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RECOGNITION OF SOME FAMILIES OF FINITE SIMPLE GROUPS BY ORDER AND SET OF ORDERS OF VANISHING ELEMENTS

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Abstract. Let G be a finite group. An element $g \in G$ is called a vanishing element if there exists an irreducible complex character χ of G such that $\chi(g) = 0$. Denote by $\mathrm{Vo}(G)$ the set of orders of vanishing elements of G. Ghasemabadi, Iranmanesh, Mavadatpour (2015), in their paper presented the following conjecture: Let G be a finite group and M a finite nonabelian simple group such that $\mathrm{Vo}(G) = \mathrm{Vo}(M)$ and $|G| = |M|$. Then $G \cong M$. We answer in affirmative this conjecture for $M = Sz(q)$, where $q = 2^{2n+1}$ and either $q-1$, $q - \sqrt{2q} + 1$ or $q + \sqrt{2q} + 1$ is a prime number, and $M = F_4(q)$, where $q = 2^n$ and either $q^4 + 1$ or $q^4 - q^2 + 1$ is a prime number.

Keywords: finite simple groups; vanishing element; vanishing prime graph

MSC 2010: 20C15, 20D05

1. Introduction

Let G be a finite group. Denote by $\mathrm{Irr}(G)$ the set of all irreducible complex characters of G. An element $g \in G$ is called a vanishing element, if $\chi(g) = 0$ for some irreducible complex character χ of G. The set of all vanishing elements of G is denoted by $\mathrm{Van}(G)$, and the set of orders of all vanishing elements of G is denoted by $\mathrm{Vo}(G)$. It is well-known that from the set $\mathrm{Vo}(G)$ we can get some information about the structure of the group G. In [4], it is proved that if G is a finite group such that $p \in \pi(G)$ and G has no vanishing element whose order is divisible by p, then G has a normal Sylow p-subgroup. Also in [13], it is shown that if G is a finite group such that $\mathrm{Vo}(G) = \mathrm{Vo}(A_5)$, then $G \cong A_5$. But not all finite simple groups are characterizable by the set of orders of their vanishing elements. For example, it is easy to see that $\mathrm{Vo}(L_3(5)) = \mathrm{Vo}(\mathrm{Aut}(L_3(5)))$, but $L_3(5) \ncong \mathrm{Aut}(L_3(5))$. Therefore in [7], the authors put forward the following conjecture:

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Conjecture. *Let G be a finite group and M a finite nonabelian simple group such that* $\mathrm{Vo}(G) = \mathrm{Vo}(M)$ *and* $|G| = |M|$. *Then* $G \cong M$.

In [7], the conjecture was proved for simple groups $L_2(q)$, where $q \in \{5, 7, 8, 9, 17\}$, $L_3(4)$, A_7, $Sz(8)$ and $Sz(32)$. Then in [6], it is proved that sporadic simple groups, alternating groups, projective special linear groups $L_2(p)$ where p is an odd prime, and finite simple K_n-groups where $n \in \{3, 4\}$, satisfy this conjecture. This has motivated us to prove this conjecture for some other simple groups as follows:

Main theorem. *If G is a finite group such that* $\mathrm{Vo}(G) = \mathrm{Vo}(M)$ *and* $|G| = |M|$, *where M is $Sz(q)$ for $q = 2^{2n+1}$ and either $q-1$, $q-\sqrt{2q}+1$ or $q+\sqrt{2q}+1$ is prime, or M is $F_4(q)$ for $q = 2^n$ and either q^4+1 or q^4-q^2+1 is prime, then $G \cong M$.*

Although the problem is group theoretic, the language of graph theory can sometimes improve the understanding of the results. Let X be a finite set of positive integers. The prime graph $\Pi(X)$ is a graph whose vertices are the prime divisors of elements of X, and two distinct vertices p and q are adjacent if there exists an element of X divisible by pq. For a finite group G, we denote by $\omega(G)$ the set of element orders of G, and by $\pi(G)$ the set of prime divisors of $|G|$. The graph $\Pi(\omega(G))$ is denoted by $GK(G)$ and is called the Gruenberg-Kegel graph of G. We denote by $t(G)$ the number of connected components of $GK(G)$, and by $\pi_i(G)$, $i = 1, \ldots, t(G)$, the vertex set of the ith connected component of $GK(G)$. If $2 \in \pi(G)$, we always assume that $2 \in \pi_1(G)$.

The prime graph $\Pi(\mathrm{Vo}(G))$ is denoted by $\Gamma(G)$ and is called the vanishing prime graph of G. Obviously, the vanishing prime graph of G is a subgraph of Gruenberg-Kegel graph of G.

Throughout this paper, we denote by $\pi(n)$ the set of prime divisors of integer n. All further notation can be found in [2], for instance.

2. Main results

A 2-Frobenius group is a group G which has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, where K and G/H are Frobenius groups with kernels H and K/H, respectively. It is a well-known result that 2-Frobenius groups are solvable.

A group G is said to be a nearly 2-Frobenius group if there exist normal subgroups F and L of G such that F is nilpotent, $F = F_1 \times F_2$ for normal subgroups F_1 and F_2 of G, G/F is a Frobenius group with kernel L/F, G/F_1 is a Frobenius group with kernel L/F_1, and G/F_2 is a 2-Frobenius group.

Theorem 2.1 ([12])**.** *Let G be a finite group such that $t(G) \geqslant 2$. Then one of the following conditions holds:*

(1) *G is either a Frobenius or a 2-Frobenius group.*
(2) *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\pi(H) \cup \pi(G/K) \subseteq \pi_1(G)$, H is nilpotent, K/H is a nonabelian simple group, and $G/H \leqslant \mathrm{Aut}(K/H)$.*

Theorem 2.2 ([1])**.** *Let G be a Frobenius group of even order with Frobenius kernel K and Frobenius complement H. Then $t(G) = 2$, and the prime graph components of G are $\pi(H)$ and $\pi(K)$.*

Lemma 2.3 ([4], [5])**.** (1) *If G is a finite nonabelian simple group except A_7, then $GK(G) = \Gamma(G)$.*

(2) *If G is a solvable group, then $\Gamma(G)$ has at most 2 connected components. Moreover, if $\Gamma(G)$ is disconnected, then G is either a Frobenius group, or a nearly 2-Frobenius group.*

Theorem 2.4 ([4], Theorem B)**.** *Let G be a finite nonsolvable group. If $\Gamma(G)$ is disconnected, then G has a unique nonabelian composition factor S, and $t(S)$ is greater than or equal to the number of connected components of $\Gamma(G)$, unless G is isomorphic to A_7.*

Lemma 2.5 ([4], Corollary 2.6)**.** *Let G be a group and K a nilpotent normal subgroup of G. If $K \cap \mathrm{Van}(G) \neq \emptyset$, then there exists $g \in K \cap \mathrm{Van}(G)$ whose order is divisible by every prime in $\pi(K)$.*

The following lemma is an easy consequence of [9], Corollary 22.26.

Lemma 2.6. *If $\chi \in \mathrm{Irr}(G)$ vanishes on a p-element for some prime p, then $p \mid \chi(1)$.*

Let p be a prime number. A character $\chi \in \mathrm{Irr}(G)$ is said to be of p-defect zero if p is not a divisor of $|G|/\chi(1)$. It is well-known that if $\chi \in \mathrm{Irr}(G)$ is of p-defect zero, then for every element $g \in G$ such that $p \mid o(g)$, we have $\chi(g) = 0$ ([8], Theorem 8.17).

In the following, we bring some well-known number theoretic theorems.

Lemma 2.7 ([3], Remark 1)**.** *The equation $p^m - q^n = 1$, where p and q are primes and $m, n > 1$, has only one solution, namely $3^2 - 2^3 = 1$.*

Lemma 2.8 ([14], Zsigmondy theorem)**.** *Let p be a prime and n a positive integer. Then one of the following assertions holds:*
(1) *there is a primitive prime p' for $p^n - 1$, that is, $p' \mid p^n - 1$ but $p' \nmid p^m - 1$, for every $1 \leqslant m < n$,*
(2) *$p = 2$, $n = 1$ or 6,*
(3) *p is a Mersenne prime and $n = 2$.*

Lemma 2.9 ([10], Lemma 8). *Assume $q > 1$ is a natural number, $s = \prod_{i=1}^{n} (q^i - 1)$, p is a prime, $p \mid s$. We denote the power of p in the standard factorization of s by s_p. $e = \min\{d\colon\ p \mid q^d - 1\}$, $q^e = 1 + p^r k$, $p \nmid k$. If $p > 2$ or $r > 2$, then $s_p < q^{np/(p-1)}$.*

Let p be a prime number and $(a, p) = 1$. Let $k \geqslant 1$ be the smallest positive integer such that $a^k \equiv 1 \pmod{n}$. Then k is called the order of a with respect to n and we denote it by $\mathrm{ord}_n(a)$. Obviously, by Euler-Fermat's theorem it follows that $\mathrm{ord}_n(a) \mid \varphi(n)$. Also, if $a^t \equiv 1 \pmod{n}$, then $\mathrm{ord}_n(a) \mid t$.

Theorem 2.10. *Let G be a finite group such that $\mathrm{Vo}(G) = \mathrm{Vo}(F_4(q))$ and $|G| = |F_4(q)|$, where $q = 2^n$ and either $q^4 + 1$ or $q^4 - q^2 + 1$ is prime. Then $G \cong F_4(q)$.*

Proof. By the assumption $\mathrm{Vo}(G) = \mathrm{Vo}(F_4(q))$, it is obvious that $\Gamma(G) = \Gamma(F_4(q))$. By Lemma 2.3, we know that $\Gamma(F_4(q)) = GK(F_4(q))$ has 3 connected components including an isolated vertex p, where $p \in \{q^4 + 1, q^4 - q^2 + 1\}$. Also, note that $|G| = |F_4(q)| = q^{24}(q^{12} - 1)(q^8 - 1)(q^6 - 1)(q^2 - 1)$. Since $p \in \mathrm{Vo}(F_4(q))$ and $\mathrm{Vo}(G) = \mathrm{Vo}(F_4(q))$, so $p \in \mathrm{Vo}(G)$. Thus there exist an element $g \in G$ and an irreducible character $\chi \in \mathrm{Irr}(G)$ such that $o(g) = p$ and $\chi(g) = 0$. So $p \mid \chi(1)$ and since $|G|_p = p$ we conclude that $p \nmid |G|/\chi(1)$. Therefore χ is of p-defect zero, and hence for every element $h \in G$ such that $p \mid o(h)$ we have $\chi(h) = 0$. So, by the fact that p is an isolated vertex in $\Gamma(G)$, we conclude that p is an isolated vertex in $GK(G)$. Hence $t(G) \geqslant 2$.

Since $\Gamma(G)$ has three connected components, Lemma 2.3 implies that G is not a solvable group, and consequently G is not a 2-Frobenius group. We also claim that G is not a Frobenius group. Suppose that G is a Frobenius group with kernel K and complement H. So $|G| = |H||K|$ and $|H| \mid |K| - 1$. Theorem 2.2 implies that $GK(G)$ has two connected components $\pi(H)$ and $\pi(K)$, and since $|H| < |K|$, it follows that $|H| = p$ and $|K| = |G|/p$. In both cases $p = q^4 + 1$ and $p = q^4 - q^2 + 1$, one can get a contradiction by the fact that $|H| \mid |K| - 1$. Therefore G is not a Frobenius group. So by Theorem 2.1, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that $\pi(H) \cup \pi(G/K) \subseteq \pi_1(G)$, H is nilpotent, K/H is a nonabelian simple group, and $G/H \leqslant \mathrm{Aut}(K/H)$. By Theorem 2.4 we have $t(K/H) \geqslant 3$. In both cases $p = q^4 + 1$ and $p = q^4 - q^2 + 1$, we use the classification of finite nonabelian simple groups with more than two Gruenberg-Kegel graph connected components to prove that K/H is isomorphic to $F_4(q)$.

Case 1. First suppose that $p = q^4 + 1 = 2^{4n} + 1$. So $\pi(n) \subseteq \{2\}$. Otherwise $n = 2^a b$ where a and b are integers and $b > 1$ is odd, and hence

$$q^4 + 1 = 2^{2^{a+2}b} + 1 = (2^{2^{a+2}} + 1)(1 - 2^{2^{a+2}} + \ldots + 2^{2^{a+2}(b-1)}),$$

which contradicts the assumption $q^4 + 1$ is prime.

▷ K/H is not a sporadic simple group.

It is easy to show that K/H is not isomorphic to a sporadic simple group. For example, if $K/H \cong Fi'_{24}$, then $p = q^4 + 1 = 17$ and consequently $q = 2$. But $|Fi'_{24}| \nmid |F_4(2)|$, a contradiction. In other cases, we can get a contradiction similarly.

▷ K/H is not an alternating group.

Let $K/H \cong A_{p'}$, where p' and $p' - 2$ are primes. If $p' = p = 2^{4n} + 1$, then $p' - 2 = 2^{4n} - 1$ is a prime number, which is impossible. If $p' - 2 = p = 2^{4n} + 1$, then $p' = 2^{4n} + 3 = q^4 + 3$ is a divisor of $|G| = q^{24}(q^{12} - 1)(q^8 - 1)(q^6 - 1)(q^2 - 1)$. Since $(p', q(q^4 + 1)) = 1$, it follows that $p' = q^4 + 3$ is a divisor of $q^{12} - 1$. One can easily get that $q^4 + 3 = 7$, which is impossible.

▷ K/H is not a simple group of Lie type, except $F_4(q)$.

If K/H is isomorphic to ${}^2A_5(2)$, $E_7(2)$, $E_7(3)$, $A_2(4)$, or ${}^2E_6(2)$, then we easily get a contradiction similar to sporadic simple groups.

Let $K/H \cong A_1(q')$, where $q' = 2^m > 2$. Therefore $q' - 1 = p$ or $q' + 1 = p$. If $q' - 1 = p = 2^{4n} + 1$, then $2^m - 2^{4n} = 2$, a contradiction. So $q' + 1 = p = 2^{4n} + 1$, and hence $m = 4n$, and $|K/H| = q'(q'^2 - 1) = q^4(q^8 - 1)$. On the other hand, $G/K \leqslant Out(K/H)$, which implies that $|G/K| \mid 4n$, so $|G/K|$ is a 2-power since $\pi(n) \subseteq \{2\}$. Therefore $2(q^{12} - 1)(q^6 - 1)(q^2 - 1) \mid |H|$. By considering $\Gamma(G)$, we conclude that there exist $g \in G$ and $\chi \in \mathrm{Irr}(G)$ such that $\pi(o(g)) \subseteq \pi(q^4 - q^2 + 1)$ and $\chi(g) = 0$. Since $\pi(o(g)) \subseteq \pi(q^4 - q^2 + 1)$, $(q^4 - q^2 + 1, 2(q^8 - 1)) = 1$ and $H \trianglelefteq G$, we conclude that $g \in H$. Therefore H is a nilpotent normal subgroup of G such that $H \cap \mathrm{Van}(G) \neq \emptyset$. Now Lemma 2.5 implies that there exists a vanishing element whose order is divisible by all prime divisors of $|H|$. So all prime divisors of $|H|$ are adjacent in $\Gamma(G)$, which is a contradiction by Table 9 of [11].

Let $K/H \cong A_1(q')$, where $3 < q' = p'^m \equiv \varepsilon \pmod 4$ for $\varepsilon = \pm 1$. Hence $q' = 2^{4n} + 1 = p$ or $(q' + \varepsilon)/2 = 2^{4n} + 1 = p$. First let $(q' + \varepsilon)/2 = 2^{4n} + 1 = p$. If $\varepsilon = 1$, then $q' - 2^{4n+1} = 1$. Now Lemma 2.7 implies that $q' = p' = 2^{4n+1} + 1$, which is impossible since $3 \mid 2^{4n+1} + 1$ and $2^{4n+1} + 1 \neq 3$. Let $\varepsilon = -1$. So $q' = 2^{4n+1} + 3 = 2q^4 + 3$ is a divisor of $|G| = q^{24}(q^{12} - 1)(q^8 - 1)(q^6 - 1)(q^2 - 1)$. Since $q' = p'^m$ and $(p', q(q^4+1)) = 1$, we conclude that $p' \mid q^{12} - 1$. On the other hand, p' is a divisor of $8q^{12} + 27$, and consequently $p' \mid 35$. But $q' \equiv -1 \pmod 4$, so $p' = 7$. Therefore $7^m = 2^{4n+1} + 3$, which is impossible because $7^m - 2^{4n+1} \equiv 2 \pmod 3$. Now let $q' = 2^{4n} + 1 = p$. So $q' = p' = q^4 + 1$, and hence $|K/H| = p'(p'^2 - 1)/2 = q^4(q^4 + 1)(q^4 + 2)/2$. So $(q^4 + 2)/2$ is a divisor of $(q^{12} - 1)(q^6 - 1)(q^4 - 1)(q^2 - 1)$. Obviously $\pi((q^4 + 2)/2) \subseteq \pi(q^{12} - 1)$. Let $r \in \pi((q^4 + 2)/2)$. So r divides $q^{12} - 1$ and $q^{12} + 8$. Therefore $r = 3$, and $2^{4n-1} + 1 = 3^t$ for some integer t. Now Lemma 2.7 implies that $n = 1$. Therefore $|G| = 2^{24} \cdot 3^6 \cdot 5^2 \cdot 7^2 \cdot 13 \cdot 17$ and $|K/H| = 2^4 \cdot 3^2 \cdot 17$ and $|G/K| \mid 2$. Hence $\{2, 3, 5, 7, 13\} \subseteq \pi(H)$, and since H is nilpotent, the Gruenberg-Kegel graph of G has two complete connected components with vertex set $\{17\}$ and

$\{2,3,5,7,13\}$. But 13 is an isolated vertex in $\Gamma(G)=GK(F_4(2))$, which implies that there exist a 13-element $g\in G$ and $\chi\in \mathrm{Irr}(G)$ such that $\chi(g)=0$. So $13\mid \chi(1)$ and consequently $13\nmid |G|/\chi(1)$. Therefore we conclude that for every $h\in G$ such that $13\mid o(h)$ we have $\chi(h)=0$. Now by the fact that 13 is an isolated vertex of $\Gamma(G)$, but 13 is connected to some other vertices in $GK(G)$, we get a contradiction.

Let $K/H\cong E_8(q')$. Therefore $p=q^4+1$ is an element of the set

$$\{q'^8\pm q'^7\mp q'^5-q'^4\mp q'^3\pm q'+1,\ q'^8-q'^6+q'^4-q'^2+1,\ q'^8-q'^4+1\}.$$

So

$$p=q^4+1<(q'^8+q'^7+q'^6+q'^5+q'^4+q'^3+q'^2+q'+1)(q'-1)=q'^9-1<q'^9+1,$$

which implies that $q^4<q'^9$. But $q'^{120}\mid |E_8(q')|$, and $|E_8(q')|$ is a divisor of $q^{24}(q^{12}-1)(q^8-1)(q^6-1)(q^2-1)$. So $q'^{120}<q^{52}=q^{4\cdot 13}<q'^{117}$, which is impossible.

Let $K/H\cong Sz(q')$, where $q'=2^{2m+1}>2$. If $2^{2m+1}-1=p=2^{4n}+1$, then $2^{2m+1}-2^{4n}=2$, a contradiction. If $2^{2m+1}\pm 2^{m+1}+1=2^{4n}+1$, then $2^{m+1}\times(2^m\pm 1)=2^{4n}$, which is impossible.

Let $K/H\cong {}^2F_4(q')$, where $q'=2^{2m+1}>2$. Then $2^{2(2m+1)}\pm 2^{3m+2}+2^{2m+1}\pm 2^{m+1}+1=2^{4n}+1$, which implies that $2^{m+1}(2^{3m+1}\pm 2^{2m+1}+2^m\pm 1)=2^{4n}$, a contradiction.

Let $K/H\cong {}^2G_2(q')$ for $q'=3^{2m+1}>3$. Therefore $3^{2m+1}\pm 3^{m+1}+1=2^{4n}+1$, and consequently $3^{m+1}(3^m\pm 1)=2^{4n}$, which is impossible. If $K/H\cong G_2(q')$, where $q'\equiv 0 \pmod 3$, one can get a contradiction similarly.

Let K/H be isomorphic to ${}^2D_{p'}(3)$, where $p'=2^m+1$. Then either $(3^{p'}+1)/4=2^{4n}+1$, or $(3^{p'-1}+1)/2=2^{4n}+1$. If $(3^{p'}+1)/4=2^{4n}+1$, then $3^{p'}-3=2^{4n+2}$, a contradiction. If $(3^{p'-1}+1)/2=2^{4n}+1$, then $3^{p'-1}-2^{4n+1}=1$, which is impossible by Lemma 2.7.

Therefore $K/H\cong F_4(q')$, where $q'=2^m$ and m is an integer. Obviously $m\leqslant n$. Since $p\in\pi(K/H)$, it follows that $p=q^4+1$ is a divisor of $q'^{24}(q'^{12}-1)(q'^8-1)\times(q'^6-1)(q'^2-1)$. Note that p is a primitive prime divisor of $2^{8n}-1$. If $m<n$, it follows that $p\in\pi(q'^{12}-1)$. So $2^{12m}\equiv 1 \pmod p$, and hence $8n\mid 12m$. Since n is a power of 2, we conclude that $2n\mid m$, a contradiction. So $m=n$, and $K/H\cong F_4(q)$.

Case 2. Now suppose that $p=q^4-q^2+1$.

▷ K/H is not a sporadic simple group.

If $K/H\cong Sz$, then $p=q^4-q^2+1=11$ or 13. The only possibility is $q=2$. But $|Sz|\nmid |F_4(2)|$, a contradiction. For other sporadic simple groups, one can get a contradiction similarly.

$\triangleright$ K/H is not an alternating group.

Let $K/H \cong A_{p'}$, where p' and $p'-2$ are primes. If $p' = q^4 - q^2 + 1$, then $p' - 2 = q^4 - q^2 - 1$ is a prime divisor of $|G| = q^{24}(q^{12}-1)(q^8-1)(q^6-1)(q^2-1)$. Since $(q^4 - q^2 - 1, q^{24}(q^4+1)) = 1$, it follows that $q^4 - q^2 - 1$ is a prime divisor of $q^{12} - 1$, which is impossible. If $p' - 2 = q^4 - q^2 + 1$, then $p' = q^4 - q^2 + 3$ is a prime divisor of $|G|$, which is a similar contradiction.

$\triangleright$ K/H is not a simple group of Lie type, except $F_4(q)$.

If K/H is isomorphic to ${}^2A_5(2)$, $E_7(2)$, $E_7(3)$, $A_2(4)$, or ${}^2E_6(2)$, then we easily get a contradiction similarly to sporadic simple groups.

Let $K/H \cong A_1(q')$, where $q' = 2^m > 2$. Therefore $q' - 1 = p$ or $q' + 1 = p$. If $q' - 1 = q^4 - q^2 + 1$, then $2^m - 2^{4n} + 2^{2n} = 2$, which is impossible because $4 \mid 2^m - 2^{4n} + 2^{2n}$. If $q' + 1 = q^4 - q^2 + 1$, then $2^m = 2^{2n}(2^{2n} - 1)$, which is again impossible.

Let $K/H \cong A_1(q')$, where $3 < q' = p'^m \equiv \varepsilon \pmod 4$ for $\varepsilon = \pm 1$. Hence $q' = p$ or $(q' + \varepsilon)/2 = p$. First, let $q' = p = q^4 - q^2 + 1$. So $|K/H| = q^2(q^2-1) \times (q^4 - q^2 + 1)(q^4 - q^2 + 2)/2$ and $|G/K| \mid 2$. Obviously $2(q^4+1) \mid |H|$. By considering $\Gamma(G)$, there exist $g \in G$ and $\chi \in \mathrm{Irr}(G)$ such that $\pi(o(g)) \subseteq \pi(q^4+1)$ and $\chi(g) = 0$. Since $(q^4 + 1, |G/H|) = 1$ and $H \trianglelefteq G$, we conclude that $g \in H$. So H is a nilpotent normal subgroup of G such that $H \cap \mathrm{Van}(G) \neq \emptyset$. Now by Lemma 2.5 there exists a vanishing element whose order is divisible by all prime divisors of $|H|$. So all prime divisors of $|H|$ are adjacent in $\Gamma(G)$, which is a contradiction. Now let $(q' + \varepsilon)/2 = p = q^4 - q^2 + 1$. If $\varepsilon = -1$, then $q' = 2q^4 - 2q^2 + 3 \equiv 0 \pmod 3$. So $p' = 3$ and $2q^4 - 2q^2 + 3 = 3^m$. Therefore $2q^2(q^2-1)/3 = 3^{m-1} - 1$. If m is even, then $|3^{m-1} - 1|_2 = 2$, a contradiction. So, m is odd and $2q^2(q^2-1)/3 = (3^{(m-1)/2} - 1) \times (3^{(m-1)/2} + 1)$. Since $(3^{(m-1)/2} - 1, 3^{(m-1)/2} + 1) = 2$, we have $q^2 \mid 3^{(m-1)/2} - \delta$ and $3^{(m-1)/2} + \delta \mid 2(q^2-1)/3$ for $\delta = \pm 1$. If $q^2 \mid 3^{(m-1)/2} + 1$ and $3^{(m-1)/2} - 1 \mid 2(q^2-1)/3$, then there exists a positive integer k such that $3^{(m-1)/2} + 1 = q^2 k$ and $2(q^2-1)/3 = (3^{(m-1)/2} - 1)k$. If $k > 1$, then

$$3^{(m-1)/2} + 1 = q^2 k \geqslant 2q^2 > 4(q^2-1)/3 = 2(3^{(m-1)/2} - 1)k > 2(3^{(m-1)/2} - 1),$$

a contradiction. So $k = 1$, hence $3^{(m-1)/2} + 1 = q^2$ and $3^{(m-1)/2} - 1 = 2(q^2-1)/3 = 2(3^{(m-1)/2})/3$, which implies that $m = 3$ and $q = 2$. So $q^4 + 1$ is prime, which satisfies Case 1. In the case $q^2 \mid 3^{(m-1)/2} - 1$ and $3^{(m-1)/2} + 1 \mid 2(q^2-1)/3$, we get a contradiction similarly. If $\varepsilon = 1$, then one can get a contradiction similarly.

Let $K/H \cong Sz(q')$, where $q' = 2^{2m+1} > 2$. If $2^{2m+1} - 1 = p = 2^{4n} - 2^{2n} + 1$, then $2^{2m+1} - 2^{4n} + 2^{2n} = 2$, a contradiction. If $2^{2m+1} \pm 2^{m+1} + 1 = 2^{4n} - 2^{2n} + 1$, then $2^{m+1}(2^m \pm 1) = 2^{2n}(2^{2n} - 1)$. The only possibility is $m = n = 1$, so $q^4 + 1$ is also prime and satisfies Case 1.

Let $K/H \cong {}^2F_4(q')$, where $q' = 2^{2m+1} > 2$. Then $2^{2(2m+1)} \pm 2^{3m+2} + 2^{2m+1} \pm 2^{m+1} + 1 = 2^{4n} - 2^{2n} + 1$, which implies that $2^{m+1}(2^{3m+1} \pm 2^{2m+1} + 2^m \pm 1) = 2^{2n}(2^{2n} - 1)$, so $m+1 = 2n$ and $2^{3m+1} \pm 2^{2m+1} + 2^m \pm 1 = 2^{m+1} - 1$, a contradiction.

Let $K/H \cong {}^2G_2(q')$ for $q' = 3^m$. Therefore $3^{2m} \pm 3^m + 1 = 2^{4n} - 2^{2n} + 1$, and consequently $3^m(3^m \pm 1) = 2^{2n}(2^{2n} - 1)$. So $2^{2n} \mid 3^m \pm 1$ and $3^m \mid 2^{2n} - 1$. Since $2^{2n} \leqslant 3^m \pm 1$ and $3^m \leqslant 2^{2n} - 1$, we conclude that $3^m = 2^{2n} - 1$. So by Lemma 2.7, we have $m = n = 1$, and hence $q^4 + 1$ is prime and satisfies Case 1. If $K/H \cong {}^2G_2(q')$, where $q' = 3^{2m+1} > 3$, one can get a contradiction similarly.

Let K/H be isomorphic to ${}^2D_{p'}(3)$, where $p' = 2^m + 1$. Then either $(3^{p'} + 1)/4 = p$, or $(3^{p'-1} + 1)/2 = p$. If $(3^{p'-1} + 1)/2 = p = q^4 - q^2 + 1$, then $(3^{p'-1} + 1)/2$ is a primitive prime divisor of $q^{12} - 1$. So 12 divides $(3^{p'-1} + 1)/2 - 1 = (3^{p'-1} - 1)/2$, a contradiction. If $(3^{p'} + 1)/4 = p = q^4 - q^2 + 1$, then $3^{p'-1} - 1 = 4q^2(q^2 - 1)/3$, and one can get a contradiction by easy calculation similar to $A_1(q')$, where $3 < q' = p'^m \equiv \varepsilon \pmod 4$ for $\varepsilon = \pm 1$.

Let $K/H \cong E_8(q')$, where $q' = p'^m$ for some prime p'. Therefore p is an element of the set

$$\{q'^8 \pm q'^7 \mp q'^5 - q'^4 \mp q'^3 \pm q' + 1,\ q'^8 - q'^6 + q'^4 - q'^2 + 1,\ q'^8 - q'^4 + 1\}.$$

So

$$\begin{aligned} q^3 + 1 < p = q^2(q^2 - 1) + 1 &< (q'^8 + q'^7 + q'^6 + q'^5 + q'^4 + q'^3 + q'^2 + q' + 1)(q' - 1) \\ &= q'^9 - 1 < q'^9 + 1, \end{aligned}$$

which implies that $q^3 < q'^9$. Let $S \in Syl_{p'}(G)$. So $q'^{120} \mid |S|$. If $p' \neq 2$, then since $p' \mid |G|$ we have $p' \mid (q^{12} - 1)(q^8 - 1)(q^6 - 1)(q^2 - 1)$. So $p' \mid \prod_{i=1}^{6}(q^{2i} - 1)$. Now by Lemma 2.9, $q'^{120} \leqslant |S| \leqslant q^{12p'/(p'-1)} \leqslant q^{18} < q'^{54}$, which is a contradiction. If $p' = 2$, then $|S| = q^{24}$. Therefore $q'^{120} \leqslant q^{24} = (q^3)^8 < q'^{72}$, which is impossible.

Therefore $K/H \cong F_4(q')$, where $q' = 2^m$ and m is an integer. Obviously $m \leqslant n$. Since $p \in \pi(K/H)$, it follows that $p = q^4 - q^2 + 1$ is a divisor of $q'^{24}(q'^{12} - 1) \times (q'^8 - 1)(q'^6 - 1)(q'^2 - 1)$. Note that p is a primitive prime divisor of $2^{12n} - 1$. So if $m < n$, then $p \nmid |G|$, a contradiction. Therefore $m = n$, and $K/H \cong F_4(q)$.

So in both cases $K/H \cong F_4(q)$ and by the fact that $|G| = |F_4(q)|$, it is obvious that $H = 1$ and $G = K$, hence $G \cong F_4(q)$ and the result is proved. □

Theorem 2.11. *If G is a finite group such that $\mathrm{Vo}(G) = \mathrm{Vo}(Sz(q))$ and $|G| = |Sz(q)|$, where $q = 2^{2n+1} > 2$ and either $q - 1$, $q - \sqrt{2q} + 1$ or $q + \sqrt{2q} + 1$ is prime, then $G \cong Sz(q)$.*

P r o o f. Since $\mathrm{Vo}(G) = \mathrm{Vo}(Sz(q))$, we have $\Gamma(G) = \Gamma(Sz(q))$. By Lemma 2.3, we know that $\Gamma(G) = GK(Sz(q))$ has four connected components including two isolated vertices 2 and p, where $p \in \{q-1, q-\sqrt{2q}+1, q+\sqrt{2q}+1\}$. Also we have $|G| = |Sz(q)| = q^2(q-1)(q^2+1)$. Since $p \in \mathrm{Vo}(Sz(q)) = \mathrm{Vo}(G)$, there exist an element $g \in G$ and an irreducible character $\chi \in \mathrm{Irr}(G)$ such that $o(g) = p$ and $\chi(g) = 0$. So by Lemma 2.6, $p \mid \chi(1)$. Therefore $p \nmid |G|/\chi(1)$, which implies that χ is of p-defect zero. So for every element $h \in G$ such that $p \mid o(h)$, we conclude that $\chi(h) = 0$. Consequently, p is also an isolated vertex of $GK(G)$, and hence $t(G) \geqslant 2$.

Since $\Gamma(G)$ has more than 2 connected components, Lemma 2.3 implies that G is not solvable. So G is not a 2-Frobenius group. Now let G be a Frobenius group with Frobenius kernel K and Frobenius complement H. So $GK(G)$ has two connected components with vertex sets $\pi(K)$ and $\pi(H)$. Also, $|G| = |H||K|$, and $|H| \mid |K|-1$. Therefore $|H| < |K|$. Since $|G| = q^2(q-1)(q^2+1)$ and p is an isolated vertex of $GK(G)$, we conclude that $|H| = p$ and $|K| = |G|/p$. So, p is a divisor of $|G|/p - 1$, which is a contradiction for every $p \in \{q-1,\ q-\sqrt{2q}+1,\ q+\sqrt{2q}+1\}$.

So G is neither a Frobenius group, nor a 2-Frobenius group. Hence Theorem 2.1 implies that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that H is a nilpotent group, K/H is a nonabelian simple group, and $\pi(H) \cup \pi(G/K) \subseteq \pi_1(G)$. Since $|K/H| \mid |G|$, we have $3 \nmid |K/H|$. So, $K/H \cong Sz(q')$, where $q' = 2^{2m+1} > 2$, and $m \leqslant n$ is an integer. We claim that $m = n$.

First, let $p = q - 1 = 2^{2n+1} - 1$. So p is a primitive prime divisor of $2^{2n+1} - 1$, by Lemma 2.8. Since $p \mid |K/H|$ and $m < n$, we conclude that $p \mid 2^{2(2m+1)} + 1$. Hence, $2^{4(2m+1)} \equiv 1 \pmod p$, and so $\mathrm{ord}_p(2) = 2n+1$ divides $4(2m+1)$. Therefore, $2n+1 \mid 2m+1$, which implies that $n \leqslant m$, and consequently $n = m$.

Now let $p = q+\sqrt{2q}+1 = 2^{2n+1}+2^{n+1}+1$. So $p \in \{2^{2m+1}-1,\ 2^{2m+1}-2^{m+1}+1,\ 2^{2m+1}+2^{m+1}+1\}$. If $p = 2^{2m+1}-1$, then p is a primitive prime divisor of $2^{2m+1}-1$. Since $p \mid 2^{4(2n+1)} - 1$, we have $2m+1 \mid 4(2n+1)$ and so $2m+1 \mid 2n+1$, hence $p \mid 2^{2n+1} - 1$, a contradiction. If $2^{2m+1} - 2^{m+1} + 1 = p = 2^{2n+1} + 2^{n+1} + 1$, then $2^{m+1}(2^m-1) = 2^{n+1}(2^n+1)$, which is impossible. So $2^{2m+1}+2^{m+1}+1 = p = 2^{2n+1}+2^{n+1}+1$, and consequently $m = n$, as required. If $p = q-\sqrt{2q}+1 = 2^{2n+1}-2^{n+1}+1$, then we can similarly get that $m = n$.

Therefore $m = n$ and $K/H \cong Sz(q)$, and by the fact that $|G| = |Sz(q)|$, we have $H = 1$, $G = K$, and $G \cong Sz(q)$ as required. □

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