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SERRIN-TYPE REGULARITY CRITERION FOR THE NAVIER-STOKES EQUATIONS INVOLVING ONE VELOCITY AND ONE VORTICITY COMPONENT

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Abstract. We consider the Cauchy problem for the three-dimensional Navier-Stokes equations, and provide an optimal regularity criterion in terms of u_3 and ω_3, which are the third components of the velocity and vorticity, respectively. This gives an affirmative answer to an open problem in the paper by P. Penel, M. Pokorný (2004).

Keywords: regularity criterion; Navier-Stokes equation

MSC 2010: 35B65, 35Q30, 76D03

1. Introduction

In this paper, we consider the Cauchy problem for the three-dimensional (3D) Navier-Stokes equations

$$\begin{cases} \mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla\pi = \mathbf{0}, \\ \nabla\cdot\mathbf{u} = 0, \\ \mathbf{u}(0) = \mathbf{u}_0, \end{cases} \tag{1.1}$$

where $\mathbf{u} = (u_1, u_2, u_3)$ is the fluid velocity field, π is a scalar pressure, $\nu > 0$ is the kinematic viscosity and is assumed to be 1 in the rest of the paper, $\mathbf{u}_0$ is the prescribed initial data satisfying $\nabla\cdot\mathbf{u}_0 = 0$ in the distributional sense, and $\mathbf{u}\cdot\nabla = \sum_{i=1}^{3} u_i\partial_i$ with $\partial_i = \partial/\partial x_i$.

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It is well-known that (1.1) possesses a global weak solution

$$\mathbf{u} \in L^\infty(0,\infty; L^2(\mathbb{R}^3)) \cap L^2(0,\infty; H^1(\mathbb{R}^3))$$

for each set of initial data $\mathbf{u}_0$ of finite energy, see Leray [8] and Hopf [5]. However, whether or not such a weak solution is regular and unique is still a challenging open problem. Consequently, various criteria ensuring the regularity of the solutions are proposed. The classical Prodi-Serrin conditions (see [4], [9], [12], [13]) state that if

$$\mathbf{u} \in L^p(0,T; L^q(\mathbb{R}^3)), \quad \frac{2}{p} + \frac{3}{q} = 1, \ 3 \leqslant q \leqslant \infty, \tag{1.2}$$

then the solution is smooth on $(0,T)$.

Later on, da Veiga in [1] showed a Serrin-type regularity criterion involving the velocity gradient (or vorticity)

$$\nabla\mathbf{u} \in L^p(0,T; L^q(\mathbb{R}^3)), \quad \frac{2}{p} + \frac{3}{q} = 2, \ \frac{3}{2} \leqslant q \leqslant \infty. \tag{1.3}$$

Due to the divergence-free condition, refinements of (1.2) and (1.3) attract many authors' attention, interested readers are referred to [3], [2], [6], [7], [11], [10], [14], [17], [18], [19], [20] and references therein.

In particular, Penel and Pokorný in [10], Theorem 1 (b) and Remark 1, discovered the following regularity criterion:

$$\begin{gathered} u_3 \in L^p(0,T; L^q(\mathbb{R}^3)); \quad \partial_2 u_1, \partial_1 u_2 \in L^r(0,T; L^s(\mathbb{R}^3)), \\ \frac{2}{p} + \frac{3}{q} = 1, \ 3 < q \leqslant \infty; \quad \frac{2}{r} + \frac{3}{s} = 2, \ 2 \leqslant s \leqslant 3; \end{gathered} \tag{1.4}$$

and remarked that it is an interesting open problem whether we could make assumptions on $\omega_3 = \partial_1 u_2 - \partial_2 u_1$ instead of $\partial_2 u_1$ and $\partial_1 u_2$, where ω_3 is the third component of the vorticity $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3) = \nabla \times \mathbf{u}$. In this direction, Zhang et al. in [18], Theorem 1.2, proved the regularity of the solution under the condition

$$\begin{gathered} \partial_3 u_3 \in L^p(0,T; L^q(\mathbb{R}^3)); \quad \omega_3 \in L^r(0,T; L^s(\mathbb{R}^3)), \\ \frac{2}{p} + \frac{3}{q} = 2, \ \frac{3}{2} < q \leqslant \infty; \quad \frac{2}{r} + \frac{3}{s} = 2, \ \frac{3}{2} < s \leqslant \infty. \end{gathered} \tag{1.5}$$

Remark 1.1. In [18], Theorem 1.4, the authors also provide a regularity criterion involving u_3 and ω_3. It reads

$$\begin{gathered} u_3 \in L^p(0,T; L^q(\mathbb{R}^3)); \quad \omega_3 \in L^r(0,T; L^s(\mathbb{R}^3)), \\ \frac{2}{p} + \frac{3}{q} = 1, \ 3 < q \leqslant \infty; \quad \frac{2}{r} + \frac{3}{s} = 2, \ \frac{3}{2} < s \leqslant 3. \end{gathered} \tag{1.6}$$

However, (1.6) is not right. We thank Dr. X. J. Jia for pointing out that [18], equation (4.1), is not correct, in fact, one more term should be added, and its correct form should be

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\mathbf{u}\|_{L^2}^2+\nu\|\Delta\mathbf{u}\|_{L^2}^2\\
&\qquad=\int_{\mathbb{R}^3}(\omega_2u_3-\omega_3u_2)\Delta u_1\,\mathrm{d}x+\int_{\mathbb{R}^3}(\omega_3u_1-\omega_1u_3)\Delta u_2\,\mathrm{d}x\\
&\qquad\quad+\int_{\mathbb{R}^3}[(\mathbf{u}\cdot\nabla)u_3]\cdot\Delta u_3\,\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^3}\partial_1|\mathbf{u}|^2\cdot\Delta u_1+\partial_2|\mathbf{u}|^2\cdot\Delta u_2\,\mathrm{d}x,
\end{aligned}$$

where the first three terms can be estimated as in [18], but the last term was neglected and was not treated. Although we can reformulate it as

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^3}(\partial_1|\mathbf{u}|^2\cdot\Delta u_1+\partial_2|\mathbf{u}|^2\cdot\Delta u_2)\,\mathrm{d}x\\
&\qquad=-\frac{1}{2}\int_{\mathbb{R}^3}(\Delta|\mathbf{u}|^2\cdot\partial_1u_1+\Delta|\mathbf{u}|^2\cdot\partial_2u_2)\,\mathrm{d}x\\
&\qquad=\frac{1}{2}\int_{\mathbb{R}^3}\Delta|\mathbf{u}|^2\cdot\partial_3u_3\,\mathrm{d}x\\
&\qquad=\int_{\mathbb{R}^3}\big(\mathbf{u}\cdot\Delta\mathbf{u}+|\nabla\mathbf{u}|^2\big)\cdot\partial_3u_3\,\mathrm{d}x\\
&\qquad=-\int_{\mathbb{R}^3}(\partial_3\mathbf{u}\cdot\Delta\mathbf{u}+\mathbf{u}\cdot\Delta\partial_3\mathbf{u}+\nabla\mathbf{u}\cdot\nabla\partial_3\mathbf{u})\cdot u_3\,\mathrm{d}x,
\end{aligned}$$

we still have no means to bound the term

$$\int_{\mathbb{R}^3}\mathbf{u}\cdot\Delta\partial_3\mathbf{u}\cdot u_3\,\mathrm{d}x.$$

The motivation of this paper is to give an affirmative answer to the above-mentioned open problem; that is, if

$$\begin{aligned}
&u_3\in L^p(0,T;L^q(\mathbb{R}^3));\quad \omega_3\in L^r(0,T;L^s(\mathbb{R}^3)),\\
&\frac{2}{p}+\frac{3}{q}=1,\ 3<q\leqslant\infty;\quad \frac{2}{r}+\frac{3}{s}=2,\ \frac{3}{2}<s\leqslant\infty;
\end{aligned}\tag{1.7}$$

then the solution is smooth on $(0,T)$.

Before stating the precise result, let us recall the weak formulation of (1.1), see [16] for instance.

Definition 1.2. *Let* $\mathbf{u}_0 \in L^2(\mathbb{R}^3)$ *with* $\nabla\cdot\mathbf{u}_0 = 0$, $T > 0$. *A measurable* $\mathbb{R}^3$*-valued function* $\mathbf{u}$ *defined in* $[0,T]\times\mathbb{R}^3$ *is said to be a weak solution to* (1.1) *if*

(1) $\mathbf{u} \in L^\infty(0,T; L^2(\mathbb{R}^3) \cap L^2(0,T;H^1(\mathbb{R}^3))$;

(2) $(1.1)_1$ *and* $(1.1)_2$ *hold in the sense of distributions, i.e.,*

$$\int_0^t\int_{\mathbb{R}^3} \mathbf{u}\cdot[\partial_t\mathbf{\Phi} + (\mathbf{u}\cdot\nabla)\,\mathbf{\Phi}]\,\mathrm{d}x\,\mathrm{d}s + \int_{\mathbb{R}^3}\mathbf{u}_0\cdot\mathbf{\Phi}(0)\,\mathrm{d}x = \int_0^T\int_{\mathbb{R}^3}\nabla\mathbf{u} : \nabla\mathbf{\Phi}\,\mathrm{d}x\,\mathrm{d}t,$$

for each $\mathbf{\Phi} \in C_c^\infty([0,T)\times\mathbb{R}^3)$ *with* $\nabla\cdot\mathbf{\Phi} = 0$, *where* $A : B = \sum\limits_{i,j=1}^{3} a_{ij}b_{ij}$ *for* 3×3 *matrices* $A = (a_{ij})$, $B = (b_{ij})$, *and*

$$\int_0^T\int_{\mathbb{R}^3}\mathbf{u}\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t = 0,$$

for each $\psi \in C_c^\infty(\mathbb{R}^3\times[0,T))$;

(3) *the energy inequality holds, that is,*

$$\|\mathbf{u}(t)\|_{L^2}^2 + 2\int_0^t \|\nabla\mathbf{u}(s)\|_{L^2}^2\,\mathrm{d}s \leqslant \|\mathbf{u}_0\|_{L^2}^2, \quad 0 \leqslant t \leqslant T.$$

Now, our main result reads as follows.

Theorem 1.3. *Let* $\mathbf{u}_0 \in L^2(\mathbb{R}^3)$ *with* $\nabla\cdot\mathbf{u}_0 = 0$, $T > 0$. *Assume that* $\mathbf{u}$ *is a weak solution to* (1.1) *in* $[0,T]$ *with initial data* $\mathbf{u}_0$. *If* (1.7) *holds, then the solution* $\mathbf{u}$ *is smooth in* $(0,T]\times\mathbb{R}^3$.

2. Proof of Theorem 1.3

In this section, we shall prove Theorem 1.3. For any $\varepsilon \in (0,T)$, due to the fact that $\nabla\mathbf{u} \in L^2(0,T;L^2(\mathbb{R}^3))$, we may find a $\delta \in (0,\varepsilon)$, such that $\nabla\mathbf{u}(\delta) \in L^2(\mathbb{R}^3)$. Taking this $\mathbf{u}(\delta)$ as initial data, there exists an $\tilde{\mathbf{u}} \in C([\delta,\Gamma^*), H^1(\mathbb{R}^3)) \cap L^2(0,\Gamma^*;H^2(\mathbb{R}^3))$, where $[\delta,\Gamma^*)$ is the life span of the unique strong solution, see [16]. Moreover, $\tilde{\mathbf{u}} \in C^\infty(\mathbb{R}^3\times(\delta,\Gamma^*))$. According to the uniqueness result, $\tilde{\mathbf{u}} = \mathbf{u}$ on $[\delta,\Gamma^*)$. If $\Gamma^* \geqslant T$, we have already that $\mathbf{u} \in C^\infty(\mathbb{R}^3\times(0,T))$, due to the arbitrariness of $\varepsilon \in (0,T)$. In the case $\Gamma^* < T$, our strategy is to show that $\|\nabla\mathbf{u}(t)\|_2$ is uniformly bounded for $t \in [\delta,\Gamma^*)$. The standard continuation argument then yields that $[\delta,\Gamma^*)$ cannot be the maximal interval of existence of $\tilde{\mathbf{u}}$, and consequently $\Gamma^* \geqslant T$. This concludes the proof.

To bound $\|\nabla \mathbf{u}(t)\|_{L^2}$, taking the inner product of $(1.1)_1$ with $-\Delta \mathbf{u}$ in $L^2(\mathbb{R}^3)$, we obtain as in [18], (3.1)–(3.2) that

$$\begin{aligned}(2.1)\quad &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\nabla \mathbf{u}\|_{L^2}^2 + \|\Delta \mathbf{u}\|_{L^2}^2 \\ &= \sum_{i,j,k,l=1}^{3} \alpha_{11ijkl}\int_{\mathbb{R}^3} \partial_1 u_1 \partial_i u_j \partial_k u_l \,\mathrm{d}x + \sum_{i,j,k,l=1}^{3} \alpha_{12ijkl}\int_{\mathbb{R}^3} \partial_1 u_2 \partial_i u_j \partial_k u_l \,\mathrm{d}x \\ &\quad + \sum_{i,j,k,l=1}^{3} \alpha_{21ijkl}\int_{\mathbb{R}^3} \partial_2 u_1 \partial_i u_j \partial_k u_l \,\mathrm{d}x + \sum_{i,j,k,l=1}^{3} \alpha_{22ijkl}\int_{\mathbb{R}^3} \partial_2 u_2 \partial_i u_j \partial_k u_l \,\mathrm{d}x \\ &\equiv I_1 + I_2 + I_3 + I_4,\end{aligned}$$

where α_{mnijkl}, $1 \leqslant m,n \leqslant 2$, $1 \leqslant i,j,k,l \leqslant 3$, are some suitable integers.

To proceed further, we need to represent $\partial_m u_n$, $1 \leqslant m,n \leqslant 2$ by u_3 and ω_3. Denoting by $\Delta_h = \partial_1\partial_1 + \partial_2\partial_2$ the horizontal Laplacian, we recall from [18], (1.4), (1.5) that

$$(2.2)\qquad \Delta_h u_1 = -\partial_2\omega_3 - \partial_1\partial_3 u_3, \quad \Delta_h u_2 = \partial_1\omega_3 - \partial_2\partial_3 u_3.$$

Consequently, for $1 \leqslant k \leqslant 2$,

$$(2.3)\quad \partial_k u_1 = \frac{\partial_2}{\sqrt{-\Delta_h}}\frac{\partial_k}{\sqrt{-\Delta_h}} + \frac{\partial_2}{\sqrt{-\Delta_h}}\frac{\partial_1}{\sqrt{-\Delta_h}}\partial_3 u_3 = \mathcal{R}_2\mathcal{R}_k\omega_3 + \mathcal{R}_1\mathcal{R}_k\partial_3 u_3,$$

$$(2.4)\qquad \partial_k u_2 = \mathcal{R}_1\mathcal{R}_k\omega_3 + \mathcal{R}_2\mathcal{R}_k\partial_3 u_3,$$

where $\mathcal{R}_k = \partial_k/\sqrt{-\Delta_h}$ is the two-dimensional Riesz transformation, see [15]. With (2.3), we may integrate by parts to dominate I_{11} as

$$\begin{aligned}(2.5)\quad I_{11} &= \sum_{i,j,k,l=1}^{3} \alpha_{11ijkl}\int_{\mathbb{R}^3} \partial_1 u_1 \cdot \partial_i u_j \cdot \partial_k u_l \,\mathrm{d}x \\ &= \sum_{i,j,k,l=1}^{3} \alpha_{11ijkl}\int_{\mathbb{R}^3} (\mathcal{R}_2\mathcal{R}_1\omega_3 + \mathcal{R}_1\mathcal{R}_1\partial_3 u_3) \cdot \partial_i u_j \cdot \partial_k u_l \,\mathrm{d}x \\ &= \sum_{i,j,k,l=1}^{3} \alpha_{11ijkl}\int_{\mathbb{R}^3} \mathcal{R}_2\mathcal{R}_1\omega_3 \cdot \partial_i u_j \cdot \partial_k u_l \,\mathrm{d}x \\ &\quad - \sum_{i,j,k,l=1}^{3} \alpha_{11ijkl}\int_{\mathbb{R}^3} \mathcal{R}_1\mathcal{R}_1 u_3 \cdot (\partial_3\partial_i u_j \cdot \partial_k u_l + \partial_i u_j \cdot \partial_3\partial_k u_l) \,\mathrm{d}x.\end{aligned}$$

Observe that the Riesz transformation is bounded in $L^q(\mathbb{R}^2)$ to $L^q(\mathbb{R}^2)$ for $1 < q < \infty$, we have for $f \in L^q(\mathbb{R}^3)$ and $1 \leqslant j \leqslant 2$,

$$
\begin{aligned}
\|\mathcal{R}_j f\|_{L^q}^q &= \int_{\mathbb{R}^3} |\mathcal{R}_j f|^q \,\mathrm{d}x \\
&= \int_{\mathbb{R}} \left[\int_{\mathbb{R}^2} |\mathcal{R}_j f|^q \,\mathrm{d}x_1 \,\mathrm{d}x_2 \right] \mathrm{d}x_3 \\
&\leqslant C \int_{\mathbb{R}} \left[\int_{\mathbb{R}^2} |f|^q \,\mathrm{d}x_1 \,\mathrm{d}x_2 \right] \mathrm{d}x_3 \\
&= C\|f\|_{L^q}^q,
\end{aligned} \tag{2.6}
$$

and thus I_{11} can be further estimated by Hölder and Gagliardo-Nirenberg inequalities,

$$
\begin{aligned}
I_{11} &\leqslant C\|\omega_3\|_{L^s} \|\nabla \mathbf{u}\|_{L^{2s/(s-1)}}^2 + C\|u_3\|_{L^q} \|\nabla \mathbf{u}\|_{L^{2q/(q-2)}} \|\nabla^2 \mathbf{u}\|_{L^2} \\
&\leqslant C\|\omega_3\|_{L^s} \|\nabla \mathbf{u}\|_{L^2}^{(2s-3)/s} \|\nabla^2 \mathbf{u}\|_{L^2}^{3/s} + C\|u_3\|_{L^q} \|\nabla \mathbf{u}\|_{L^2}^{(q-3)/q} \|\nabla^2 \mathbf{u}\|_{L^2}^{(q+3)/q} \\
&\leqslant C\big(\|\omega_3\|_{L^s}^{2s/(2s-3)} + \|u_3\|_{L^q}^{2q/(q-3)}\big) \|\nabla \mathbf{u}\|_{L^2}^2 + \frac{1}{8}\|\Delta \mathbf{u}\|_{L^2}^2.
\end{aligned} \tag{2.7}
$$

Substituting (2.7) and similar bounds for I_{12}, I_{21}, I_{22} into (2.1), we conclude by Gronwall inequality that

$$
\begin{aligned}
\|\nabla \mathbf{u}(t)\|_{L^2}^2 &\leqslant \|\nabla \mathbf{u}(\delta)\|_{L^2}^2 \\
&\quad \times \exp\left[\int_\delta^t \big(\|\omega_3\|_{L^s}^{2s/(2s-3)} + \|u_3\|_{L^q}^{2q/(q-3)}\big) \,\mathrm{d}s \right] < \infty, \quad \delta \leqslant t < \varGamma^*.
\end{aligned} \tag{2.8}
$$

This bound completes the proof of Theorem 1.3. □

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