any $P \in \operatorname{Lat}(\mathcal{A}_{\mathcal{M}})$ and $Q \in \operatorname{Lat}(\mathcal{B}_{\mathcal{M}})^{\perp}$.*
(iii) *If $\mathcal{C} \subseteq \operatorname{Lat}(\mathcal{A}_{\mathcal{M}})$ and $\mathcal{D} \subseteq \operatorname{Lat}(\mathcal{B}_{\mathcal{M}})^{\perp}$ are nonempty sets, then $\varphi(\bigvee \mathcal{C}) = \bigwedge \varphi(\mathcal{C})$ and $\theta(\bigvee \mathcal{D}) = \bigwedge \theta(\mathcal{D})$.*
(iv) *$P \leqslant \theta\varphi(P)$ and $Q \leqslant \varphi\theta(Q)$ for all $P \in \operatorname{Lat}(\mathcal{A}_{\mathcal{M}})$ and $Q \in \operatorname{Lat}(\mathcal{B}_{\mathcal{M}})^{\perp}$.*
(v) *$\varphi\theta\varphi = \varphi$ and $\theta\varphi\theta = \theta$.*

Proof. Assertions (i)–(iv) will only be proved for the map φ, since the corresponding assertions concerning the map θ can be proved similarly. For the same reason, only the first equality in (v) will be proved.

(i) If $P_1, P_2 \in \operatorname{Lat}(\mathcal{A}_{\mathcal{M}})$ are such that $P_1 \leqslant P_2$, then $P_1P_2 = P_1 = P_2P_1$. Hence, if Q is a projection in $\mathcal{P}(\mathscr{H}_2)$ with $(P_2, Q) \in \operatorname{Bil}(\mathcal{M})$, then for every $T \in \mathcal{M}$ we have $QTP_1 = QTP_2P_1 = 0$, yielding $(P_1, Q) \in \operatorname{Bil}(\mathcal{M})$. It follows that

$$\begin{aligned}\varphi(P_2) &= \bigvee\{Q \in \operatorname{Lat}(\mathcal{B}_{\mathcal{M}})^{\perp};\ (P_2, Q) \in \operatorname{Bil}(\mathcal{M})\}\\ &\leqslant \bigvee\{Q \in \operatorname{Lat}(\mathcal{B}_{\mathcal{M}})^{\perp};\ (P_1, Q) \in \operatorname{Bil}(\mathcal{M})\} = \varphi(P_1).\end{aligned}$$

(ii) Let $P \in \operatorname{Lat}(\mathcal{A}_{\mathcal{M}})$. We have to show that $\varphi(P)TP = 0$ for every $T \in \mathcal{M}$. Let $T \in \mathcal{M}$, $x \in \mathscr{H}_1$, $y \in \mathscr{H}_2$ be arbitrary, and let $Q \in \mathcal{P}(\mathscr{H}_2)$ be a projection such that $(P, Q) \in \operatorname{Bil}(\mathcal{M})$. Then $\langle TPx, Qy\rangle = \langle QTPx, y\rangle = 0$, that is to say that $TP\mathscr{H}_1 \perp Q\mathscr{H}_2$. Since $\varphi(P)$ is the orthogonal projection onto the closed linear span of all the spaces $Q\mathscr{H}_2$, where Q is an orthogonal projection in $\mathcal{P}(\mathscr{H}_2)$ such that $(P, Q) \in \operatorname{Bil}(\mathcal{M})$, we conclude that $TP\mathscr{H}_1 \perp \varphi(P)\mathscr{H}_2$, i.e. $\varphi(P)TP = 0$.

(iii) Let $\mathcal{C} \subseteq \operatorname{Lat}(\mathcal{A}_{\mathcal{M}})$ be a nonempty set. Then for all $P \in \mathcal{C}$, $P \leqslant \bigvee \mathcal{C}$ and, since $\operatorname{Lat}(\mathcal{A}_{\mathcal{M}})$ is complete, $\bigvee \mathcal{C} \in \operatorname{Lat}(\mathcal{A}_{\mathcal{M}})$. It follows that for all $P \in \mathcal{C}$, $\varphi(\bigvee \mathcal{C}) \leqslant \varphi(P)$, as φ is an order-reversing map. Therefore $\varphi(\bigvee \mathcal{C}) \leqslant \bigwedge \varphi(\mathcal{C})$.

To show that this inequality can be reversed, we shall prove firstly that $\big(\bigvee \mathcal{C}, \bigwedge \varphi(\mathcal{C})\big) \in \operatorname{Bil}(\mathcal{M})$. Let $T \in \mathcal{M}$ be arbitrary. Then for every $P \in \mathcal{C}$ we have $\bigwedge \varphi(\mathcal{C}) \leqslant \varphi(P)$, from which it follows that $\big(\bigwedge \varphi(\mathcal{C})\big)\varphi(P) = \bigwedge \varphi(\mathcal{C})$. Hence, for all $P \in \mathcal{C}$,

$$\big(\bigwedge \varphi(\mathcal{C})\big)TP = \big(\bigwedge \varphi(\mathcal{C})\big)\varphi(P)TP = 0$$

and, consequently, $\big(\bigwedge \varphi(\mathcal{C})\big)T\big(\bigvee \mathcal{C}\big) = 0$, i.e. $\big(\bigvee \mathcal{C}, \bigwedge \varphi(\mathcal{C})\big) \in \operatorname{Bil}(\mathcal{M})$. It follows, by the definition of $\varphi(\bigvee \mathcal{C})$, that $\bigwedge \varphi(\mathcal{C}) \leqslant \varphi(\bigvee \mathcal{C})$.

(iv) Let $P \in \operatorname{Lat}(\mathcal{A}_{\mathcal{M}})$. By assertion (ii) we have $(P, \varphi(P))$, $(\theta(\varphi(P)), \varphi(P)) \in \operatorname{Bil}(\mathcal{M})$. Since by the definition (3.1), the projection $\theta(\varphi(P))$ is the largest $P' \in \operatorname{Lat}(\mathcal{A}_{\mathcal{M}})$ such that $(P', \varphi(P)) \in \operatorname{Bil}(\mathcal{M})$, we conclude that $P \leqslant \theta\bigl(\varphi(P)\bigr)$.

(v) Let $P \in \operatorname{Lat}(\mathcal{A}_{\mathcal{M}})$ be arbitrary. By assertion (iv), we know that $\varphi(P) \leqslant \varphi\theta\varphi(P)$. Moreover, since by (ii) of this proposition, $(P, \varphi(P))$ and $(\theta\varphi(P), \varphi\theta\varphi(P))$ lie in the bilattice $\operatorname{Bil}(\mathcal{M})$, we have $\bigl(P \wedge \theta\varphi(P), \varphi(P) \vee \varphi\theta\varphi(P)\bigr) \in \operatorname{Bil}(\mathcal{M})$. Notice however that (iv) implies $P \wedge \theta\varphi(P) = P$ and $\varphi(P) \vee \varphi\theta\varphi(P) = \varphi\theta\varphi(P)$. Thus, $(P, \varphi\theta\varphi(P)) \in \operatorname{Bil}(\mathcal{M})$. By the definition of φ, the projection $\varphi(P)$ is the largest $Q \in \operatorname{Lat}(\mathcal{B}_{\mathcal{M}})^{\perp}$ having the property $(P, Q) \in \operatorname{Bil}(\mathcal{M})$. Hence, $\varphi\theta\varphi(P) \leqslant \varphi(P)$. Consequently, for all $P \in \operatorname{Lat}(\mathcal{A}_{\mathcal{M}})$ we have $\varphi\theta\varphi(P) = \varphi(P)$. $\square$

Let $\Psi_1, \Psi_2\colon \operatorname{Bil}(\mathcal{M}) \to \operatorname{Bil}(\mathcal{M})$ be defined by

$$\begin{aligned} \Psi_1(P,Q) &= (\theta\varphi(P), \varphi(P)) \quad \text{and} \\ \Psi_2(P,Q) &= (\theta(Q), \varphi\theta(Q)), \quad (P,Q) \in \operatorname{Bil}(\mathcal{M}). \end{aligned} \tag{3.3}$$

Observe that Proposition 7 (ii) guarantees that the maps Ψ_1 and Ψ_2 are well defined.

Corollary 2. *Let $\mathcal{M}$ be a linear subspace of $\mathcal{B}(\mathscr{H}_1, \mathscr{H}_2)$ and let $\Psi_1, \Psi_2\colon \operatorname{Bil}(\mathcal{M}) \to \operatorname{Bil}(\mathcal{M})$ be the maps defined in* (3.3). *Then Ψ_1, Ψ_2 are order-preserving maps and $\Psi_1(\operatorname{Bil}(\mathcal{M})) = \Psi_2(\operatorname{Bil}(\mathcal{M}))$.*

Proof. It easily follows from Proposition 7 (i) that Ψ_1 and Ψ_2 are order-preserving maps. The coincidence of the images of Ψ_1 and Ψ_2 is an immediate consequence of Proposition 7 (v). $\square$

We are now able to prove our main result.

Theorem 1. *Let $\mathcal{M}$ be a linear subspace of $\mathcal{B}(\mathscr{H}_1, \mathscr{H}_2)$ and let $\mathcal{A}_{\mathcal{M}}$, $\mathcal{B}_{\mathcal{M}}$ be the algebras defined in* (2.1)–(2.2). *The following assertions are equivalent.*

(i) *$\mathcal{M}$ is a reflexive space.*

(ii) *There exists a map $\Psi = (\psi_1, \psi_2)\colon \operatorname{Bil}(\mathcal{M}) \to \operatorname{Bil}(\mathcal{M})$ such that $P \leqslant \psi_1(P,Q)$ and $Q \leqslant \psi_2(P,Q)$ for any pair $(P,Q) \in \operatorname{Bil}(\mathcal{M})$, and*

$$\mathcal{M} = \{T \in \mathcal{B}(\mathscr{H}_1, \mathscr{H}_2);\ \psi_2(P,Q) T \psi_1(P,Q) = 0 \text{ for all } (P,Q) \in \operatorname{Bil}(\mathcal{M})\}.$$

(iii) *There exists a map $\psi_1\colon \operatorname{Lat}(\mathcal{B}_{\mathcal{M}})^{\perp} \to \operatorname{Lat}(\mathcal{A}_{\mathcal{M}})$ such that $P \leqslant \psi_1(Q)$ for any pair $(P,Q) \in \operatorname{Bil}(\mathcal{M})$, and*

$$\mathcal{M} = \{T \in \mathcal{B}(\mathscr{H}_1, \mathscr{H}_2);\ QT\psi_1(Q) = 0 \text{ for all } Q \in \operatorname{Lat}(\mathcal{B}_{\mathcal{M}})^{\perp}\}.$$

(iv) *There exists a map* $\psi_2\colon \operatorname{Lat}(\mathcal{A}_{\mathcal{M}}) \to \operatorname{Lat}(\mathcal{B}_{\mathcal{M}})^{\perp}$ *such that* $Q \leqslant \psi_2(P)$ *for any pair* $(P,Q) \in \operatorname{Bil}(\mathcal{M})$, *and*

$$\mathcal{M} = \{T \in \mathcal{B}(\mathscr{H}_1, \mathscr{H}_2);\ \psi_2(P)TP = 0 \textit{ for all } P \in \operatorname{Lat}(\mathcal{A}_{\mathcal{M}})\}.$$

P r o o f. Firstly we show that (i) $\iff$ (ii). Assume that $\mathcal{M}$ is a reflexive space. Let Ψ be the map Ψ_1 defined in (3.3), and let $\mathcal{F} = \Psi(\operatorname{Bil}(\mathcal{M}))$. Clearly, $\mathcal{F} \subseteq \operatorname{Bil}(\mathcal{M})$ and therefore $\operatorname{Op}(\mathcal{F}) \supseteq \operatorname{Op}\operatorname{Bil}(\mathcal{M}) = \mathcal{M}$.

To reverse the inclusion, fix $T \in \operatorname{Op}(\mathcal{F})$. Observe that by Proposition 7 (iv) for any pair $(P,Q) \in \operatorname{Bil}(\mathcal{M})$, $P \leqslant \theta\varphi(P) = \psi_1(P,Q)$ and, by the definition of the map φ, $Q \leqslant \varphi(P) = \psi_2(P,Q)$. Hence, for all $(P,Q) \in \operatorname{Bil}(\mathcal{M})$, $P = \theta\varphi(P)P$, $Q = Q\varphi(P)$ and, consequently,

$$QTP = Q\varphi(P)T\theta\varphi(P)P = 0.$$

It follows that $T \in \operatorname{Op}\operatorname{Bil}(\mathcal{M}) = \mathcal{M}$, as required.

Conversely, suppose that there exists a map $\Psi = (\psi_1, \psi_2)$ as stated in (ii). It has to be shown that $\mathcal{M} = \operatorname{Op}\operatorname{Bil}(\mathcal{M})$. Since it is clear that $\mathcal{M} \subseteq \operatorname{Op}\operatorname{Bil}(\mathcal{M})$, it remains to show that $\mathcal{M} \supseteq \operatorname{Op}\operatorname{Bil}(\mathcal{M})$. Let $S \in \operatorname{Op}\operatorname{Bil}(\mathcal{M})$ be arbitrary. Hence, for any pair $(P', Q') \in \operatorname{Bil}(\mathcal{M})$ we have $Q'SP' = 0$. In particular, since for $(P,Q) \in \operatorname{Bil}(\mathcal{M})$, the image $(\psi_1(P,Q), \psi_2(P,Q))$ lies also in $\operatorname{Bil}\mathcal{M}$, it follows that $\psi_2(P,Q)S\psi_1(P,Q) = 0$. Finally, this yields that S lies in the set

$$\{T \in \mathcal{B}(\mathscr{H}_1, \mathscr{H}_2);\ \psi_2(P,Q)T\psi_1(P,Q) = 0 \text{ for all } (P,Q) \in \operatorname{Bil}(\mathcal{M})\},$$

which coincides with $\mathcal{M}$, by the assumption.

The remaining equivalences are proved similarly. Notice that to prove the implication (i) $\Rightarrow$ (iii) and (i) $\Rightarrow$ (iv), we set $\psi_1 = \theta$ and $\psi_2 = \varphi$, respectively. $\square$

Observe that the maps appearing in the equalities characterizing a reflexive space $\mathcal{M}$ in Theorem 9 need not be unique (see [3], Remark, page 223). In particular, the map Ψ in Theorem 9 (ii) can be chosen to be order-preserving.

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