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A HIGHER RANK SELBERG SIEVE AND APPLICATIONS

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Abstract. We develop an axiomatic formulation of the higher rank version of the classical Selberg sieve. This allows us to derive a simplified proof of the Zhang and Maynard-Tao result on bounded gaps between primes. We also apply the sieve to other subsequences of the primes and obtain bounded gaps in various settings.

Keywords: Selberg sieve; bounded gaps; prime k-tuples

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1. Introduction

A higher rank version of the Selberg sieve was first suggested by Selberg in [11] (see page 351 of [11] and page 245 of [10]). Recently, a special case of this new sieve method was applied by Maynard in [5] and the Polymath project in [9] to give a simplified proof of the breakthrough result of Zhang in [14] regarding bounded gaps between primes. In addition, they obtained better numerical results than Zhang.

A comparison of their methods reveals that the method of the Polymath project is Fourier analytic whereas Maynard's method is combinatorial. In this paper, we develop a general higher rank Selberg sieve, in the derivation of which we have opted to apply the Fourier analytic method since it seems to lead to the general result quickly. We formulate an axiomatic treatment of a general higher rank Selberg sieve in a manner which can then be applied to an assortment of problems. In this context, our main result is Theorem 3.6.

We then discuss various applications of the sieve. Applying the sieve to prime k-tuples with the characteristic function of the primes chosen as the weight, one obtains a conceptually clear and simplified proof of the Zhang-Maynard-Tao result

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on bounded gaps. This is contained in Lemmas 4.2 and 4.3. Finally, we make some comments about the application of the method to other subsequences of primes. In particular, we consider the set of primes satisfying certain Chebotarev conditions, as well as primes having a given primitive root. We highlight the ingredients that make the general sieve work in these situations.

In a forthcoming paper [13], we build upon the higher rank sieve established in this work to generalize it to the ring of integers of an imaginary quadratic field with class number one. This yields better results than those previously obtained for gaps between primes in the corresponding number rings.

2. Notation and setting

It will be convenient to introduce notation and terminology to study k-tuples. We denote the k-tuple of integers $(d_1, \dots, d_k)$ by $\underline{d}$. A tuple is said to be square-free if the product of its components is square-free. For a real number R, the inequality $\underline{d} \leqslant R$ means that $\prod_i d_i \leqslant R$. The notion of divisibility among tuples is defined component-wise, that is,

$$\underline{d} \mid \underline{n} \Leftrightarrow d_i \mid n_i \quad \forall\, 1 \leqslant i \leqslant k.$$

It follows that the notion of congruence among tuples, modulo a tuple, is also defined component-wise. On the other hand, we say a scalar q divides the tuple $\underline{d}$ if q divides the product $\prod_i d_i$. However, when we explicitly write the congruence relation $\underline{d} \equiv \underline{e} \pmod{q}$, we mean that it holds for each component. When we say that a tuple $\underline{d}$ divides a scalar q, we mean that $\prod_i d_i$ divides q. For a square-free tuple, this is equivalent to each component dividing q.

We do not invoke any special notation for vector functions, that is, functions acting on k-tuples. It will be evident from its argument whether a function is a vector or scalar function. Most of the functions that we deal with are multiplicative. A vector function is said to be multiplicative if all its component functions are multiplicative. In this context, we define the function $f(\underline{d})$ to mean the product of its component (multiplicative) functions acting on the corresponding components of the tuple, that is,

$$f(\underline{d}) = \prod_{i=1}^{k} f_i(d_i).$$

For example, if μ is the Möbius function, all its components are the same, hence

$$\mu(\underline{d}) = \prod_{i=1}^{k} \mu(d_i).$$

The identity function acting on a tuple $\underline{d}$ is denoted by $\underline{d}$ itself. In this case, $\underline{d}$ would represent the product $\prod_{i=1}^{k} d_i$. It will be clear from the context whether we mean the above product or the vector tuple itself. Similarly, when we write a tuple raised to some power, we interpret it as the appropriate function acting on the tuple. For example,

$$\underline{d}^2 = \prod_{i=1}^{k} d_i^2.$$

Furthermore, we define for k-tuples $\underline{d}$ and $\underline{\alpha}$,

$$\underline{d}^{(\underline{\alpha})} = \prod_{i=1}^{k} d_i^{\alpha_i}.$$

Some more vector functions that will be used by us are the Euler phi function, as well as the lcm and gcd functions.

We use the convention $n \sim N$ to denote $N \leqslant n < 2N$. Alternatively, $f(x) \sim g(x)$ may also denote that $\lim_{x\to\infty} f(x)/g(x) = 1$. The meaning will be clear from the context. Moreover, if we have an expression of the form $f(x) = (1 + o(1))cg(x)$, where c is a constant independent of x, it is understood that the case $c = 0$ implies that $f(x) = o(g(x))$.

We use the standard notation $[a, b]$ and (a, b) to denote the lcm and gcd of a and b, respectively. In the case of tuples, this means the product of the lcms (or gcds) of the corresponding components. For example,

$$[\underline{d}, \underline{e}] := \prod_{i=1}^{k} [d_i, e_i].$$

We also use the notation $[\underline{d}, \underline{e}] \mid \underline{n}$ to mean $[d_i, e_i] \mid n_i$ for $1 \leqslant i \leqslant k$. When written as the argument of a vector function, $[\underline{d}, \underline{e}]$ will denote the tuple whose components are $[d_i, e_i]$. Once again, the meaning of the use will be evident from the context.

Furthermore, we let $\tau(n)$ denote the number of divisors of the integer n and $\omega(n)$ the number of distinct prime factors of n. The greatest integer less than or equal to x is denoted as $\lfloor x \rfloor$. Throughout this paper, δ denotes a positive quantity which can be made as small as needed.

We employ the following multi-index notation to denote mixed derivatives of a function on k-tuples, $\mathcal{F}(\underline{t})$:

$$\mathcal{F}^{(\underline{\alpha})}(\underline{t}) := \frac{\partial^{\alpha} \mathcal{F}(t_1, \ldots, t_k)}{(\partial t_1)^{\alpha_1} \ldots (\partial t_k)^{\alpha_k}}, \tag{2.1}$$

for any k-tuple $\underline{\alpha}$ with $\alpha := \sum_{j=1}^{k} \alpha_j$.

3. The higher rank Selberg sieve

In the classical Selberg sieve, one considers sums of the form

$$\sum_{n} w_n \Bigl(\sum_{d|n} \lambda_d\Bigr)^2,$$

where the outer sum is over integers n belonging to a certain set $\mathcal{S}$, w_n are weights, and the λ_d's are parameters to be chosen so as to minimize the value of this expression. Here we proceed similarly replacing integers with tuples.

A Fourier analytic approach to optimize the parameters appearing in Selberg's sieve, as demonstrated by the Polymath project in [9], seems to be the most conducive for this purpose. This method in fact gives us asymptotic formulas for the sums involved.

3.1. Preliminary results. We begin by setting up some preliminary results. The following proposition will be a useful tool in our estimation of error terms throughout the paper.

Proposition 3.1. *Let $\Omega_k(r)$ denote the number of k-tuples $\underline{d}$, $\underline{e}$ satisfying $[d_i, e_i]$, $[d_j, e_j]$ relatively prime for $i \neq j$, such that $[\underline{d}, \underline{e}] = r$. Then*

$$\Omega_k(r) = k^{\omega(r)}\tau(r^2),$$

where $\omega(r)$ denotes the number of distinct prime factors of r.

In particular, $\Omega_k(r) \leqslant \tau_{3k}(r)$, where $\tau_{3k}(r)$ is the number of ways of writing r as a product of $3k$ positive integers.

Proof. The number of k-tuples $\underline{d}$, $\underline{e}$ such that $[\underline{d}, \underline{e}] = r$ is a multiplicative function of r. It can be checked that $\Omega_k(r)$ is also a multiplicative function of r and hence it is enough to compute $\Omega_k(p^a)$ for a prime p. Consider the equation

$$[d_1, e_1] \dots [d_k, e_k] = p^a.$$

As the terms on the left hand side are pairwise co-prime, there are k choices of the component i such that $[d_i, e_i] = p^a$. Once i is fixed, one of d_i, e_i must be exactly p^a while the other must equal p^b for some $0 \leqslant b \leqslant a$. There are then $2a+1$ choices, taking care that the choice $d_i, e_i = p^a$ is counted only once. This gives $\Omega_k(p^a) = k\tau(p^{2a})$, and hence $\Omega_k(r) = k^{\omega(r)}\tau(r^2)$.

The inequality $\Omega_k(r) \leqslant \tau_{3k}(r)$ can be shown for prime powers by induction using the identity $\tau_m(r) = \sum_{d|r} \tau_{m-1}(d)$ and thus holds for all r. $\square$

We derive an Euler product for a very special kind of series.

Lemma 3.2. *Let $\underline{d}$ and $\underline{e}$ be k-tuples, and f, g, h be multiplicative functions acting on k-tuples. We denote by S the sum*

$$\sum_{\underline{d},\underline{e}}{}' \frac{\mu(\underline{d})\mu(\underline{e})g(\underline{d})h(\underline{e})}{f([\underline{d},\underline{e}])},$$

where the dash over the sum means that we are summing over square-free tuples $\underline{d}$ and $\underline{e}$ with $[d_i, e_i]$, $[d_j, e_j]$ mutually co-prime for all $i \neq j$. Then

$$S = \prod_p \Big(1 - \sum_{j=1}^{k} \Big(\frac{g_j(p)}{f_j(p)} + \frac{h_j(p)}{f_j(p)} - \frac{g_j(p)h_j(p)}{f_j(p)}\Big)\Big),$$

assuming that both the series and the product are absolutely convergent.

Proof. We denote by $S(p)$ the sum

$$\sum_{\substack{\underline{d},\underline{e} \\ p\nmid[\underline{d},\underline{e}]}}{}' \frac{\mu(\underline{d})\mu(\underline{e})g(\underline{d})h(\underline{e})}{f([\underline{d},\underline{e}])}. \tag{3.1}$$

Then as $[d_i, e_i]$, $[d_j, e_j]$ are co-prime for all $i \neq j$, p can divide only one of the $[d_j, e_j]$'s if it divides the tuple $[\underline{d}, \underline{e}]$. Hence,

$$\begin{aligned} S &= S(p) + \sum_{\substack{\underline{d},\underline{e} \\ p\mid[\underline{d},\underline{e}]}}{}' \frac{\mu(\underline{d})\mu(\underline{e})g(\underline{d})h(\underline{e})}{f([\underline{d},\underline{e}])} \\ &= S(p) + \sum_{j=1}^{k} \sum_{\substack{\underline{d},\underline{e} \\ p\mid[d_j,e_j]}}{}' \frac{\mu(\underline{d})\mu(\underline{e})g(\underline{d})h(\underline{e})}{f([\underline{d},\underline{e}])}. \end{aligned}$$

Now, for each j, the condition $p \mid [d_j, e_j]$ leads to three cases:
(a) $p \mid d_j$, $p \nmid e_j$,
(b) $p \nmid d_j$, $p \mid e_j$,
(c) $p \mid d_j$, $p \mid e_j$.
For each of these cases, note that the dash over the sum indicates that p cannot divide $[d_i, e_i]$ for any $i \neq j$. Then case (a) gives, upon writing $d_j = pd'_j$ and noting

that d_j is square-free,

$$\sum_{j=1}^{k} \sum_{\substack{\underline{d},\underline{e} \\ p \mid d_j, p \nmid e_j}} \frac{\mu(\underline{d})\mu(\underline{e})g(\underline{d})h(\underline{e})}{f([\underline{d},\underline{e}])} = \sum_{j=1}^{k} \frac{\mu(p)g_j(p)}{f_j(p)} \sum_{\substack{\underline{d'},\underline{e} \\ p \nmid d'_j, p \nmid e_j}} \frac{\mu(\underline{d'})\mu(\underline{e})g(\underline{d'})h(\underline{e})}{f([\underline{d'},\underline{e}])}$$
$$= -\sum_{j=1}^{k} \frac{g_j(p)}{f_j(p)} S(p),$$

with obvious notation (since in the innermost sum of the penultimate step, we have $p \nmid [\underline{d'},\underline{e}]$).

Similarly, we get $S(p)$ times a factor of $-\sum_{j=1}^{k} h_j(p)/f_j(p)$ from (b) and a factor of $\sum_{j=1}^{k} g_j(p)h_j(p)/f_j(p)$ from (c), respectively. Thus

$$S = \left(1 - \sum_{j=1}^{k} \left(\frac{g_j(p)}{f_j(p)} + \frac{h_j(p)}{f_j(p)} - \frac{g_j(p)h_j(p)}{f_j(p)} \right) \right) S(p). \tag{3.2}$$

Thus, S is $S(p)$ multiplied by the 'Euler factor' coming from p. As $S(p)$ is simply the sum S with the prime p eliminated, we can now repeat this process for the sum $S(p)$, by taking some prime $q \neq p$. We keep getting Euler factors of the above form for each prime, thereby proving the result. □

For future reference, the notion of a function $g(x)$ being integrated r times with respect to x is defined as

$$\int^{(r)} g(x)\,\mathrm{d}x := \int_{x_r=0}^{\infty} \int_{x_{r-1} \geqslant x_r} \cdots \int_{x_1 \geqslant x_2} g(x_1)\,\mathrm{d}x_1 \ldots \mathrm{d}x_r.$$

Remark. In particular, note that for $\mathrm{Re}(\alpha) > 0$,

$$\int^{(r)} \exp(-\alpha x)\,\mathrm{d}x = \frac{1}{\alpha^r}.$$

We state the following lemma which will give a convenient way of reducing such multiple integrals to single integrals.

Lemma 3.3. *Let F be a function with compact support in $[0,\infty)$. Then*

$$\int^{(r)} F(x)\,\mathrm{d}x = \frac{1}{(r-1)!} \int_0^{\infty} x^{r-1} F(x)\,\mathrm{d}x.$$

Proof. Repeated use of Fubini's theorem gives us

$$\int^{(r)} F(x)\,\mathrm{d}x = \int_{x_1=0}^{\infty} \int_{x_2 \leqslant x_1} \dots \int_{x_r \leqslant x_{r-1}} F(x_1)\,\mathrm{d}x_r \dots \mathrm{d}x_1.$$

Now,

$$\int_{x_r \leqslant x_{r-1}} F(x_1)\,\mathrm{d}x_r \dots \mathrm{d}x_1 = x_{r-1} F(x_1),$$

$$\int_{x_{r-1} \leqslant x_{r-2}} x_{r-1} F(x_1)\,\mathrm{d}x_{r-1} = \frac{x_{r-2}^2}{2!} F(x_1),$$

and so on. We continue this until we are left with only the integral with respect to x_1. □

3.2. The sieve. We now proceed to what we call the higher rank Selberg sieve. Consider a set $\mathcal{S}$ of k-tuples (not necessarily finite),

$$\mathcal{S} = \{\underline{n} = (n_1, \dots, n_k)\}.$$

In our applications, we consider a sequence of finite sets S_N whose size tends to infinity with N. We wish to estimate sums of the form

$$\sum_{\underline{n} \in \mathcal{S}} w_{\underline{n}} \Big(\sum_{\underline{d} \mid \underline{n}} \lambda_{\underline{d}} \Big)^2, \tag{3.3}$$

where $w_{\underline{n}}$ is a 'weight' attached to the tuples $\underline{n}$, and $\lambda_{\underline{d}}$'s are parameters to be chosen. Henceforth throughout this section, the condition $\underline{n} \in \mathcal{S}$ is understood to be imposed without being explicitly stated. We impose the following hypotheses on our sum. These assumptions are in line with those in Halberstam-Richert, see [2], and Friedlander-Iwaniec, see [1].

(H1) If a prime p divides a tuple $\underline{n}$ such that p divides n_i and n_j, with $i \neq j$, then p must lie in some fixed finite set of primes $\mathcal{P}_0$.

Note that this condition is empty when $k = 1$. This hypothesis allows us to perform what is called the 'W trick'. That is, we can fix some $W = \prod_{p < D_0} p$, with D_0 depending on $\mathcal{S}$, such that $p \in \mathcal{P}_0$ implies that $p \mid W$. We then fix some tuple of residue classes $\underline{b} \pmod W$ with $(b_i, W) = 1$ for all i and restrict $\underline{n}$ to be congruent to $\underline{b}$ in the sum we are concerned with.

(H2) With $W, \underline{b}$ as in (H1), the function $w_{\underline{n}}$ satisfies

$$\sum_{\substack{\underline{d} \mid \underline{n} \\ \underline{n} \equiv \underline{b} \pmod W}} w_{\underline{n}} = \frac{X}{f(\underline{d})} + r_{\underline{d}}$$

for some multiplicative function f and some quantity X depending on the set $\mathcal{S}$.

In practice,

$$X = \frac{1}{\varphi(W)} \sum_{(\underline{n}, W)=1} w_{\underline{n}},$$

$1/f(\underline{d})$ is heuristically a measure of the "probability" of $\underline{d}$ dividing $\underline{n}$'s in our sequence and $r_{\underline{d}}$ is the "error" of using this approximation. The condition f being multiplicative can be understood as the probabilities being independent.

(H3) The components of f satisfy

$$f_j(p) = \frac{p}{\alpha_j} + O(p^t), \qquad \text{with } t < 1$$

for some fixed $\alpha_j \in \mathbb{N}$, α_j independent of X, k.

We denote the tuple $(\alpha_1, \dots, \alpha_k)$ as $\underline{\alpha}$ and the sum of the components $\sum_{j=1}^{k} \alpha_j$ as α.

(H4) There exist $\theta > 0$ and $Y \ll X$ such that

$$\sum_{[\underline{d}, \underline{e}] < Y^\theta} |r_{[\underline{d}, \underline{e}]}| \ll \frac{Y}{(\log Y)^A}$$

for any $A > 0$, as $Y \to \infty$.

Henceforth, we assume D_0 (and hence W) $\to \infty$ as $X \to \infty$. We now prove a general result, which lies at the very heart of this sieve and animates all of our subsequent discussion. All asymptotics involving R or W in this and future results are with respect to $X \to \infty$ unless explicitly stated otherwise.

Lemma 3.4. *Set R to be some fixed power of X and let $D_0 = o(\log \log R)$. Let f be a multiplicative function satisfying* (H3) *and*

$$\mathcal{G}, \mathcal{H} \colon [0, \infty)^k \to \mathbb{R}$$

be smooth functions with compact support. We denote

$$\mathcal{G}\Big(\frac{\log \underline{d}}{\log R}\Big) := \mathcal{G}\Big(\frac{\log d_1}{\log R}, \dots, \frac{\log d_k}{\log R}\Big)$$

and similarly for $\mathcal{H}$. Let the dash over the sum mean that we sum over k-tuples $\underline{d}$ and $\underline{e}$ with $[\underline{d}, \underline{e}]$ square-free and co-prime to W. Then

$$\sum_{\underline{d}, \underline{e}}{}' \frac{\mu(\underline{d})\mu(\underline{e})}{f([\underline{d}, \underline{e}])} \mathcal{G}\Big(\frac{\log \underline{d}}{\log R}\Big) \mathcal{H}\Big(\frac{\log \underline{e}}{\log R}\Big) = (1 + o(1)) C(\mathcal{G}, \mathcal{H})^{(\underline{\alpha})} \frac{c(W)}{(\log R)^\alpha},$$

where

$$C(\mathcal{G},\mathcal{H})^{(\underline{\alpha})} = \int_0^\infty \dots \int_0^\infty \Big(\prod_{j=1}^k \frac{t_j^{\alpha_j-1}}{(\alpha_j-1)!}\Big)\mathcal{G}(\underline{t})^{(\underline{\alpha})}\mathcal{H}(\underline{t})^{(\underline{\alpha})}\,\mathrm{d}\underline{t},$$

with $\mathcal{G}(\underline{t})^{(\underline{\alpha})}$ *and* $\mathcal{H}(\underline{t})^{(\underline{\alpha})}$ *as in the notation of* (2.1). *Furthermore,*

$$c(W) := \prod_{p\mid W}\frac{p^\alpha}{\varphi(p)^\alpha}.$$

P r o o f. We extend the functions $\mathcal{G}$ and $\mathcal{H}$ to smooth compactly supported functions on $\mathbb{R}^k$. Let $\eta_{\mathcal{G}}$, $\eta_{\mathcal{H}}$ be shifted Fourier transforms of $\mathcal{G}$ and $\mathcal{H}$, respectively. More precisely, let

$$\eta_{\mathcal{G}}(\underline{u}) = \int_{\mathbb{R}^k}(\mathcal{G}(\underline{t})\exp(\underline{t}))\exp(\mathrm{i}\underline{u}\cdot\underline{t})\,\mathrm{d}\underline{t},$$

where $\exp(\underline{t}) = \prod_{j=1}^n \mathrm{e}^{t_j}$ and the dot denotes the dot product of tuples. We have a similar expression for $\eta_{\mathcal{H}}(\underline{u})$. Then by Fourier inversion we have

$$\mathcal{G}(\underline{t}) = \int_{\mathbb{R}^k}\eta_{\mathcal{G}}(\underline{u})\exp(-(\underline{1}+\mathrm{i}\underline{u})\cdot\underline{t})\,\mathrm{d}\underline{u},\quad \mathcal{H}(\underline{t}) = \int_{\mathbb{R}^k}\eta_{\mathcal{H}}(\underline{v})\exp(-(\underline{1}+\mathrm{i}\underline{v})\cdot\underline{t})\,\mathrm{d}\underline{v}. \tag{3.4}$$

Since $\eta_{\mathcal{G}}$ and $\eta_{\mathcal{H}}$ are Fourier transforms of smooth functions with compact support, they are rapidly decaying smooth functions, satisfying the bounds

$$|\eta_{\mathcal{G}}(\underline{t})| \ll (1+|\underline{t}|)^{-A_1},\quad |\eta_{\mathcal{H}}(\underline{t})| \ll (1+|\underline{t}|)^{-A_2}, \tag{3.5}$$

for any $A_1, A_2 > 0$.

The required sum can be written as

$$\int_{\mathbb{R}^k}\int_{\mathbb{R}^k}\eta_{\mathcal{G}}(\underline{u})\eta_{\mathcal{H}}(\underline{v})Z(\underline{u},\underline{v})\,\mathrm{d}\underline{u}\,\mathrm{d}\underline{v}, \tag{3.6}$$

where

$$Z(\underline{u},\underline{v}) = \sum_{\underline{d},\underline{e}}\frac{\mu(\underline{d})\mu(\underline{e})}{f([\underline{d},\underline{e}])}\frac{1}{\underline{d}^{(1+\mathrm{i}\underline{u})/\log R}}\frac{1}{\underline{e}^{(1+\mathrm{i}\underline{v})/\log R}}.$$

Using Lemma 3.2, we see that

$$Z(\underline{u},\underline{v}) = \prod_{p\nmid W}\Big(1-\sum_{j=1}^k\frac{1}{f_j(p)}\Big(\frac{1}{p^{(1+\mathrm{i}u_j)/\log R}}+\frac{1}{p^{(1+\mathrm{i}v_j)/\log R}} - \frac{1}{p^{(1+\mathrm{i}u_j)/\log R+(1+\mathrm{i}v_j)/\log R}}\Big)\Big).$$

Here W is the product of primes below D_0. The condition (H3) on f means that for each f_j,

$$\frac{1}{f_j(p)} = \frac{\alpha_j}{p} + O\Big(\frac{1}{p^{2-t}}\Big).$$

As $t < 1$, we have $2 - t > 1$. Substituting this into the expression for $Z(\underline{u}, \underline{v})$ gives

$$\text{(3.7)} \qquad Z(\underline{u}, \underline{v}) = (1 + o(1)) \prod_{p > D_0} \Bigg(1 - \sum_{j=1}^{k} \frac{\alpha_j}{p} \Big(\frac{1}{p^{(1+\mathrm{i}u_j)/\log R}} + \frac{1}{p^{(1+\mathrm{i}v_j)/\log R}} - \frac{1}{p^{(1+\mathrm{i}u_j)/\log R + (1+\mathrm{i}v_j)/\log R}}\Big)\Bigg).$$

Notice that for complex numbers w_1, w_2, w_3 with $\mathrm{Re}(w_i) > 1$ we have

$$\begin{aligned}\frac{(1 - 1/p^{w_1})(1 - 1/p^{w_2})}{(1 - 1/p^{w_3})} &= \Big(1 - \frac{1}{p^{w_1}} - \frac{1}{p^{w_2}} + \frac{1}{p^{w_3}} + O\Big(\frac{1}{p^2}\Big)\Big) \\ &= \Big(1 - \frac{1}{p^{w_1}} - \frac{1}{p^{w_2}} + \frac{1}{p^{w_3}}\Big)\Big(1 + O\Big(\frac{1}{p^2}\Big)\Big),\end{aligned}$$

using the expansion $(1 - p^{-w_3})^{-1} = \sum_{j=0}^{\infty} p^{-jw_3}$, which is absolutely convergent since $\mathrm{Re}(w_3) > 1$.

Since $\prod_{p > D_0} (1 + O(p^{-2})) = (1 + o(1))$ as $D_0 \to \infty$, a convenient approximation for $Z(\underline{u}, \underline{v})$ is then easily seen to be

$$\text{(3.8)} \qquad Z(\underline{u}, \underline{v}) = (1 + o(1)) \prod_{j=1}^{k} \prod_{p > D_0} \frac{(1 - \alpha_j p^{-1-(1+\mathrm{i}u_j)/\log R})}{1 - \alpha_j p^{-1-(1+\mathrm{i}u_j + 1 + \mathrm{i}v_j)/\log R}} \times (1 - \alpha_j p^{-1-(1+\mathrm{i}v_j)/\log R}).$$

Now we are ready to obtain the required sum. Note that the expression (3.7) allows us to bound $Z(\underline{u}, \underline{v})$ as follows:

$$\text{(3.9)} \qquad |Z(\underline{u}, \underline{v})| \ll \prod_{p} \Bigg(1 + \sum_{j=1}^{k} \frac{3\alpha_j}{p^{1+1/\log R}}\Bigg) \ll (\log R)^{O(1)},$$

applying the well-known estimate for the Riemann-zeta function: $\zeta(\sigma) \ll (\sigma - 1)^{-1}$ for σ near 1.

Consider the case $|\underline{u}| \geqslant (\log R)^{\varepsilon}$ for some small $\varepsilon > 0$. Then by the bounds (3.5) we have

$$|\eta_{\mathcal{G}}(\underline{u})| \ll \frac{1}{(1 + |u|)^{2A}} \ll (\log R)^{-A'} \frac{1}{(1 + |u|)^{A}}$$

for any $A, A' > 0$. This along with the rapid decay bound (3.5) for $\eta_{\mathcal{H}}(\underline{v})$ and the bound (3.9) for $Z(\underline{u}, \underline{v})$ ensures that the integral (3.6) in the region $|\underline{u}| \geqslant (\log R)^{\varepsilon}$ is then of the order of

$$(\log R)^{-A_1} \iint \frac{1}{(1+|\underline{u}|)^{A}} \frac{1}{(1+|\underline{v}|)^{A}} \,\mathrm{d}\underline{u}\,\mathrm{d}\underline{v}$$

for any $A_1, A > 0$. As the integral above is absolutely convergent because of the rapid decay of the integrand, the contribution to (3.6) from this region is $O((\log R)^{-A})$ for any $A > 0$. Similarly, the contribution from the region $|\underline{v}| \geqslant (\log R)^{\varepsilon}$ is also $O((\log R)^{-A})$.

We are going to estimate (3.6) in the region $|\underline{u}|, |\underline{v}| < (\log R)^{\varepsilon}$. To this end, we first rewrite the general Euler product

$$\prod_{p > D_0} \Big(1 - \frac{\alpha_j}{p^{1+(1+\mathrm{i}s_j)/\log R}}\Big)$$

as

$$\prod_{p > D_0} \Big(1 - \frac{1}{p^{1+(1+\mathrm{i}s_j)/\log R}}\Big)^{\alpha_j} \Big(1 - \frac{1}{p^{1+(1+\mathrm{i}s_j)/\log R}}\Big)^{-\alpha_j} \Big(1 - \frac{\alpha_j}{p^{1+(1+\mathrm{i}s_j)/\log R}}\Big),$$

which equals

$$\zeta\Big(1 + \frac{1+\mathrm{i}s_j}{\log R}\Big)^{-\alpha_j} \prod_{p \mid W} \Big(1 - \frac{1}{p^{1+(1+\mathrm{i}s_j)/\log R}}\Big)^{-\alpha_j} D_j\Big(1 + \frac{1+\mathrm{i}s_j}{\log R}\Big),$$

where

$$D_j(s) = \prod_{p > D_0} \Big(1 - \frac{1}{p^s}\Big)^{-\alpha_j} \Big(1 - \frac{\alpha_j}{p^s}\Big)$$

is an Euler product supported on primes $p > D_0$ and absolutely convergent for $\mathrm{Re}(s) > 1/2$. For $\mathrm{Re}(s) = 1$, the above expression gives

$$D_j(s) = 1 + O\Big(\sum_{p > D_0} \frac{1}{p^2}\Big) = 1 + O\Big(\frac{1}{D_0}\Big)$$

as R (and hence D_0) goes to infinity. We will use this with $s = 1 + (1 + \mathrm{i}s_j)/\log R$.

As $\zeta(s)^{-t}$ has a zero of order t at $s = 1$, we have the asymptotic formula in the region $|s_j| < (\log R)^{\varepsilon}$

$$\begin{aligned}\zeta\Big(1 + \frac{1+\mathrm{i}s_j}{\log R}\Big)^{-\alpha_j} &= \Big(\frac{1+\mathrm{i}s_j}{\log R}\Big)^{\alpha_j} \Big(1 + O\Big(\Big|\frac{1+\mathrm{i}s_j}{\log R}\Big|\Big)\Big) \\ &= \Big(\frac{1+\mathrm{i}s_j}{\log R}\Big)^{\alpha_j} (1 + O((\log R)^{\varepsilon-1}))\end{aligned}$$

as $R \to \infty$. Moreover, it can be seen that

$$\prod_{p|W}\Big(1-\frac{1}{p^{1+(1+s_j)/\log R}}\Big)=\Big(1+O\Big(\frac{D_0}{(\log R)^{1-\varepsilon}}\Big)\Big)\prod_{p|W}\Big(1-\frac{1}{p}\Big),$$

by observing that

$$p^{(1+s_j)/\log R}=1+O\Big(\frac{\log p}{(\log R)^{1-\varepsilon}}\Big),$$

and using elementary estimates. Hence, in the region $|s_j|<(\log R)^{\varepsilon}$, as R goes to ∞, our Euler product becomes

$$(1+o(1))\Big(\frac{1+\mathrm{i}s_j}{\log R}\Big)^{\alpha_j}\prod_{p|W}\Big(1-\frac{1}{p}\Big)^{-\alpha_j}.$$

We have thus obtained in this region

$$\prod_{p\nmid W}\Big(1-\frac{\alpha_j}{p^{1+(1+\mathrm{i}s_j)/\log R}}\Big)=(1+o(1))\frac{W^{\alpha_j}}{\varphi(W)^{\alpha_j}}\Big(\frac{1+\mathrm{i}s_j}{\log R}\Big)^{\alpha_j}.$$

Applying this to (3.8) we conclude

$$Z(\underline{u},\underline{v})=(1+o(1))c(W)\frac{1}{(\log R)^{\alpha}}\prod_{j=1}^{k}\frac{(1+\mathrm{i}u_j)^{\alpha_j}(1+\mathrm{i}v_j)^{\alpha_j}}{(1+\mathrm{i}u_j+1+\mathrm{i}v_j)^{\alpha_j}},\tag{3.10}$$

where $c(W)$ is as given in the statement of the lemma. This expression for $Z(\underline{u},\underline{v})$ holds in the region $|\underline{u}|,|\underline{v}|<(\log R)^{\varepsilon}$. As discussed, the main contribution to the integral (3.6) should be from this region. From (3.6), it is clear that in this region, the required sum is now given by

$$\begin{aligned}\frac{c(W)}{(\log R)^{\alpha}}&\int_{\mathbb{R}^k}'\int_{\mathbb{R}^k}'\eta_{\mathcal{G}}(\underline{u})\eta_{\mathcal{H}}(\underline{v})\prod_{j=1}^{k}\frac{(1+\mathrm{i}u_j)^{\alpha_j}(1+\mathrm{i}v_j)^{\alpha_j}}{(1+\mathrm{i}u_j+1+\mathrm{i}v_j)^{\alpha_j}}\,\mathrm{d}\underline{u}\,\mathrm{d}\underline{v}\\&+\frac{c(W)}{(\log R)^{\alpha}}o\Big(\int_{\mathbb{R}^k}'\int_{\mathbb{R}^k}'|\eta_{\mathcal{G}}(\underline{u})||\eta_{\mathcal{H}}(\underline{v})|\prod_{j=1}^{k}\Big|\frac{(1+\mathrm{i}u_j)^{\alpha_j}(1+\mathrm{i}v_j)^{\alpha_j}}{(1+\mathrm{i}u_j+1+\mathrm{i}v_j)^{\alpha_j}}\Big|\,\mathrm{d}\underline{u}\,\mathrm{d}\underline{v}\Big),\end{aligned}\tag{3.11}$$

where the prime over the integrals means that they are restricted to the region $|\underline{u}|,|\underline{v}|<(\log R)^{\varepsilon}$. We first deal with the error term. All the asymptotics discussed are with respect to R (and hence D_0) going to ∞. As $\eta_{\mathcal{G}}$ and $\eta_{\mathcal{H}}$ are rapidly decreasing, the integral in the error term is convergent. This gives for (3.11)

$$\frac{c(W)}{(\log R)^{\alpha}}\Big(\int_{\mathbb{R}^k}'\int_{\mathbb{R}^k}'\eta_{\mathcal{G}}(\underline{u})\eta_{\mathcal{H}}(\underline{v})\prod_{j=1}^{k}\frac{(1+\mathrm{i}u_j)^{\alpha_j}(1+\mathrm{i}v_j)^{\alpha_j}}{(1+\mathrm{i}u_j+1+\mathrm{i}v_j)^{\alpha_j}}\,\mathrm{d}\underline{u}\,\mathrm{d}\underline{v}+o(1)\Big).$$

In the complementary region $|\underline{u}|$ or $|\underline{v}| \geqslant (\log R)^{\varepsilon}$, as before, the contribution to the integral can be seen to be $O((\log R)^{-A})$ for any $A > 0$. The integral above can thus be extended to the whole of $\mathbb{R}^k \times \mathbb{R}^k$, absorbing the error into the $o(1)$ term.

This takes care of the contribution to (3.6) from the region $|\underline{u}|, |\underline{v}| \leqslant (\log R)^{\varepsilon}$. The contributions from the complementary regions are absorbed into the $o(1)$ term to finally give for the sum (3.6),

$$(1+o(1))\frac{c(W)}{(\log R)^{\alpha}} \int_{\mathbb{R}^k}\int_{\mathbb{R}^k} \eta_{\mathcal{G}}(\underline{u})\eta_{\mathcal{H}}(\underline{v}) \prod_{j=1}^{k} \frac{(1+\mathrm{i}u_j)^{\alpha_j}(1+\mathrm{i}v_j)^{\alpha_j}}{(1+\mathrm{i}u_j+1+\mathrm{i}v_j)^{\alpha_j}}\,\mathrm{d}\underline{u}\,\mathrm{d}\underline{v}.$$

The lemma will be proved if one can show that

$$\int_{\mathbb{R}^k}\int_{\mathbb{R}^k} \eta_{\mathcal{G}}(\underline{u})\eta_{\mathcal{H}}(\underline{v}) \prod_{j=1}^{k} \frac{(1+\mathrm{i}u_j)^{\alpha_j}(1+\mathrm{i}v_j)^{\alpha_j}}{(1+\mathrm{i}u_j+1+\mathrm{i}v_j)^{\alpha_j}}\,\mathrm{d}\underline{u}\,\mathrm{d}\underline{v} = C(\mathcal{G},\mathcal{H})^{\alpha}. \tag{3.12}$$

Consider the expressions (3.4) for $\mathcal{G}$ and $\mathcal{H}$. Now, differentiate $\mathcal{G}$ and $\mathcal{H}$, α_j times with respect to each u_j and v_j, respectively. With notation as in the statement of the lemma, this gives

$$\mathcal{G}(\underline{t})^{(\underline{\alpha})} = (-1)^{\alpha}\int_{\mathbb{R}^k} \eta_{\mathcal{G}}(\underline{u}) \prod_{j} \exp\left(-(1+\mathrm{i}u_j)t_j\right)(1+\mathrm{i}u_j)^{\alpha_j}\,\mathrm{d}\underline{u},$$

$$\mathcal{H}(\underline{t})^{(\underline{\alpha})} = (-1)^{\alpha}\int_{\mathbb{R}^k} \eta_{\mathcal{H}}(\underline{v}) \prod_{j} \exp\left(-(1+\mathrm{i}v_j)t_j\right)(1+\mathrm{i}v_j)^{\alpha_j}\,\mathrm{d}\underline{v}.$$

We multiply these two expressions and recall the remark preceding Lemma 3.3 to see that the left hand side of (3.12) is nothing but the product $\mathcal{G}(\underline{t})^{(\underline{\alpha})} \cdot \mathcal{H}(\underline{t})^{(\underline{\alpha})}$ integrated α_j times with respect to t_j for all j. Now, applying Lemma 3.3 for each t_j in turn gives (3.12). This completes the proof. $\square$

We record for future use a general result, a special case of which was used in the proof of the previous lemma.

Lemma 3.5. *Let $\underline{a}$ denote the tuple $(a_1, \ldots, a_k)$ and let $a = \sum_j a_j$. We follow the same notation for $\underline{b}$ and $\underline{c}$ and the notation of* (2.1) *for the relevant mixed derivatives. Then the integral*

$$\int_{\mathbb{R}^k}\int_{\mathbb{R}^k} \eta_{\mathcal{G}}(\underline{u})\eta_{\mathcal{H}}(\underline{v}) \prod_{j=1}^{k} \frac{(1+\mathrm{i}u_j)^{a_j}(1+\mathrm{i}v_j)^{b_j}}{(1+\mathrm{i}u_j+1+\mathrm{i}v_j)^{c_j}}\,\mathrm{d}\underline{u}\,\mathrm{d}\underline{v}$$

is given by

$$C(\mathcal{G},\mathcal{H})^{(\underline{a},\underline{b},\underline{c})} := (-1)^{a+b}\int_0^{\infty}\!\!\cdots\!\int_0^{\infty} \left(\prod_{j=1}^{k} \frac{t_j^{c_j-1}}{(c_j-1)!}\right) \mathcal{G}(\underline{t})^{(\underline{a})}\mathcal{H}(\underline{t})^{(\underline{b})}\,\mathrm{d}\underline{t}.$$

P r o o f. The proof follows the same argument that we used to show (3.12). The basic observation is that we need to differentiate $\mathcal{G}$, a_j times with respect to each u_j; $\mathcal{H}$, b_j times with respect to each v_j, and multiply the two expressions. This gives the required numerators in the integrals but with a factor of $(-1)^{(a+b)}$. Then integrating the resulting expression c_j times with respect to each t_j in the sense of Lemma 3.3 gives the desired result. $\square$

Note that when the tuples $\underline{a}$, $\underline{b}$, $\underline{c}$ are all equal, say equal to $\underline{\alpha}$, we will denote $C(\mathcal{G},\mathcal{H})^{(\underline{a},\underline{b},\underline{c})}$ by $C(\mathcal{G},\mathcal{H})^{(\underline{\alpha})}$ as was done in Lemma 3.4.

We now choose our parameters $\lambda_{\underline{d}}$ and turn to specific sums involved in this sieve. The choice of $\lambda_{\underline{d}}$ will be made as in [9]. Let $\mathcal{F}\colon [0,\infty)^k \to \mathbb{R}$ be a fixed symmetric smooth function supported on the simplex

$$\Delta_k(1) := \{(t_1,\ldots,t_k) \in [0,\infty)^k\colon\ t_1+\ldots+t_k \leqslant 1\}.$$

We choose

$$\lambda_{\underline{d}} = \mu(\underline{d})\mathcal{F}\Big(\frac{\log d_1}{\log R}\Big) := \mu(d_1)\ldots\mu(d_k)\mathcal{F}\Big(\frac{\log d_1}{\log R},\ldots,\frac{\log d_k}{\log R}\Big). \tag{3.13}$$

Observe that as $\mathcal{F}$ is a smooth function with compact support, it is bounded and so is $|\lambda_{\underline{d}}|$. We have the following result which we call the higher rank Selberg sieve.

Theorem 3.6. *Let $\lambda_{\underline{d}}$'s be as chosen above. Suppose hypotheses* (H1) *to* (H3) *hold and* (H4) *holds with $Y = X$. Set $R = X^{\theta/2-\delta}$ for small $\delta > 0$ and let $D_0 = o(\log\log R)$. Then*

$$\sum_{\underline{n}\equiv\underline{b} \pmod W} w_{\underline{n}}\Big(\sum_{\underline{d}|\underline{n}} \lambda_{\underline{d}}\Big)^2 = (1+o(1))C(\mathcal{F},\mathcal{F})^{(\underline{\alpha})}c(W)\frac{X}{(\log R)^{\alpha}},$$

with

$$c(W) := \frac{W^{\alpha}}{\varphi(W)^{\alpha}}$$

and

$$C(\mathcal{F},\mathcal{F})^{(\underline{\alpha})} = \int_0^{\infty}\!\!\cdots\int_0^{\infty}\Big(\prod_{j=1}^{k}\frac{t_j^{\alpha_j-1}}{(\alpha_j-1)!}\Big)(\mathcal{F}^{(\underline{\alpha})}(\underline{t}))^2\,\mathrm{d}\underline{t}.$$

P r o o f. Expanding out the square, interchanging the order of summation gives us

$$\sum_{\underline{n}\equiv\underline{b} \pmod W} w_{\underline{n}}\Big(\sum_{\underline{d}|\underline{n}} \lambda_{\underline{d}}\Big)^2 = \sum_{\underline{d},\underline{e}} \lambda_{\underline{d}}\lambda_{\underline{e}}\Bigg(\sum_{\substack{[\underline{d},\underline{e}]|\underline{n}\\ \underline{n}\equiv\underline{b} \pmod W}} w_{\underline{n}}\Bigg)$$

For the above expression, one can use (H1) along with the W-trick to conclude that for $i \neq j$, $[d_i, e_i]$ and $[d_j, e_j]$ must be co-prime. Indeed, if a prime p divides both $[d_i, e_i]$ and $[d_j, e_j]$, then p divides n_i and n_j, which means that $p \in \mathcal{P}_0$ and hence $p \mid W$. But each n_i is co-prime to W, so this cannot happen. From the support of $\lambda_{\underline{d}}$'s as defined in (3.13), it is clear that $\underline{d}, \underline{e}$ must be square-free tuples. Moreover, $[\underline{d}, \underline{e}] \mid \underline{n}$ and each component of $\underline{n}$ co-prime to W, implies that $[d_i, e_i]$ is co-prime to W for each i. These restrictions on the tuples $\underline{d}, \underline{e}$ along with the hypothesis (H2) give that the required sum is

$$X \sum_{\underline{d},\underline{e}<R}' \frac{\lambda_{\underline{d}}\lambda_{\underline{e}}}{f([\underline{d},\underline{e}])} + O\Big(\sum_{\underline{d},\underline{e}<R}' |\lambda_{\underline{d}}||\lambda_{\underline{e}}||r_{[\underline{d},\underline{e}]}|\Big), \tag{3.14}$$

where the dash over the sum has the same meaning as in Lemma 3.4. The first term is viewed as the main term of our estimate, while the O-sum is the error term. With our choice of $\lambda_{\underline{d}}$'s, the main term becomes

$$X \sum_{\underline{d},\underline{e}} \frac{\mu(\underline{d})\mu(\underline{e})}{f([\underline{d},\underline{e}])} \mathcal{F}\Big(\frac{\log \underline{d}}{\log R}\Big)\mathcal{F}\Big(\frac{\log \underline{e}}{\log R}\Big).$$

Using Lemma 3.4, this is equal to

$$(1+o(1))C(\mathcal{F},\mathcal{F})^{(\underline{\alpha})}c(W)\frac{X}{(\log R)^{\alpha}},$$

as required.

The choice of R along with condition (H4) means that

$$\sum_{\underline{d},\underline{e}<R} |\lambda_{\underline{d}}||\lambda_{\underline{e}}||r_{[\underline{d},\underline{e}]}| \ll \sum_{[\underline{d},\underline{e}]<X^{\theta}} |r_{[\underline{d},\underline{e}]}| \ll \frac{X}{(\log X)^A}.$$

We can thus neglect the error term by taking A sufficiently large. □

4. Applications

4.1. Bounded gaps between primes. In this section, we apply the higher rank sieve discussed above to the well-known prime k-tuples problem. A set $\mathscr{H}$ of distinct nonnegative integers is said to be admissible if for every prime p there is a residue class $b_p \pmod p$ such that $b_p \notin \mathscr{H} \pmod p$. We will work with a fixed admissible k-tuple

$$\mathscr{H} = \{h_1, \ldots, h_k\}.$$

We apply the sieve to the set $S_N = \{\underline{n} = (n+h_1, \dots, n+h_k)\colon\ n \sim N\}$. We use the '$W$ trick' to remove the effect of small primes, that is we restrict n to be in a fixed residue class b modulo W, where $W = \prod\limits_{p<D_0} p$ and b is chosen so that $b + h_i$ is co-prime to W for each h_i. This choice of b is possible because of admissibility of the set $\mathscr{H}$. One can choose $D_0 = \log\log\log N$, so that $W \sim (\log\log N)^{(1+o(1))}$ by an application of the prime number theorem.

Let $\chi_{\mathbb{P}}$ denote the characteristic function of the primes. Consider two expressions

$$Q_1 = \sum_{\substack{n\sim N\\ n\equiv b \pmod W}} a_n$$

and

$$Q_2 = \sum_{\substack{n\sim N\\ n\equiv b \pmod W}} \Bigl(\sum_{m=1}^{k} \chi_{\mathbb{P}}(n+h_m)\Bigr) a_n,$$

where a_n are nonnegative parameters given by

$$a_n = \Bigl(\sum_{\underline{d}\mid\underline{n}} \lambda_{\underline{d}}\Bigr)^2,$$

with $\lambda_{\underline{d}}$'s chosen as in (3.13). For ϱ positive, we denote by $Q(N, \varrho)$ the quantity

$$Q_2 - \varrho Q_1 = \sum_{\substack{n\sim N\\ n\equiv b \pmod W}} \Bigl(\sum_{j=1}^{k} \chi_{\mathbb{P}}(n+h_j) - \varrho\Bigr) a_n. \tag{4.1}$$

The key idea then used is the following proposition.

Proposition 4.1. *Given a positive number ϱ, if*

$$Q(N, \varrho) > 0$$

for all large N, then there are infinitely many integers n such that at least $\lfloor \varrho \rfloor + 1$ of $n + h_1, \dots, n + h_k$ are primes.

Proof. The definition of $Q(N, \varrho)$ gives that

$$\sum_{\substack{n\sim N\\ n\equiv b \pmod W}} \Bigl(\sum_{j=1}^{k} \chi_{\mathbb{P}}(n+h_j) - \varrho\Bigr) a_n > 0.$$

As a_n are nonnegative parameters, we must have

$$\sum_{j=1}^{k} \chi_{\mathbb{P}}(n+h_j) - \varrho > 0$$

for some $n \sim N$. As this happens for all large N,

$$\sum_{j=1}^{k} \chi_{\mathbb{P}}(n+h_j) > \varrho$$

holds for infinitely many integers n. As each $\chi_{\mathbb{P}}(n+h_j)$ is an integer, this completes the proof. $\square$

As an application of the higher rank sieve, we derive asymptotic formulas for Q_1 and Q_2 that agree with those obtained by Maynard in [5] and the Polymath project in [9].

Let $\pi(x)$ denote the number of primes upto x. For $(a, q) = 1$, put

$$E_{\mathbb{P}}(x, q, a) = \sum_{\substack{n \leqslant x \\ n \equiv a \pmod{q}}} \chi_{\mathbb{P}}(n) - \frac{\pi(x)}{\varphi(q)}. \tag{4.2}$$

Then for any $A > 0$ and any $\theta < 1/2$, the Bombieri-Vinogradov theorem establishes that

$$\sum_{q \leqslant x^{\theta}} \max_{(a,q)=1} |E_{\mathbb{P}}(x, q, a)| \ll \frac{x}{(\log x)^A}. \tag{4.3}$$

The higher rank sieve gives us the following asymptotic formula for Q_1.

Lemma 4.2. *Choose $\theta < 1$, $\lambda_{\underline{d}}$'s as in (3.13) in terms of the function $\mathcal{F}$, and $R = N^{\theta/2-\delta}$. Then, as $N \to \infty$,*

$$Q_1 := \sum_{\substack{n \sim N \\ n \equiv b \pmod{W}}} \Bigg(\sum_{d_j | (n+h_j) \ \forall j} \lambda_{\underline{d}} \Bigg)^2 = (1 + o(1)) \frac{W^{k-1}}{\varphi(W)^k} \frac{N}{(\log R)^k} I(\mathcal{F}),$$

where

$$I(\mathcal{F}) = \int_{\Delta_k(1)} (\mathcal{F}^{(\underline{1})}(\underline{t}))^2 \, \mathrm{d}t_1 \dots \mathrm{d}t_k,$$

with

$$\mathcal{F}^{(\underline{1})}(\underline{t}) = \frac{\partial^k \mathcal{F}(t_1, \dots, t_k)}{\partial t_1 \dots \partial t_k}. \tag{4.4}$$

P r o o f. We will prove this as a special case of Theorem 3.6 to facilitate better understanding of the general sieve at work here.

We begin by establishing the setting of the sieve. The tuple $\underline{n}$ in this case is $(n+h_1,\dots,n+h_k)$, where h_i are elements of the fixed set $\mathscr{H}$. The set $\mathcal{S}$ is given by

$$\mathcal{S}=\{\underline{n}=(n+h_1,\dots,n+h_k)\colon\ n\sim N\}.$$

The choice of W was stated at the beginning of this section. The tuple $\underline{b}$ is then $(b+h_1,\dots,b+h_k)$, each component of which is co-prime to W. The weights $w_{\underline{n}}$ are all equal to 1. For this choice of weights, any $\theta<1$ suffices, as we will see.

We verify hypotheses (H1) to (H4). It is clear that (H1) holds as $\mathscr{H}$ is a fixed set. To obtain (H2) with the weights $w_{\underline{n}}=1$, we consider the sum

$$\sum_{\substack{n\sim N,\\ d_j\mid(n+h_j)\ \forall j\\ n\equiv b \pmod W}} 1.$$

We first note that in the above expression, $n+h_j\equiv b+h_j \pmod W$ means that $n+h_j$ is co-prime to W for all j. Hence we must have $(d_j,W)=1$ for all j. Indeed, if a prime p divides both, then p divides both $n+h_j$ and W, which cannot happen. Moreover, if p divides $n+h_i$ and $n+h_j$, for $i\neq j$, then $p\mid(h_j-h_i)$ and hence $p\mid W$. This again gives $p\mid(n+h_j)$ and $p\mid W$, which is a contradiction.

Thus, all the $(n+h_j)$'s and in particular all the d_j's are mutually co-prime and each d_j is co-prime to W. Then we can write the resulting sum as a sum over a single residue class modulo $W\prod\limits_j d_j$ and use the Chinese remainder theorem to get

$$\frac{N}{Wd_1\dots d_k}+O(1).$$

Thus $X=N/W$ and $r_{\underline{d}}=O(1)$. Furthermore, $f_j(p)=p$ for all j, thereby showing that (H3) holds with $\alpha_j=1$ for each j. In order to verify (H4), observe that invoking Proposition 3.1, we have

$$\sum_{[\underline{d},\underline{e}]\leqslant N^\theta} 1=\sum_{r\leqslant N^\theta}\Omega_k(r)\leqslant\sum_{r\leqslant N^\theta}\tau_{3k}(r).$$

As the average order of $\tau_{3k}(r)$ is $(\log r)^{3k-1}$ and $0<\theta<1$, we see that

$$\sum_{[\underline{d},\underline{e}]\leqslant N^\theta}|r_{[\underline{d},\underline{e}]}|\ll N^\theta(\log N^\theta)^{3k-1}\ll\frac{N}{(\log N)^A}$$

for any $A>0$. Hence, (H4) is satisfied.

The proof is then a straightforward application of Theorem 3.6 as follows. As $\underline{\alpha}$ is the tuple $(1, \dots, 1)$ and $\alpha = \sum_j \alpha_j = k$, one sees that the factor $C(\mathcal{F}, \mathcal{F})^{(\underline{\alpha})}$ is nothing but the integral η given above. We also have

$$c(W) = \prod_{p|W} \frac{p^k}{\varphi(p)^k} = \frac{W^k}{\varphi(W)^k}$$

in this case. Letting $X = N/W$ completes the proof. □

We now derive the asymptotic formula for Q_2.

Lemma 4.3. *With θ chosen so that* (4.3) *holds, $\lambda_{\underline{d}}$'s chosen as in* (3.13) *in terms of $\mathcal{F}$, and $R = N^{\theta/2-\delta}$, we have as $N \to \infty$,*

$$\begin{aligned} Q_2^{(m)} &:= \sum_{\substack{n \sim N \\ n \equiv b \pmod W}} \chi_{\mathbb{P}}(n + h_m) \Big(\sum_{d_j | (n+h_j)\ \forall j} \lambda_{\underline{d}} \Big)^2 \\ &= (1 + o(1)) \frac{W^{k-1}}{\varphi(W)^k} \frac{(\pi(2N) - \pi(N))}{(\log R)^{k-1}} J_m(\mathcal{F}), \end{aligned}$$

with $J_m(\mathcal{F})$ given by the integral

$$\int_{\Delta_{k-1}(1)} (\mathcal{F}_m^{(\underline{1})}(t_1, \dots, t_{m-1}, t_{m+1}, \dots, t_k))^2 \, \mathrm{d}t_1 \dots \mathrm{d}t_{m-1} \, \mathrm{d}t_{m+1} \dots \mathrm{d}t_k.$$

We use the usual notation of (2.1)*, but here $\mathcal{F}_m$ is the function $\mathcal{F}$ restricted to tuples with mth component zero, more precisely,*

$$\mathcal{F}_m(t_1, \dots, t_{m-1}, t_{m+1}, \dots, t_k) = \mathcal{F}(t_1, \dots, t_{m-1}, 0, t_{m+1}, \dots, t_k). \tag{4.5}$$

P r o o f. The proof is an application of Theorem 3.6 after verification of the required hypotheses. (H1) holds by construction. To obtain (H2), we consider the sum

$$\sum_{\substack{n \sim N, \\ d_j | (n+h_j)\ \forall j \\ n \equiv b \pmod W}} \chi_{\mathbb{P}}(n + h_m).$$

This sum is nonzero if and only if $d_m = 1$ and hence this additional condition must be imposed. Denoting $n + h_m$ as n', and $\chi(n)$ as the function taking the value 1 if $n = 1$ and 0 otherwise, we can rewrite the above sum as

$$\chi(d_m) \sum_{\substack{n \sim N, \\ n' \equiv (h_m - h_j) \pmod{d_j}\ \forall j \neq m \\ n' \equiv b + h_m \pmod W}} \chi_{\mathbb{P}}(n').$$

As done in the proof of Lemma 4.2, one can show that the residue classes above are co-prime to the corresponding moduli and the moduli are themselves mutually co-prime, thereby facilitating the use of the Chinese remainder theorem to rewrite the inner sum as the sum over some co-prime residue class a modulo $q = W \prod_{j\neq m} d_j$. Using (4.2), this gives for the inner sum above

$$\frac{\pi(2N)-\pi(N)}{\varphi(W)\prod_{j\neq m}\varphi(d_j)} + E_{\mathbb{P}}(N,q,a).$$

We must exercise caution because of the constraint $d_m = 1$. This condition means that the sieve has now collapsed to a $(k-1)$ rank sieve. Hence fixing $d_m = 1$ in the sum that appears in (H2), this hypothesis holds with $X = (\pi(2N)-\pi(N))/\varphi(W)$ and $f_j(d_j) = \varphi(d_j)$ for all $j \neq m$. The error $r_{\underline{d}}$ is given by $E_{\mathbb{P}}(N,q,a)$. Furthermore, this shows that (H3) holds with $\alpha_j = 1$ for each $j \neq m$.

Keeping the additional constraint on the mth component in mind, to check (H4) we must check that

$$\sum_{\substack{[\underline{d},\underline{e}]<N^\theta\\ d_m=e_m=1}} |E_{\mathbb{P}}(N,q,a)| \ll \frac{N}{(\log N)^A}$$

for any $A > 0$. Here $q = W \prod_{j\neq m}[d_j, e_j]$. There exists $\varepsilon > 0$ such that (4.3) holds for $\theta' = \theta + \varepsilon$. As $W \sim \log\log N$, the above sum is bounded by

$$\sum_{\substack{q<N^{\theta'}\\ d_m=e_m=1}} |E_{\mathbb{P}}(N,q,a)|,$$

which is $\ll N/(\log N)^A$ by (4.3). The proof is then an application of Theorem 3.6 with some modification. In the sum

$$\sum_{\underline{n}\equiv\underline{b} \pmod W} w_n \Big(\sum_{\underline{d}|\underline{n}} \lambda_{\underline{d}}\Big)^2$$

appearing in Theorem 3.6, we must restrict d_m to be 1. By (3.13), this means that the function $\mathcal{F}$ is only evaluated on k-tuples whose mth component is 0. The above mentioned theorem can then be applied, with $\alpha_j = 1$ for all $j \neq m$ and $\alpha = \sum_{j\neq m}\alpha_j = k-1$. Viewing the function $\mathcal{F}(\underline{t})$ as effectively a function on $(k-1)$-tuples, we obtain

$$C(\mathcal{F},\mathcal{F})^{(\underline{\alpha})} = J_m(\mathcal{F})$$

in this case, as required. □

As done in [5], we can restrict $\mathcal{F}$ to be a symmetric function without any loss of generality due to the symmetry of the integrals $I(\mathcal{F})$ and $\sum_{m=1}^{k} J_m(\mathcal{F})$. This allows us to set $m = k$ in the result above enabling us to state it in the form given by the Polymath project [9]. In particular, then

$$J(\mathcal{F}) := \sum_{m=1}^{k} J_m(\mathcal{F}) = kJ_k(\mathcal{F}). \tag{4.6}$$

Choosing some $\theta < 1$ for which (4.3) holds, so that θ is admissible in the derivation of the asymptotic formula for Q_1 as well as Q_2, one obtains the following after using the prime number theorem.

Lemma 4.4. *Let $J(\mathcal{F})$ be as in (4.6). Then, with $\theta < 1/2$, $\lambda_{\underline{d}}$'s chosen as in (3.13) in terms of $\mathcal{F}$, and $R = N^{\theta/2-\delta}$, we have as $N \to \infty$*

$$\begin{aligned} Q(N, \varrho) &:= Q_2 - \varrho Q_1 \\ &= \frac{W^{k-1}}{\varphi(W)^k} \frac{N}{(\log R)^k} \Big(\Big(\frac{\theta}{2} - \delta \Big) J(\mathcal{F}) - \varrho I(\mathcal{F}) \Big) + o\Big(\frac{W^{k-1}}{\varphi(W)^k} \frac{N}{(\log R)^k} \Big). \end{aligned}$$

Dropping the notation $J(\mathcal{F})$, $I(\mathcal{F})$ in favor of the less unwieldy J, I, we see from the above lemma that we need

$$\varrho < \Big(\frac{\theta}{2} - \delta \Big) \frac{J}{I}, \tag{4.7}$$

and hence we wish to maximize the ratio $M_k := J/I = kJ_k/I$. We will aim at finding a lower bound for this ratio and then maximize this lower bound in order to find the optimal ϱ. We write $\mathcal{F}^{(\underline{1})}(\underline{t})$ as $G(\underline{t})$, where G is a smooth function supported on the simplex $\Delta_k(1)$. Writing G as

$$G(\underline{t}) = \frac{\partial}{\partial t_m} \Big(\frac{\partial^{k-1} \mathcal{F}(\underline{t})}{\partial t_1 \dots \partial t_{m-1} \partial t_m \dots \partial t_k} \Big),$$

we see from the fundamental theorem of calculus that the function $\mathcal{F}_m^{(\underline{1})}$ that appears in the definition of J in Lemma 4.3 is simply the anti-derivative of G with respect to the mth component, evaluated at $t_m = 0$. It is also clear that the anti-derivative of G with respect to the mth component has the same support as $\mathcal{F}$. Hence, we can write

$$\int_0^\infty G(\underline{t}) \, \mathrm{d}t_m = - \Big(\frac{\partial^{k-1} \mathcal{F}(\underline{t})}{\partial t_1 \dots \partial t_{m-1} \partial t_m \dots \partial t_k} \Big) \Big|_{t_m = 0} = -\mathcal{F}_m^{(\underline{1})}(\underline{t}).$$

This allows us to recast M_k in terms of $G(\underline{t})$ as

$$M_k = k\frac{\int_{\Delta_{k-1}(1)}\left(\int_0^\infty G(\underline{t})\,\mathrm{d}t_k\right)^2 \mathrm{d}t_1\dots dt_{k-1}}{\int_{\Delta_k(1)}(G(\underline{t}))^2\,\mathrm{d}\underline{t}}. \tag{4.8}$$

Observe that the choice of $G(\underline{t})$ corresponding to the GPY test function, namely

$$G(\underline{t}) = \left(1 - \sum_{i=1}^{k} t_i\right)^l,$$

yields

$$M_k = \frac{k}{(l+1)^2}\frac{I_{2l+2,k-1}}{I_{2l,k}},$$

where the integral $I_{l,k}$ is defined as

$$I_{l,k} = \int_{\Delta_k(1)}\left(1 - \sum_{i=1}^{k} t_i\right)^l \mathrm{d}\underline{t}.$$

After some routine calculations based on induction, one can show $I_{l,k} = l!/(l+k)!$, so that

$$M_k = \frac{2k(2l+1)}{(l+1)(k+2l+1)}.$$

The limiting value of M_k for large k (assuming l also becomes large, for example $l = \lfloor\sqrt{k}\rfloor$) is 4. The inequality (4.7) then becomes $\varrho < 2\theta - 2\delta$, which is satisfied for some ϱ with $\lfloor\varrho\rfloor = 1$, provided we assume $\theta > 1/2$. Applying Proposition 4.1, we see that the GPY choice of a test function yields bounded gaps between primes *conditionally*, under the assumption that the primes have a level of distribution $\theta > 1/2$. This observation, made by GPY, played a key role in Zhang's result on bounded gaps between primes.

This was treated unconditionally by Maynard in [5], who chose G on its support $\Delta_k(1)$ to be the product of one dimensional functions

$$G(\underline{t}) = \prod_{i=1}^{k} g(kt_i),$$

with $g(u)$ given by $g(u) := (1 + Au)^{-1}$ for some fixed $A > 0$, g supported on $u \in [0, T]$. He arrived at this choice of functions via an optimization argument and then evaluated the integrals involved by suitably choosing A and T in terms of k. After some work, this gives

$$M_k \geqslant \log k - 2\log\log k - 2,$$

when k is sufficiently large. As M_k can be made as large as needed by taking k large enough, applying Proposition 4.1 proves that there are infinitely many bounded gaps between primes. One can also show that $M_k \geqslant \log k$ as indicated in [9].

4.2. Bounded gaps between primes in Chebotarev sets. It is worth noting that the higher rank sieve can be applied to prove bounded gaps for any sequence of primes satisfying the axioms (H1) to (H4). In particular, these axioms hold for primes in arithmetic progressions and more generally, primes satisfying certain Chebotarev conditions.

The corresponding result on bounded gaps between primes satisfying certain Chebotarev conditions has been obtained by Thorner in [12]. It is evident that our formulation gives a clean and simplified proof of this result. For instance, in this situation, (H1) can be seen to hold with an appropriate choice of W, and the Chebotarev density theorem allows one to fulfill (H2), (H3). The crucial hypothesis (H4) is satisfied by invoking the variant of the Bombieri-Vinogradov theorem established by M. Ram Murty and V. Kumar Murty in [7].

4.3. Bounded gaps between primes with a given primitive root. Given an integer $g \neq \pm 1$ which is not a perfect square, Artin conjectured in 1927 that there are infinitely many primes for which g is a primitive root. This conjecture was proved under the Generalized Riemann hypothesis (GRH) by Hooley in [3], who obtained an asymptotic formula for the number of such primes up to x.

Fixing g as above, let $\mathscr{P}$ denote the set of primes p for which g is a primitive root $\pmod p$. Assuming GRH, Pollack in [8] showed that there are infinitely many bounded gaps between primes contained in $\mathscr{P}$. In order to do this, the sums considered are S_1 and S_2, where S_1 is the same as Q_1 considered in Section 4.1 and S_2 is given by

$$S_2 = \sum_{\substack{n \sim N \\ n \equiv b \pmod W}} \Bigg(\sum_{m=1}^{k} \chi_{\mathscr{P}}(n+h_m)\Bigg)\Bigg(\sum_{\underline{d} | \underline{n}} \lambda_{\underline{d}}\Bigg)^2,$$

with $\lambda_{\underline{d}}$'s chosen as before. It is possible to show that $S_2 \sim Q_2$ by observing that

$$0 \leqslant |\mathbb{P} - \mathscr{P}| \leqslant \sum_{q} \#\{p\colon\ p \equiv 1 \pmod q,\ g \pmod p \text{ has order } (p-1)/q\},$$

where the sum runs over all primes q. One then splits the sum over q into different ranges in order to show that the corresponding difference between S_2 and Q_2 is small.

By a theorem of Dedekind, we have for $p \nmid g$

$$p \equiv 1 \pmod q,\ g^{(p-1)/q} \equiv 1 \pmod p \Leftrightarrow p \text{ splits completely in } L_q = \mathbb{Q}(\zeta_q, \sqrt[q]{g}),$$

where ζ_q is a primitive qth root of unity. Let n_q denote the degree of the finite extension $L_q/\mathbb{Q}$. Then this problem is essentially an application of our higher rank sieve with the hypothesis (H4) satisfied by the following special case of an effective Chebotarev density theorem by Lagarias and Odlyzko in [4], conditional upon GRH (for each of the Dedekind zeta functions $\zeta(s, L_q)$):

$$\pi_q(x) = \frac{\mathrm{li}(x)}{n_q} + O\Big(\frac{x^{1/2}}{n_q}\log(d_q x^{n_q})\Big),$$

where $\mathrm{li}(x)$ denotes the logarithmic integral $\int_2^x \mathrm{d}t/\log t$, π_q denotes the number of primes $p \leqslant x$ which split completely in L_q and d_q is the discriminant, $\mathrm{disc}(L_q/\mathbb{Q})$. In particular, as indicated by M. Ram Murty in [6], an average result of the following form would suffice for this purpose: Given any $B > 0$, there exists $A > 0$ (depending on B) such that

$$\sum_{q \leqslant \frac{x^{1/2}}{(\log x)^B}} \Big|\pi_q(x) - \frac{\mathrm{li}(x)}{n_q}\Big| \ll \frac{x}{(\log x)^A}.$$

This also allows one to fulfill (H2), (H3). The hypothesis (H1) entails a careful choice of W and the residue class $\underline{b} \pmod{W}$, as discussed in [8].

5. Concluding remarks

As mentioned in the introduction, we have taken a Fourier analytic approach to the higher rank Selberg sieve rather than the combinatorial approach of Selberg and Maynard. In the latter approach, the problem reduces to a comparison of two Selberg sieves in that we would consider

$$\sum_{\underline{n}\in\mathcal{S}} (a_{\underline{n}} - b_{\underline{n}}) \Big(\sum_{\underline{d}|\underline{n}} \lambda_{\underline{d}}\Big)^2,$$

with two different sets of weights $a_{\underline{n}}$ and $b_{\underline{n}}$, and attempt to derive a lower bound. This would allow us to deduce that we have $a_{\underline{n}} > b_{\underline{n}}$ for infinitely many $\underline{n}$. As in the classical Selberg sieve, combinatorial considerations admit a choice of $\lambda_{\underline{d}}$'s so as to minimize a quadratic form and thus optimize one of the terms, and then one needs to study how this choice affects the other term. One could develop this along classical lines and derive analogous results.

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