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THE EXCEPTIONAL SET FOR DIOPHANTINE INEQUALITY WITH UNLIKE POWERS OF PRIME VARIABLES

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Abstract. Suppose that $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are nonzero real numbers, not all negative, $\delta > 0$, $\mathcal{V}$ is a well-spaced set, and the ratio λ_1/λ_2 is algebraic and irrational. Denote by $E(\mathcal{V}, N, \delta)$ the number of $v \in \mathcal{V}$ with $v \leqslant N$ such that the inequality

$$|\lambda_1 p_1^2 + \lambda_2 p_2^3 + \lambda_3 p_3^4 + \lambda_4 p_4^5 - v| < v^{-\delta}$$

has no solution in primes p_1, p_2, p_3, p_4. We show that

$$E(\mathcal{V}, N, \delta) \ll N^{1+2\delta-1/72+\varepsilon}$$

for any $\varepsilon > 0$.

Keywords: Davenport-Heilbronn method; prime varaible; exceptional set; Diophantine inequality

MSC 2010: 11D75, 11P32, 11P55

1. Introduction

Let λ_1, λ_2, λ_3, λ_4, λ_5 be nonzero real numbers, not all negative. Suppose that λ_1/λ_2 is irrational. Recently, Yang and Li in [14] considered the inequality

$$\Big|\lambda_1 x_1^2 + \lambda_2 x_2^3 + \lambda_3 x_3^4 + \lambda_4 x_4^5 - p - \frac{1}{2}\Big| < \frac{1}{2} \tag{1.1}$$

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and proved that the inequality (1.1) has infinite solutions with prime p and integers x_1, x_2, x_3, x_4. In [4], Ge and Li replaced the integer variables by prime variables and proved that for any real number η, the inequality

$$(1.2)\qquad |\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^3 + \lambda_4 p_4^4 + \lambda_5 p_5^5 + \eta| < \Bigl(\max_{1\leqslant j\leqslant 5} p_j^j\Bigr)^{-1/720+\varepsilon}$$

has infinite solutions with primes p_1, p_2, p_3, p_4, p_5. In this paper, we drop the linear prime variable in (1.2) and consider the exceptional set for the inequality

$$(1.3)\qquad |\lambda_1 p_1^2 + \lambda_2 p_2^3 + \lambda_3 p_3^4 + \lambda_4 p_4^5 - v| < v^{-\delta}.$$

First, we recall a definition in [5]. We call a set of positive real numbers $\mathcal{V}$ a *well-spaced set* if there is a $c > 0$ such that

$$u, v \in \mathcal{V},\ u \neq v \ \Rightarrow\ |u - v| > c.$$

In order to get the full strength of the results quoted here one must also assume that

$$|\{v \in \mathcal{V}\colon\ 0 \leqslant v \leqslant N\}| \gg N^{1-\varepsilon}.$$

Let λ_1, λ_2, λ_3 be nonzero real numbers, not all negative, let $\mathcal{V}$ be a well-spaced set, and let $\delta > 0$. We introduce the notation $E_1(\mathcal{V}, X, \delta)$ to denote the number of $v \in \mathcal{V}$ with $v \leqslant X$ such that the inequality

$$|\lambda_1 p_1^2 + \lambda_2 p_2^2 + \lambda_3 p_3^2 - v| < v^{-\delta}$$

has no solution in primes p_1, p_2, p_3. Harman in [5] showed that if λ_1/λ_2 is irrational and algebraic, then one has

$$E_1(\mathcal{V}, X, \delta) \ll X^{7/8+2\delta+\varepsilon},$$

for any $\varepsilon > 0$. He also proved that 7/8 can be replaced by 6/7 using his sieve method.

Recently, under similar conditions, Mu and Lü in [9] proved that for any $\varepsilon > 0$

$$E_2(\mathcal{V}, X, \delta) \ll X^{67/72+2\delta+\varepsilon},$$

where $E_2(\mathcal{V}, X, \delta)$ is the number of $v \in \mathcal{V}$ with $v \leqslant X$ such that the inequality

$$|\lambda_1 p_1^2 + \lambda_2 p_2^2 + \lambda_3 p_3^3 + \lambda_4 p_4^3 - v| < v^{-\delta}$$

has no solution in primes p_1, p_2, p_3, p_4.

In this paper, we consider the inequality (1.3). Denote by $\mathbb{E}(\mathcal{V}, N, \delta)$ the set of $v \in \mathcal{V}$ with $v \leqslant N$ such that the inequality (1.3) has no solution in primes p_1, p_2, p_3, p_4. Let $E(\mathcal{V}, N, \delta) = |\mathbb{E}(\mathcal{V}, N, \delta)|$. Using some ingredients from [5] and [8], we establish the following results.

Theorem 1.1. *Suppose that λ_1, λ_2, λ_3, λ_4 are nonzero real numbers, not all negative, $\delta > 0$, $\mathcal{V}$ is a well-spaced set, the ratio λ_1/λ_2 is algebraic and irrational. Then*

$$E(\mathcal{V}, N, \delta) \ll N^{1+2\delta-1/72+\varepsilon} \tag{1.4}$$

for any $\varepsilon > 0$.

Theorem 1.2. *Let λ_1, λ_2, λ_3, λ_4 be nonzero real numbers, not all negative. Suppose that λ_1/λ_2 is irrational. Let $\mathcal{V}$ be a well-spaced set. Let $\delta > 0$. Then there is a sequence $N_j \to \infty$ such that*

$$E(\mathcal{V}, N_j, \delta) \ll N_j^{1+2\delta-1/72+\varepsilon} \tag{1.5}$$

for any $\varepsilon > 0$. Moreover, if the convergent denominators q_j for λ_1/λ_2 satisfy

$$q_{j+1}^{1-\omega} \ll q_j \qquad \text{for some } \omega \in [0, 1), \tag{1.6}$$

then, for all $N \geqslant 1$,

$$E(\mathcal{V}, N, \delta) \ll N^{1+2\delta-(1/6)\chi+\varepsilon} \tag{1.7}$$

for any $\varepsilon > 0$ with

$$\chi = \min\Big(\frac{6-6\omega}{29-20\omega}, \frac{1}{12}\Big). \tag{1.8}$$

Remark. In the case of λ_1/λ_2 algebraic, we claim that we can take $\omega = \varepsilon$ in (1.6), where ε is an arbitrarily small positive real number. If not, there are infinitely many q_j satisfying $q_{j+1}^{1-\varepsilon} \gg q_j$. Since q_j is a convergent denominator for λ_1/λ_2, by the proof of Lemma 4D on page 14 of [11], we have that there are infinitely many q_j satisfying

$$\Big\| q_j \frac{\lambda_1}{\lambda_2} \Big\| < \frac{1}{q_{j+1}} \ll \frac{1}{q_j^{1/(1-\varepsilon)}}.$$

This is in contradiction with Roth's theorem (Theorem 2A on page 116 of [11]). Thus Theorem 1.1 follows immediately from Theorem 1.2.

Notation. Throughout, the symbols p and p_j denote primes. Nonzero real numbers λ_j, $j=1,2,3,4$, are given constants, and Vinogradov symbols may depend on λ_j, $j=1,2,3,4$. The number ε is an arbitrarily small positive real number, but not necessarily the same, we may write $X^{2\varepsilon}\ll X^{\varepsilon}$. Denote by $\|y\|$ the distance from y to the nearest integer. We write $e(x)=\exp(2\pi \mathrm{i} x)$.

2. Outline of the proof and preliminary lemmas

Suppose that N is some large positive quantity which we will choose later. Let

$$\mathbb{E}(\mathfrak{I})=\mathbb{E}(\mathcal{V},N,\delta)\cap\mathfrak{I},\quad E(\mathfrak{I})=|\mathbb{E}(\mathfrak{I})|, \tag{2.1}$$

where $\mathfrak{I}$ is any subset of $[0,N]$. By the definition of the well-spaced set $\mathcal{V}$ and the definition of $\mathbb{E}(\mathcal{V},N,\delta)$, it is easy to show that

$$E(\mathcal{V},N,\delta)=E([N^{71/72},N])+E([0,N^{71/72}))\ll E([N^{71/72},N])+N^{71/72}. \tag{2.2}$$

So we just need to estimate $E([N^{71/72},N])$. We use the standard dyadic argument. Let $N^{71/72}\leqslant X\leqslant N$. We will estimate $E([X/2,X])$. So we shall restrict our attention to those v satisfying

$$\frac{X}{2}\leqslant v\leqslant X. \tag{2.3}$$

We use the Hardy-Littlewood circle method which was first stated by Davenport-Heilbronn in [3]. Suppose that $\mathcal{I}_j=[(X/16|\lambda_{j-1}|)^{1/j},(X/|\lambda_{j-1}|)^{1/j}]$ for $j=2,3,4,5$, and $0<\tau<1$. We define

$$K(\alpha)=\begin{cases}\left(\dfrac{\sin\pi\tau\alpha}{\pi\alpha}\right)^2 & \text{if } \alpha\neq 0;\\ \tau^2 & \text{if } \alpha=0,\end{cases} \tag{2.4}$$

$$S_j(\alpha)=\sum_{p\in\mathcal{I}_j}(\log p)e(\alpha p^j), \tag{2.5}$$

$$F_j(\alpha)=\int_{\mathcal{I}_j}e(\alpha u^j)\,\mathrm{d}u,\qquad U_j(\alpha)=\sum_{n\in\mathcal{I}_j}e(\alpha n^j), \tag{2.6}$$

for $j=2,3,4,5$.

Then we can easily get (also see [12] or [13])

$$K(\alpha)\ll\min(\tau^2,|\alpha|^{-2}),\qquad \int_{-\infty}^{\infty}K(\alpha)e(\alpha x)\,\mathrm{d}\alpha=\max(0,\tau-|x|). \tag{2.7}$$

For convenience, we define

$$\mathcal{I}(v,X;\mathfrak{X}) = \int_{\mathfrak{X}} S_2(\lambda_1\alpha)S_3(\lambda_2\alpha)S_4(\lambda_3\alpha)S_5(\lambda_4\alpha)K(\alpha)e(-v\alpha)\,\mathrm{d}\alpha, \tag{2.8}$$

where $\mathfrak{X}$ is a measurable subset of $\mathbb{R}$. Then by (2.7), we have

$$\begin{aligned}
&\mathcal{I}(v,X;\mathbb{R})\\
&\quad= \sum_{p_j\in\mathcal{I}_{j+1}} (\log p_1)\dots(\log p_4)\int_{-\infty}^{\infty} e((\lambda_1p_1^2+\lambda_2p_2^3+\lambda_3p_3^4+\lambda_4p_4^5-v)\alpha)K(\alpha)\,\mathrm{d}\alpha\\
&\quad= \sum_{p_j\in\mathcal{I}_{j+1}} (\log p_1)\dots(\log p_4)\max(0,\tau-|\lambda_1p_1^2+\lambda_2p_2^3+\lambda_3p_3^4+\lambda_4p_4^5-v|)\\
&\quad\leqslant (\log X)^4 \sum_{p_j\in\mathcal{I}_{j+1}} \max(0,\tau-|\lambda_1p_1^2+\lambda_2p_2^3+\lambda_3p_3^4+\lambda_4p_4^5-v|).
\end{aligned}$$

Thus we have

$$0 \leqslant \mathcal{I}(v,X;\mathbb{R}) \leqslant \tau(\log X)^4\mathcal{N}(v,X), \tag{2.9}$$

where $\mathcal{N}(v,X)$ is the number of solutions to the inequality

$$|\lambda_1p_1^2+\lambda_2p_2^3+\lambda_3p_3^4+\lambda_4p_4^5-v| < \tau, \qquad p_j\in\mathcal{I}_{j+1},\ j=1,2,3,4.$$

In order to estimate the integral $\mathcal{I}(v,X;\mathbb{R})$, we divide the region of integration into four parts: the major region $\mathfrak{C}$, the middle region $\mathfrak{D}$, the minor region $\mathfrak{c}$ and the trivial region $\mathfrak{t}$, which are defined as follows

$$\begin{aligned}
\mathfrak{C} &= \{\alpha\colon\ |\alpha|<\varphi\}, &\qquad \mathfrak{D} &= \{\alpha\colon\ \varphi\leqslant|\alpha|<\xi\},\\
\mathfrak{c} &= \{\alpha\colon\ \xi\leqslant|\alpha|<\kappa\}, &\qquad \mathfrak{t} &= \{\alpha\colon\ |\alpha|\geqslant\kappa\},
\end{aligned} \tag{2.10}$$

where $\varphi = X^{-13/15-\varepsilon}$, $\xi = X^{-7/9-\varepsilon}$ and $\kappa = \tau^{-2}X^{7/12+2\varepsilon}$.

We assemble here the important results we need for the analysis of all the regions for α.

Lemma 2.1 ([5], Lemma 3). *Suppose that $(a,q)=1$ and $|q\alpha-a|\leqslant q^{-1}$, then for any $\varepsilon>0$ we have*

$$\sum_{1\leqslant p\leqslant P} (\log p)e(\alpha p^2) \ll P^{1+\varepsilon}\big(q^{-1}+P^{-1/2}+qP^{-2}\big)^{1/4}.$$

Corollary 2.2 ([5], Corollary 1)**.** *Let* $X^{1/2} \geqslant Y \geqslant X^{1/2-1/16+\varepsilon}$*. If* $|S_2(\lambda_1\alpha)| > Y$*, then there are coprime integers* a*,* q *satisfying*

$$1 \leqslant q \ll \Big(\frac{X^{1/2+\varepsilon}}{Y}\Big)^4, \qquad |q\lambda_1\alpha - a| \ll \Big(\frac{X^{1/2+\varepsilon}}{Y}\Big)^4 X^{-1}. \tag{2.11}$$

Lemma 2.3. *Suppose that* α *is a real number, and there exist* $a \in \mathbb{Z}$ *and* $q \in \mathbb{N}$ *with*

$$(a,q) = 1, \quad 1 \leqslant q \leqslant P^{3/2} \quad \text{and} \quad |q\alpha - a| < P^{-3/2}.$$

Then one has

$$\sum_{P<p\leqslant 2P} (\log p) e(p^3\alpha) \ll P^{11/12+\varepsilon} + \frac{P^{1+\varepsilon}}{q^{1/2}(1+P^3|\alpha - a/q|)^{1/2}}.$$

Proof. This follows from Lemma 8.5 in [15] and Theorem 1.1 in [10] (one can also see Lemma 2.3 in [16]). □

Corollary 2.4. *Suppose that* $X^{1/3} \geqslant Y \geqslant X^{1/3-1/36+\varepsilon}$ *and* $|S_3(\lambda_2\alpha)| \geqslant Y$*. Then there are two coprime integers* a, q *satisfying*

$$1 \leqslant q \ll \Big(\frac{X^{1/3+\varepsilon}}{Y}\Big)^2, \qquad |q\lambda_2\alpha - a| \ll \Big(\frac{X^{1/3+\varepsilon}}{Y}\Big)^2 X^{-1}.$$

Proof. Let $P = X^{1/3}$ and $Q = P^{3/2}$. By Dirichlet's approximation theorem, there exist two coprime integers a, q with $1 \leqslant q \leqslant Q$ and $|q\lambda_2\alpha - a| \leqslant Q^{-1}$. By Lemma 2.3 and the hypothesis $Y \geqslant X^{1/3-1/36+\varepsilon}$, we have

$$\begin{aligned} X^{1/3-1/36+\varepsilon} &\leqslant Y \leqslant |S_3(\lambda_2\alpha)| \\ &\ll X^{1/3-1/36+\varepsilon/2} + \frac{X^{1/3+\varepsilon/2}}{q^{1/2}(1+X|\lambda_2\alpha - a/q|)^{1/2}}. \end{aligned}$$

Thus we have $1 \leqslant q \ll (X^{1/3+\varepsilon}/Y)^2$, $|q\lambda_2\alpha - a| \ll (X^{1/3+\varepsilon}/Y)^2 X^{-1}$. □

Lemma 2.5 ([6], Theorem 1)**.** *Suppose that* α *is a real number, and there exist* $a \in \mathbb{Z}$ *and* $q \in \mathbb{N}$ *with*

$$(a,q) = 1, \qquad |q\alpha - a| < q^{-1}. \tag{2.12}$$

Let k *be a positive integer with* $k \geqslant 2$*. Then for any* $\varepsilon > 0$*, one has*

$$\sum_{1\leqslant p\leqslant P} (\log p) e(p^k\alpha) \ll P^{1+\varepsilon}\Big(\frac{1}{q} + \frac{1}{P^{1/2}} + \frac{q}{P^k}\Big)^{4^{1-k}}. \tag{2.13}$$

Corollary 2.6. *Suppose that* $X^{-13/15-\varepsilon} \leqslant |\alpha| \leqslant X^{-7/9-\varepsilon}$. *Then*

$$|S_5(\lambda_4\alpha)| \ll X^{1/5-1/2560+\varepsilon}. \tag{2.14}$$

Proof. Without loss of generality, we need only consider $\lambda_4\alpha > 0$. If we take $q = [|\lambda_4\alpha|^{-1}]$, $a = 1$ in (2.12), where $[x]$ is the integral part of a real number x, then (2.14) follows from (2.13) immediately. □

Lemma 2.7. *With the previous notation, for* $k = 2, 3, 4, 5$ *and any positive integer* m *with* $1 \leqslant m \leqslant k$, *one has*

$$\int_{-\infty}^{\infty} |S_k(\alpha)|^{2^m} K(\alpha)\,\mathrm{d}\alpha \ll \tau X^{(2^m-m)/k+\varepsilon}, \quad \int_{-1}^{1} |S_k(\alpha)|^{2^m}\,\mathrm{d}\alpha \ll X^{(2^m-m)/k+\varepsilon}.$$

Proof. These results follow from Hua's lemma, one can also see Lemma 2.5 in [12] for details. □

Lemma 2.8 ([4], Lemma 4.2)**.** *With the previous notation, one has*

$$\int_{-\infty}^{\infty} |S_2(\lambda_1\alpha)^2 S_4(\lambda_3\alpha)^4| K(\alpha)\,\mathrm{d}\alpha \ll \tau X^{1+\varepsilon}, \quad \int_{-1}^{1} |S_2(\lambda_1\alpha)^2 S_4(\lambda_3\alpha)^4|\,\mathrm{d}\alpha \ll X^{1+\varepsilon}.$$

Lemma 2.9. *Suppose that* λ_1/λ_2 *is an irrational number, and* a/q *is a continued fraction convergent to* λ_1/λ_2. *If a positive integer* m *satisfies* $\|m\lambda_1/\lambda_2\| < 1/2q$, *then we have* $m \geqslant q$.

Proof. We prove by contradiction. Suppose that $m < q$, then by Legendre's law of best approximation for continued fractions (Theorem 5E on page 21 of [11]), we have

$$\Big\|q\frac{\lambda_1}{\lambda_2}\Big\| < \Big\|m\frac{\lambda_1}{\lambda_2}\Big\| < \frac{1}{2q}.$$

Since a/q is a continued fraction convergent to λ_1/λ_2, we have $(a, q) = 1$ and $\lambda_1/\lambda_2 = a/q + \theta/q^2$ with $|\theta| < 1$. Then

$$\Big\|q\frac{\lambda_1}{\lambda_2}\Big\| = \frac{|\theta|}{q} < \frac{1}{2q},$$

and

$$\frac{1}{2q} > \Big\|m\frac{\lambda_1}{\lambda_2}\Big\| = \Big\|\frac{ma}{q} + \frac{m\theta}{q^2}\Big\| \geqslant \frac{1}{q} - \frac{m|\theta|}{q^2}.$$

Thus we have $|\theta| < 1/2 < m|\theta|/q$; this is in contradiction with the hypothesis $m < q$. □

3. The major region $\mathfrak{C}$

In this section, we evaluate the integral on the major region $\mathfrak{C}$. The next theorem is our main result.

Theorem 3.1. *We have*

$$\mathcal{I}(v, X; \mathfrak{C}) \gg \tau^2 X^{17/60}. \tag{3.1}$$

First, we need some lemmas prepared to get the assertion of Theorem 3.1. For any real number $r \geqslant 1$, we set

$$J_r(X, h) = \int_{X/2}^{X} \big(\theta((x+h)^{1/r}) - \theta(x^{1/r}) - ((x+h)^{1/r} - x^{1/r})\big)^2 \,\mathrm{d}x,$$

where $\theta(x) = \sum\limits_{1 \leqslant p \leqslant x} \log p$ is the usual Chebyshev function.

Lemma 3.2 ([8], Theorem 2)**.** *For* $0 < Y \leqslant 1/2$ *and* $k = 2, 3, 4, 5$, *one has*

$$\int_{-Y}^{Y} |S_k(\alpha) - U_k(\alpha)|^2 \,\mathrm{d}\alpha \ll \frac{X^{2/k-2}\log^2 X}{Y} + Y^2 X + Y^2 J_k\Big(X, \frac{1}{2Y}\Big).$$

Lemma 3.3 ([8], Theorem 3)**.** *Let* ε *be an arbitrarily small positive constant. For* $k = 2, 3, 4, 5$, *there exists a positive constant* $c_1 = c_1(\varepsilon)$, *such that*

$$J_k(X, h) \ll h^2 X^{2/k-1} \exp\Big(-c_1\Big(\frac{\log X}{\log\log X}\Big)^{1/3}\Big)$$

holds uniformly for $X^{1-5/(6k)+\varepsilon} \leqslant h \leqslant X$.

From Lemmas 3.2 and 3.3, it is easy to deduce the following corollary.

Corollary 3.4. *For* $k = 2, 3, 4, 5$, *we have*

$$\int_{|\alpha| \leqslant X^{-1+5/(6k)-\varepsilon}} |S_k(\alpha) - U_k(\alpha)|^2 \,\mathrm{d}\alpha \ll X^{2/k-1} \exp\Big(-c_1\Big(\frac{\log X}{\log\log X}\Big)^{1/3}\Big).$$

Lemma 3.5. *For* $k = 2, 3, 4, 5$, *we have*

$$F_k(\alpha) \ll \min\Big(X^{1/k}, \frac{X^{1/k}}{|\alpha| X}\Big), \tag{3.2}$$

$$\int_{-1/2}^{1/2} |F_k(\alpha)|^2 \,\mathrm{d}\alpha \ll X^{2/k-1}. \tag{3.3}$$

P r o o f. This result follows from Lemma 1 of [1] (one can also see [5] or [12]). □

Lemma 3.6. *For $k=2,3,4,5$, we have*

$$\int_{|\alpha|\leqslant X^{-1+2/(3k)-\varepsilon}}|S_k(\alpha)-F_k(\alpha)|^2\,\mathrm{d}\alpha \ll X^{2/k-1}\exp\Bigl(-c_1\Bigl(\frac{\log X}{\log\log X}\Bigr)^{1/3}\Bigr), \tag{3.4}$$

$$\int_{|\alpha|\leqslant X^{-1+2/(3k)-\varepsilon}}|S_k(\alpha)|^2\,\mathrm{d}\alpha \ll X^{2/k-1}. \tag{3.5}$$

P r o o f. It is easy to show that

$$\begin{aligned}\int_{|\alpha|\leqslant X^{-1+2/(3k)-\varepsilon}}&|S_k(\alpha)-F_k(\alpha)|^2\,\mathrm{d}\alpha\\ &\ll \int_{|\alpha|\leqslant X^{-1+2/(3k)-\varepsilon}}|S_k(\alpha)-U_k(\alpha)|^2\,\mathrm{d}\alpha\\ &\quad+\int_{|\alpha|\leqslant X^{-1+2/(3k)-\varepsilon}}|U_k(\alpha)-F_k(\alpha)|^2\,\mathrm{d}\alpha.\end{aligned} \tag{3.6}$$

By Euler-Maclaurin summation formula, we have

$$|U_k(\alpha)-F_k(\alpha)| \ll 1+|\alpha|X. \tag{3.7}$$

Then

$$\begin{aligned}\int_{|\alpha|\leqslant X^{-1+2/(3k)-\varepsilon}}&|U_k(\alpha)-F_k(\alpha)|^2\,\mathrm{d}\alpha\\ &\ll \int_{|\alpha|\leqslant X^{-1+2/(3k)-\varepsilon}}(1+|\alpha|X)^2\,\mathrm{d}\alpha\\ &\ll \Bigl(\int_{|\alpha|\leqslant X^{-1}}+\int_{X^{-1}<|\alpha|\leqslant X^{-1+2/(3k)-\varepsilon}}\Bigr)(1+|\alpha|X)^2\,\mathrm{d}\alpha\\ &\ll X^{-1}+X^2(X^{-1+2/(3k)-\varepsilon})^3 \ll X^{2/k-1-\varepsilon}.\end{aligned} \tag{3.8}$$

Hence, (3.4) follows immediately from (3.6), (3.8) and Corollary 3.4. Then (3.5) follows from (3.4) and Lemma 3.5. □

Using the previous lemmas, we prove Theorem 3.1.

P r o o f of Theorem 3.1. Similarly to Lemma 3.3 in [4], we have

$$\int_{\mathfrak{C}} |S_2(\lambda_1\alpha)S_3(\lambda_2\alpha)S_4(\lambda_3\alpha)S_5(\lambda_4\alpha) - F_2(\lambda_1\alpha)F_3(\lambda_2\alpha)F_4(\lambda_3\alpha)F_5(\lambda_4\alpha)|K(\alpha)\,\mathrm{d}\alpha$$
$$\leqslant \tau^2|S_3(\lambda_2\alpha)S_4(\lambda_3\alpha)|\Big(\int_{-\varphi}^{\varphi}|S_2(\lambda_1\alpha)-F_2(\lambda_1\alpha)|^2\,\mathrm{d}\alpha\Big)^{1/2}\Big(\int_{-\varphi}^{\varphi}|S_5(\lambda_4\alpha)|^2\,\mathrm{d}\alpha\Big)^{1/2}$$
$$+\tau^2|F_2(\lambda_1\alpha)S_4(\lambda_3\alpha)|\Big(\int_{-\varphi}^{\varphi}|S_3(\lambda_2\alpha)-F_3(\lambda_2\alpha)|^2\,\mathrm{d}\alpha\Big)^{1/2}\Big(\int_{-\varphi}^{\varphi}|S_5(\lambda_4\alpha)|^2\,\mathrm{d}\alpha\Big)^{1/2}$$
$$+\tau^2|F_2(\lambda_1\alpha)F_3(\lambda_2\alpha)|\Big(\int_{-\varphi}^{\varphi}|S_4(\lambda_3\alpha)-F_4(\lambda_3\alpha)|^2\,\mathrm{d}\alpha\Big)^{1/2}\Big(\int_{-\varphi}^{\varphi}|S_5(\lambda_4\alpha)|^2\,\mathrm{d}\alpha\Big)^{1/2}$$
$$+\tau^2|F_2(\lambda_1\alpha)F_3(\lambda_2\alpha)|\Big(\int_{-\varphi}^{\varphi}|S_5(\lambda_4\alpha)-F_5(\lambda_4\alpha)|^2\,\mathrm{d}\alpha\Big)^{1/2}\Big(\int_{-\varphi}^{\varphi}|F_4(\lambda_3\alpha)|^2\,\mathrm{d}\alpha\Big)^{1/2}.$$

Note that $\varphi = X^{-13/15-\varepsilon}$. Then, by Lemmas 3.5, 3.6 and the trivial estimates of $S_k(\alpha)$, we easily obtain

$$\begin{aligned}\int_{\mathfrak{C}} |S_2(\lambda_1\alpha)&S_3(\lambda_2\alpha)S_4(\lambda_3\alpha)S_5(\lambda_4\alpha)\\ &- F_2(\lambda_1\alpha)F_3(\lambda_2\alpha)F_4(\lambda_3\alpha)F_5(\lambda_4\alpha)|K(\alpha)\,\mathrm{d}\alpha\\ &\ll \tau^2X^{17/60}\exp\Big(-\frac{1}{2}c_1\Big(\frac{\log X}{\log\log X}\Big)^{1/3}\Big).\end{aligned}\tag{3.9}$$

Then, by (2.7) and Lemma 3.5, we have

$$\begin{aligned}\int_{|\alpha|\geqslant\varphi}&|F_2(\lambda_1\alpha)F_3(\lambda_2\alpha)F_4(\lambda_3\alpha)F_5(\lambda_4\alpha)|K(\alpha)\,\mathrm{d}\alpha\\ &\ll \tau^2X^{77/60-4}\int_{|\alpha|\geqslant\varphi}|\alpha|^{-4}\,\mathrm{d}\alpha \ll \tau^2X^{17/60-1/4+3\varepsilon}.\end{aligned}\tag{3.10}$$

Finally, we will show that

$$\int_{-\infty}^{\infty}F_2(\lambda_1\alpha)F_3(\lambda_2\alpha)F_4(\lambda_3\alpha)F_5(\lambda_4\alpha)K(\alpha)e(-v\alpha)\,\mathrm{d}\alpha \gg \tau^2X^{17/60}.\tag{3.11}$$

Now we prove (3.11) as in Lemma 51 of Davenport, see [3]. We have

$$\begin{aligned}\int_{-\infty}^{\infty}&F_2(\lambda_1\alpha)F_3(\lambda_2\alpha)F_4(\lambda_3\alpha)F_5(\lambda_4\alpha)K(\alpha)e(-v\alpha)\,\mathrm{d}\alpha\\ &=\int_{\mathcal{I}_2}\int_{\mathcal{I}_3}\int_{\mathcal{I}_4}\int_{\mathcal{I}_5}\int_{-\infty}^{\infty}e(\alpha(\lambda_1x_1^2+\lambda_2x_2^3+\lambda_3x_3^4+\lambda_4x_4^5-v))\\ &\qquad\times K(\alpha)\,\mathrm{d}\alpha\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}x_4\end{aligned}$$

$$= \int_{\mathcal{I}_2}\int_{\mathcal{I}_3}\int_{\mathcal{I}_4}\int_{\mathcal{I}_5} \max(0, \tau - |\lambda_1 x_1^2 + \lambda_2 x_2^3 + \lambda_3 x_3^4 + \lambda_4 x_4^5 - v|)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}x_4$$

$$= \frac{1}{120\lambda_1^{1/2}\lambda_2^{1/3}\lambda_3^{1/4}\lambda_4^{1/5}} \int_{X/16}^{X}\int_{X/16}^{X}\int_{X/16}^{X}\int_{X/16}^{X} \max(0, \tau - |y_1 + y_2 + y_3 + y_4 - v|) \times y_1^{-1/2} y_2^{-2/3} y_3^{-3/4} y_4^{-4/5}\,\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}y_3\,\mathrm{d}y_4,$$

where $y_j = \lambda_j x_j^{j+1}$, $j = 1, 2, 3, 4$. We may take $X/16 \leqslant y_1, y_2, y_3 \leqslant X/8$. Note that $X/2 \leqslant v \leqslant X$ by the hypothesis (2.3), and $0 < \tau < 1$ (indeed, we will take $\tau = X^{-\delta}$). Then we have

$$v - (y_1 + y_2 + y_3) - \frac{\tau}{2} \geqslant \frac{X}{2} - \frac{3X}{8} - \frac{\tau}{2} > \frac{X}{16}$$

and

$$v - (y_1 + y_2 + y_3) + \frac{\tau}{2} \leqslant X - \frac{3X}{16} + \frac{\tau}{2} < X.$$

So we can take $v - (y_1 + y_2 + y_3) - \tau/2 < y_4 < v - (y_1 + y_2 + y_3) + \tau/2$. Thus we have $|y_1 + y_2 + y_3 + y_4 - v| < \tau/2$. Therefore, we obtain that the lower bound of the above integral is

$$\gg \tau X^{-1/2-2/3-3/4-4/5} \int_{X/16}^{X/8}\int_{X/16}^{X/8}\int_{X/16}^{X/8}\int_{v-(y_1+y_2+y_3)-\tau/2}^{v-(y_1+y_2+y_3)+\tau/2} \mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}y_3\,\mathrm{d}y_4$$

$$\gg \tau^2 X^{-163/60} \int_{X/16}^{X/8}\int_{X/16}^{X/8}\int_{X/16}^{X/8} \mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}y_3 \gg \tau^2 X^{17/60}.$$

Combining (3.9), (3.10) and (3.11), Theorem 3.1 follows immediately. □

4. The middle region $\mathfrak{D}$

In this section we show that

$$\mathcal{I}(v, X; \mathfrak{D}) = o(\tau^2 X^{17/60}). \tag{4.1}$$

By Corollary 2.6, for any $\alpha \in \mathfrak{D} = \{\alpha\colon X^{-13/15-\varepsilon} = \varphi \leqslant |\alpha| < \xi = X^{-7/9-\varepsilon}\}$ we have

$$|S_5(\lambda_4\alpha)| \ll X^{1/5-1/2560+\varepsilon}. \tag{4.2}$$

Then by Cauchy's inequality and (2.7) we have

$$\begin{aligned} \mathcal{I}(v, X; \mathfrak{D}) &= \int_{\mathfrak{D}} S_2(\lambda_1\alpha)S_3(\lambda_2\alpha)S_4(\lambda_3\alpha)S_5(\lambda_4\alpha)K(\alpha)e(-v\alpha)\,\mathrm{d}\alpha \\ &\ll \tau^2 X^{1/4} \max_{\alpha\in\mathfrak{D}} |S_5(\lambda_4\alpha)| \Big(\int_{\mathfrak{D}} |S_2(\lambda_1\alpha)|^2\,\mathrm{d}\alpha\Big)^{1/2} \Big(\int_{\mathfrak{D}} |S_3(\lambda_2\alpha)|^2\,\mathrm{d}\alpha\Big)^{1/2} \\ &\ll \tau^2 X^{9/20-1/2560+\varepsilon} \Big(\int_{\mathfrak{D}} |S_2(\lambda_1\alpha)|^2\,\mathrm{d}\alpha\Big)^{1/2} \Big(\int_{\mathfrak{D}} |S_3(\lambda_2\alpha)|^2\,\mathrm{d}\alpha\Big)^{1/2}. \end{aligned} \tag{4.3}$$

Note that $\xi = X^{-7/9-\varepsilon} = X^{-1+2/(3\cdot 3)-\varepsilon}$. Then by Lemma 3.6, we get

$$\int_{\mathfrak{D}} |S_2(\lambda_1\alpha)|^2 \,\mathrm{d}\alpha \ll 1, \qquad \int_{\mathfrak{D}} |S_3(\lambda_2\alpha)|^2 \,\mathrm{d}\alpha \ll X^{-1/3}. \tag{4.4}$$

Hence, (4.1) follows immediately from (4.3) and (4.4).

5. The trivial region $\mathfrak{t}$

In this section we show that

$$\mathcal{I}(v, X; \mathfrak{t}) = o(\tau^2 X^{17/60}). \tag{5.1}$$

By Cauchy's inequality and the trivial bounds of $S_4(\lambda_3\alpha)$, $S_5(\lambda_4\alpha)$, we have

$$\begin{aligned}\mathcal{I}(v, X; \mathfrak{t}) &= \int_{\mathfrak{t}} S_2(\lambda_1\alpha)S_3(\lambda_2\alpha)S_4(\lambda_3\alpha)S_5(\lambda_4\alpha)K(\alpha)e(-v\alpha)\,\mathrm{d}\alpha \\ &\ll X^{1/4+1/5}\left(\int_{\mathfrak{t}} |S_2(\lambda_1\alpha)|^2 K(\alpha)\,\mathrm{d}\alpha\right)^{1/2}\left(\int_{\mathfrak{t}} |S_3(\lambda_2\alpha)|^2 K(\alpha)\,\mathrm{d}\alpha\right)^{1/2}.\end{aligned} \tag{5.2}$$

By (2.7), the periodicity of $S_2(\alpha)$ and Lemma 2.7, we have

$$\begin{aligned}\int_{\mathfrak{t}} |S_2(\lambda_1\alpha)|^2 K(\alpha)\,\mathrm{d}\alpha &\ll \int_{\kappa}^{\infty} |S_2(\lambda_1\alpha)|^2 \frac{1}{|\alpha|^2}\,\mathrm{d}\alpha \ll \int_{|\lambda_1|\kappa}^{\infty} |S_2(\alpha)|^2 \frac{1}{|\alpha|^2}\,\mathrm{d}\alpha \\ &\ll \sum_{m=[|\lambda_1|\kappa]}^{\infty} \int_m^{m+1} |S_2(\alpha)|^2 \frac{1}{|\alpha|^2}\,\mathrm{d}\alpha \\ &\ll \sum_{m=[|\lambda_1|\kappa]}^{\infty} \int_m^{m+1} |S_2(\alpha)|^2 \frac{1}{m^2}\,\mathrm{d}\alpha \\ &\ll \int_0^1 |S_2(\alpha)|^2\,\mathrm{d}\alpha \sum_{m=[|\lambda_1|\kappa]}^{\infty} \frac{1}{m^2} \ll X^{1/2+\varepsilon}\kappa^{-1}.\end{aligned}$$

Similarly, we have

$$\int_{\mathfrak{t}} |S_3(\lambda_2\alpha)|^2 K(\alpha)\,\mathrm{d}\alpha \ll X^{1/3+\varepsilon}\kappa^{-1}. \tag{5.3}$$

Combining (5.2)–(5.4), we have

$$\begin{aligned}&\int_{\mathfrak{t}} S_2(\lambda_1\alpha)S_3(\lambda_2\alpha)S_4(\lambda_3\alpha)S_5(\lambda_4\alpha)K(\alpha)e(-v\alpha)\,\mathrm{d}\alpha \\ &\qquad \ll X^{26/30+\varepsilon}\kappa^{-1} \ll X^{26/30+\varepsilon}\tau^2 X^{-7/12-2\varepsilon} \ll \tau^2 X^{17/60-\varepsilon}.\end{aligned}$$

Hence, (5.1) follows immediately.

6. The minor region $\mathfrak{c}$

Our argument is similar to that used in [2], [5]. First we fix our parameter N. Let a/q be a continued fraction convergent to λ_1/λ_2 and put

$$N = q^{(72/71)\cdot(18/13)}. \tag{6.1}$$

In this section, we take

$$\sigma = \frac{1}{12}. \tag{6.2}$$

Let $\tilde{\mathfrak{c}} = \tilde{\mathfrak{c}}_1 \cup \tilde{\mathfrak{c}}_2$ and $\hat{\mathfrak{c}} = \mathfrak{c} \setminus \tilde{\mathfrak{c}}$, where

$$\tilde{\mathfrak{c}}_1 = \{\alpha \in \mathfrak{c} \colon\ |S_2(\lambda_1\alpha)| < X^{1/2-(2/3)\sigma+2\varepsilon}\},$$
$$\tilde{\mathfrak{c}}_2 = \{\alpha \in \mathfrak{c} \colon\ |S_3(\lambda_2\alpha)| < X^{1/3-(1/3)\sigma+2\varepsilon}\}.$$

Lemma 6.1. *We have*

$$\int_{\tilde{\mathfrak{c}}} \Big|\prod_{j=1}^{4} S_{j+1}(\lambda_j\alpha)\Big|^2 K(\alpha)\,\mathrm{d}\alpha \ll \tau X^{47/30-1/72+2\varepsilon}. \tag{6.3}$$

Proof. By Hölder's inequality and Lemmas 2.7, 2.8, we have

$$\begin{aligned}
&\int_{\tilde{\mathfrak{c}}_1} \Big|\prod_{j=1}^{4} S_{j+1}(\lambda_j\alpha)\Big|^2 K(\alpha)\,\mathrm{d}\alpha \\
&\quad\ll \Big(\max_{\tilde{\mathfrak{c}}_1} |S_2(\lambda_1\alpha)|^{1/4}\Big)\Big(\int_{\tilde{\mathfrak{c}}_1} |S_2(\lambda_1\alpha)|^4 K(\alpha)\,\mathrm{d}\alpha\Big)^{3/16} \\
&\quad\quad\times \Big(\int_{\tilde{\mathfrak{c}}_1} |S_3(\lambda_2\alpha)|^8 K(\alpha)\,\mathrm{d}\alpha\Big)^{1/4} \\
&\quad\quad\times \Big(\int_{\tilde{\mathfrak{c}}_1} |S_2(\lambda_1\alpha)|^2 |S_4(\lambda_3\alpha)|^4 K(\alpha)\,\mathrm{d}\alpha\Big)^{1/2} \\
&\quad\quad\times \Big(\int_{\tilde{\mathfrak{c}}_1} |S_5(\lambda_4\alpha)|^{32} K(\alpha)\,\mathrm{d}\alpha\Big)^{1/16} \\
&\quad\ll \tau X^{173/120+\varepsilon} \max_{\alpha\in\tilde{\mathfrak{c}}_1} |S_2(\lambda_1\alpha)|^{1/4} \\
&\quad\ll \tau X^{47/30-(1/6)\sigma+2\varepsilon} = \tau X^{47/30-1/72+2\varepsilon},
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\tilde{\mathfrak{c}}_2}\Big|\prod_{j=1}^{4}S_{j+1}(\lambda_j\alpha)\Big|^2K(\alpha)\,\mathrm{d}\alpha\\
&\quad\ll\Big(\max_{\alpha\in\tilde{\mathfrak{c}}_2}|S_3(\lambda_2\alpha)|^{1/2}\Big)\Big(\int_{\tilde{\mathfrak{c}}_2}|S_2(\lambda_1\alpha)|^4|K(\alpha)\,\mathrm{d}\alpha\Big)^{1/4}\Big(\int_{\tilde{\mathfrak{c}}_2}|S_3(\lambda_2\alpha)|^8K(\alpha)\,\mathrm{d}\alpha\Big)^{3/16}\\
&\qquad\times\Big(\int_{\tilde{\mathfrak{c}}_2}|S_2(\lambda_1\alpha)|^2|S_4(\lambda_3\alpha)|^4K(\alpha)\,\mathrm{d}\alpha\Big)^{1/2}\Big(\int_{\tilde{\mathfrak{c}}_2}|S_5(\lambda_4\alpha)|^{32}K(\alpha)\,\mathrm{d}\alpha\Big)^{1/16}\\
&\quad\ll\tau X^{7/5+\varepsilon}\max_{\alpha\in\tilde{\mathfrak{c}}_2}|S_3(\lambda_2\alpha)|^{1/2}\ll\tau X^{47/30-(1/6)\sigma+2\varepsilon}=\tau X^{47/30-1/72+2\varepsilon}.
\end{aligned}$$

Then Lemma 6.1 follows immediately. □

Lemma 6.2. *We have*

$$\int_{\hat{\mathfrak{c}}}\Big|\prod_{j=1}^{4}S_{j+1}(\lambda_j\alpha)\Big|^2K(\alpha)\,\mathrm{d}\alpha\ll\tau X^{47/30+1/9-(71/72)\cdot(13/18)+5\varepsilon}. \tag{6.4}$$

P r o o f. We divide the region $\hat{\mathfrak{c}}$ into disjoint sets $A(Y_1,Y_2,u)$, where

$$A(Y_1,Y_2,u)=\{\alpha\in\hat{\mathfrak{c}}\colon\ Y_1\leqslant|S_2(\lambda_1\alpha)|<2Y_1,\ Y_2\leqslant|S_3(\lambda_2\alpha)|<2Y_2,\ u\leqslant|\alpha|<2u\}$$

and $Y_1=X^{1/2-(2/3)\sigma+2\varepsilon}2^{l_1}$, $Y_2=X^{1/3-(1/3)\sigma+2\varepsilon}2^{l_2}$, $u=\xi 2^r$ for some positive integers l_1, l_2, r. Thus by Corollaries 2.2 and 2.4, there are integers a_1, q_1, a_2, q_2 satisfying

$$(a_1,q_1)=1,\quad(a_2,q_2)=1,$$

$$1\leqslant q_1\ll\Big(\frac{X^{1/2+\varepsilon}}{Y_1}\Big)^4,\quad|q_1\lambda_1\alpha-a_1|\ll\Big(\frac{X^{1/2+\varepsilon}}{Y_1}\Big)^4X^{-1}, \tag{6.5}$$

$$1\leqslant q_2\ll\Big(\frac{X^{1/3+\varepsilon}}{Y_2}\Big)^2,\quad|q_2\lambda_2\alpha-a_2|\ll\Big(\frac{X^{1/3+\varepsilon}}{Y_2}\Big)^2X^{-1}. \tag{6.6}$$

Here we have $a_1a_2\neq0$. If $a_1=0$, by (6.5) we have

$$|\alpha|\ll|q_1\lambda_1\alpha-a_1|\ll\Big(\frac{X^{1/2+\varepsilon}}{Y_1}\Big)^4X^{-1}\ll X^{-7/9-4\varepsilon}. \tag{6.7}$$

This contradicts $|\alpha|\geqslant\xi=X^{-7/9-\varepsilon}$, thus we have $a_1\neq0$. Similarly, we also have $a_2\neq0$.

Note that $|\alpha| \geqslant \xi = X^{-7/9-\varepsilon}$, hence we have

$$\left|\frac{a_1}{\alpha}\right| \ll \left|\frac{a_1}{\lambda_1\alpha}\right| \ll q_1 + \left(\frac{X^{1/2+\varepsilon}}{Y_1}\right)^4 X^{-1}\frac{1}{|\lambda_1\alpha|} \ll q_1 + X^{-\varepsilon} \ll q_1, \tag{6.8}$$

$$\left|\frac{a_2}{\alpha}\right| \ll \left|\frac{a_2}{\lambda_2\alpha}\right| \ll q_2 + \left(\frac{X^{1/3+\varepsilon}}{Y_2}\right)^2 X^{-1}\frac{1}{|\lambda_2\alpha|} \ll q_2 + X^{-1/6-\varepsilon} \ll q_2. \tag{6.9}$$

We further divide the set $A(Y_1, Y_2, u)$ into sets $A(Y_1, Y_2, u, Q_1, Q_2)$ with α satisfying (6.5) and (6.6), where $Q_j \leqslant q_j < 2Q_j$ on each set. Then we have

$$\begin{aligned}\left|a_2 q_1\frac{\lambda_1}{\lambda_2} - a_1 q_2\right| &= \left|\frac{a_1(a_2 - q_2\lambda_2\alpha) + a_2(q_1\lambda_1\alpha - a_1)}{\lambda_2\alpha}\right| \\ &\ll Q_1\left(\frac{X^{1/3+\varepsilon}}{Y_2}\right)^2 X^{-1} + Q_2\left(\frac{X^{1/2+\varepsilon}}{Y_1}\right)^4 X^{-1} \\ &\ll \left(\frac{X^{1/2+\varepsilon}}{Y_1}\right)^4\left(\frac{X^{1/3+\varepsilon}}{Y_2}\right)^2 X^{-1} \ll X^{-1+(10/3)\sigma-\varepsilon} \\ &\ll X^{-13/18-\varepsilon} \ll N^{-(71/72)\cdot(13/18)-\varepsilon},\end{aligned} \tag{6.10}$$

since $N^{71/72} \leqslant X \leqslant N$. Also, by (6.9) and $\alpha \in A(Y_1, Y_2, u, Q_1, Q_2)$, we have

$$|a_2 q_1| \ll |q_1 q_2 \alpha| \ll u Q_1 Q_2. \tag{6.11}$$

Since $N^{(71/72)\cdot(13/18)} = q$ is sufficiently large, where a/q is a continued fraction convergent to λ_1/λ_2, then by (6.10) we have

$$\left\|a_2 q_1\frac{\lambda_1}{\lambda_2}\right\| < \frac{1}{4q}. \tag{6.12}$$

Then by Lemma 2.9, we have $|a_2 q_1| \geqslant q$. Suppose that $|a_2 q_1|$ only takes L distinct values, then we have $L \leqslant |\mathfrak{T}|$, where

$$\mathfrak{T} = \left\{m \in \mathbb{N}\colon\ q \leqslant m \ll uQ_1Q_2,\ \left\|m\frac{\lambda_1}{\lambda_2}\right\| < \frac{1}{4q}\right\}. \tag{6.13}$$

For any two positive integers $m_1, m_2 \in \mathfrak{T}$, we have

$$\left\||m_1 - m_2|\frac{\lambda_1}{\lambda_2}\right\| \leqslant \left\|m_1\frac{\lambda_1}{\lambda_2}\right\| + \left\|m_2\frac{\lambda_1}{\lambda_2}\right\| < \frac{1}{2q}.$$

Then by Lemma 2.9, we have $|m_1 - m_2| \geqslant q$. Thus by the pigeon-hole principle, we have

$$L \leqslant |\mathfrak{T}| \ll \frac{uQ_1Q_2}{q}. \tag{6.14}$$

Each value of $|a_2q_1|$ corresponds to much less than X^{ε} values of a_2, q_1, by the well-known bound on the divisor function. For every fixed a_2 and q_1, by (6.10), the value of $|a_1q_2|$ is the integral part of $a_2q_1\lambda_1/\lambda_2$, so there are much less than X^{ε} values of a_1, q_2. Therefore, by (6.5) and (6.6) we get that the length of $A(Y_1, Y_2, u, Q_1, Q_2)$ is

$$\begin{aligned}
&\ll LX^{2\varepsilon}\min\left(\frac{1}{Q_1X}\left(\frac{X^{1/2+\varepsilon}}{Y_1}\right)^4, \frac{1}{Q_2X}\left(\frac{X^{1/3+\varepsilon}}{Y_2}\right)^2\right)\\
&\ll \frac{X^{2\varepsilon}uQ_1Q_2}{q}\frac{1}{Q_1^{1/2}Q_2^{1/2}X}\left(\frac{X^{1/2+\varepsilon}}{Y_1}\right)^2\frac{X^{1/3+\varepsilon}}{Y_2}\\
&\ll \frac{X^{1/3+5\varepsilon}uQ_1^{1/2}Q_2^{1/2}}{qY_1^2Y_2} \ll \frac{X^{5/3+8\varepsilon}u}{qY_1^4Y_2^2}.
\end{aligned}$$

Here we have used $Q_1 \ll (X^{1/2+\varepsilon}/Y_1)^4$, $Q_2 \ll (X^{1/3+\varepsilon}/Y_2)^2$ and the inequality $\min(M, N) \leqslant M^{1/2}N^{1/2}$ for any positive numbers M, N.

Now we evaluate the integral over the set $A(Y_1, Y_2, u, Q_1, Q_2)$. By (2.7) and the trivial bounds of $S_4(\lambda_3\alpha)$, $S_5(\lambda_4\alpha)$, we have

$$\begin{aligned}
(6.15)\quad &\int_{A(Y_1,Y_2,u,Q_1,Q_2)}\left|\prod_{j=1}^{4}S_{j+1}(\lambda_j\alpha)\right|^2K(\alpha)\,\mathrm{d}\alpha\\
&\ll \min(\tau^2, u^{-2})Y_1^2Y_2^2X^{2/4}X^{2/5}\int_{A(Y_1,Y_2,u,Q_1,Q_2)}\mathrm{d}\alpha\\
&\ll \tau u^{-1}Y_1^2Y_2^2X^{9/10}\frac{X^{5/3+8\varepsilon}u}{qY_1^4Y_2^2} \ll \frac{\tau X^{77/30+8\varepsilon}}{qY_1^2}\\
&\ll \frac{\tau X^{47/30+(4/3)\sigma+4\varepsilon}}{q} \ll \tau X^{47/30+1/9+4\varepsilon}N^{-(71/72)\cdot(13/18)}\\
&\ll \tau X^{47/30+1/9-(71/72)\cdot(13/18)+4\varepsilon}.
\end{aligned}$$

Now we sum over all possible values of Y_1, Y_2, u, Q_1, Q_2, and we obtain that

$$(6.16)\qquad \int_{\hat{\mathfrak{c}}}\left|\prod_{j=1}^{4}S_{j+1}(\lambda_j\alpha)\right|^2K(\alpha)\,\mathrm{d}\alpha \ll \tau X^{47/30+1/9-(71/72)\cdot(13/18)+5\varepsilon}.$$

□

7. The proof of Theorem 1.2

We take $\tau = X^{-\delta}$. Let $\mathcal{V}$ be a well-spaced set. Then by (2.8) and (2.9), we have $\mathcal{I}(v, X; \mathbb{R}) = 0$ for every $v \in \mathbb{E}([X/2, X])$, where $\mathbb{E}([X/2, X])$ is defined by (2.1). Hence we have

$$(7.1) \qquad \sum_{v\in\mathbb{E}([X/2,X])} \mathcal{I}(v, X; \mathbb{R}) = \sum_{v\in\mathbb{E}([X/2,X])} (\mathcal{I}(v, X; \mathfrak{C}) + \mathcal{I}(v, X; \mathfrak{D}) + \mathcal{I}(v, X; \mathfrak{c}) + \mathcal{I}(v, X; \mathfrak{t})) = 0.$$

This together with (3.1), (4.1) and (5.1) yields

$$(7.2) \qquad \Big| \sum_{v\in\mathbb{E}([X/2,X])} \int_{\mathfrak{c}} \prod_{j=1}^{4} S_{j+1}(\lambda_j\alpha) e(-v\alpha) K(\alpha)\,\mathrm{d}\alpha \Big| \gg \tau^2 X^{17/60} E\Big(\Big[\frac{X}{2}, X\Big]\Big).$$

By Cauchy's inequality and (2.7) we get

$$\begin{aligned}
(7.3) \qquad & \Big| \sum_{v\in\mathbb{E}([X/2,X])} \int_{\mathfrak{c}} \prod_{j=1}^{4} S_{j+1}(\lambda_j\alpha) e(-v\alpha) K(\alpha)\,\mathrm{d}\alpha \Big| \\
&\ll \Big(\int_{-\infty}^{\infty} \Big| \sum_{v\in\mathbb{E}([X/2,X])} e(-v\alpha) \Big|^2 K(\alpha)\,\mathrm{d}\alpha \Big)^{1/2} \\
&\quad \times \Big(\int_{\mathfrak{c}} \Big| \prod_{j=1}^{4} S_{j+1}(\lambda_j\alpha) \Big|^2 K(\alpha)\,\mathrm{d}\alpha \Big)^{1/2} \\
&= \Big(\sum_{v_1,v_2\in\mathbb{E}([X/2,X])} \int_{-\infty}^{\infty} e((v_1 - v_2)\alpha) K(\alpha)\,\mathrm{d}\alpha \Big)^{1/2} \\
&\quad \times \Big(\int_{\mathfrak{c}} \Big| \prod_{j=1}^{4} S_{j+1}(\lambda_j\alpha) \Big|^2 K(\alpha)\,\mathrm{d}\alpha \Big)^{1/2} \\
&= \Big(\sum_{v_1,v_2\in\mathbb{E}([X/2,X])} \max(0, \tau - |v_1 - v_2|) \Big)^{1/2} \\
&\quad \times \Big(\int_{\mathfrak{c}} \Big| \prod_{j=1}^{4} S_{j+1}(\lambda_j\alpha) \Big|^2 K(\alpha)\,\mathrm{d}\alpha \Big)^{1/2} \\
&\ll \tau^{1/2} \Big(E\Big(\Big[\frac{X}{2}, X\Big]\Big)\Big)^{1/2} \Big(\int_{\mathfrak{c}} \Big| \prod_{j=1}^{4} S_{j+1}(\lambda_j\alpha) \Big|^2 K(\alpha)\,\mathrm{d}\alpha \Big)^{1/2}.
\end{aligned}$$

Here we have used that for every $v_1, v_2 \in \mathbb{E}([X/2, X]) \subset \mathcal{V}$, if $v_1 \neq v_2$, then there is a constant c such that $|v_1 - v_2| > c$, since $\mathcal{V}$ is a well-spaced set. Note that $\tau = X^{-\delta}$, $\delta > 0$ and X is a sufficiently large quantity. Thus we have

$$\sum_{v_1, v_2 \in \mathbb{E}([X/2, X])} \max(0, \tau - |v_1 - v_2|) = \tau E\Big(\Big[\frac{X}{2}, X\Big]\Big).$$

Combining (7.2) and (7.3), we obtain

$$E\Big(\Big[\frac{X}{2}, X\Big]\Big) \ll \tau^{-3} X^{-17/30} \int_{\mathfrak{c}} \Big|\prod_{j=1}^{4} S_{j+1}(\lambda_j \alpha)\Big|^2 K(\alpha)\, \mathrm{d}\alpha. \tag{7.4}$$

Now we begin to prove the first part of Theorem 1.2. Note that a/q is convergent to λ_1/λ_2, $N = q^{(72/71)\cdot(18/13)}$ by (6.1), and $N^{71/72} \leqslant X \leqslant N$. By (7.4), Lemmas 6.1 and 6.2, we have

$$E([X/2, X]) \ll \tau^{-3} X^{-17/30} \tau X^{47/30 - 1/72 + 2\varepsilon} \ll X^{1 + 2\delta - 1/72 + 2\varepsilon}. \tag{7.5}$$

Thus, by (2.1) and (7.5), we conclude that

$$\begin{aligned} E(\mathcal{V}, N, \delta) &\ll E([N^{71/72}, N]) + N^{71/72} \\ &\ll \sum_{l=1}^{[(1/72)\log_2 N]+1} E\big([2^{-l}N, 2^{1-l}N]\big) + N^{71/72} \\ &\ll \sum_{l=1}^{[(1/72)\log_2 N]+1} (2^{1-l}N)^{1+2\delta-1/72+2\varepsilon} + N^{71/72} \\ &\ll N^{1+2\delta-1/72+3\varepsilon}. \end{aligned} \tag{7.6}$$

Obviously, there are infinitely many q we could have taken since λ_1/λ_2 is irrational, and this gives the sequence $N_j \to \infty$. This completes the proof of the first part of Theorem 1.2.

Next, we begin to prove the second part of Theorem 1.2. Now, if the convergent denominators for λ_1/λ_2 satisfy (1.6), then we can modify our work in Section 6. Now let N be a sufficiently large number, $N^{71/72} \leqslant X \leqslant N$, and assume that

$$\sigma = \chi, \tag{7.7}$$

where χ is given by (1.8).

Then the expression corresponding to (6.3) in Lemma 6.1 is

$$\int_{\tilde{\mathfrak{c}}} \left| \prod_{j=1}^{4} S_{j+1}(\lambda_j \alpha) \right|^2 K(\alpha)\, \mathrm{d}\alpha \ll \tau X^{47/30-(1/6)\chi+\varepsilon}. \tag{7.8}$$

Now we modify our argument in Lemma 6.2, argument obtaining

$$\left| a_2 q_1 \frac{\lambda_1}{\lambda_2} - a_2 q_1 \right| \ll X^{-1+(10/3)\chi-\varepsilon}.$$

However, we know from (1.6) that there is a convergent a/q to λ_1/λ_2 with

$$X^{(1-\omega)(1-(10/3)\chi)} \ll q \ll X^{1-(10/3)\chi}.$$

Indeed, if not, there would be two convergent denominators q_j and q_{j+1} satisfying

$$q_j \ll X^{(1-\omega)(1-(10/3)\chi)}, \quad q_{j+1} \gg X^{1-(10/3)\chi}.$$

Thus we have $q_j \ll q_{j+1}^{1-\omega}$, which is in contradiction with (1.6).

Then, the expression corresponding to (6.15) is

$$\begin{aligned}
&\int_{A(Y_1,Y_2,u,Q_1,Q_2)} \left| \prod_{j=1}^{4} S_{j+1}(\lambda_j \alpha) \right|^2 K(\alpha)\, \mathrm{d}\alpha \\
&\qquad \ll \frac{\tau X^{47/30+(4/3)\chi+4\varepsilon}}{q} \ll \tau X^{47/30+(4/3)\chi+4\varepsilon-(1-\omega)(1-(10/3)\chi)} \\
&\qquad \ll \tau X^{47/30-(1/6)\chi+4\varepsilon-(1-\omega)+(1/6)\chi(29-20\omega)} \ll \tau X^{47/30-(1/6)\chi+4\varepsilon}
\end{aligned}$$

by our choice of χ. Thus the expression corresponding to (6.4) in Lemma 6.2 is

$$\int_{\hat{\mathfrak{c}}} \left| \prod_{j=1}^{4} S_{j+1}(\lambda_j \alpha) \right|^2 K(\alpha)\, \mathrm{d}\alpha \ll \tau X^{47/30-(1/6)\chi+5\varepsilon}. \tag{7.9}$$

Recall $\mathfrak{c} = \tilde{\mathfrak{c}} \cup \hat{\mathfrak{c}}$. Combining (7.4), (7.8) and (7.9), the second part of Theorem 1.2 follows immediately by modifying our argument in (7.6). □

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