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GENERALIZED DERIVATIONS ACTING ON MULTILINEAR POLYNOMIALS IN PRIME RINGS

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Abstract. Let R be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, let F, G and H be three generalized derivations of R, I an ideal of R and $f(x_1, \ldots, x_n)$ a multilinear polynomial over C which is not central valued on R. If

$$F(f(r))G(f(r)) = H(f(r)^2)$$

for all $r = (r_1, \ldots, r_n) \in I^n$, then one of the following conditions holds:

(1) there exist $a \in C$ and $b \in U$ such that $F(x) = ax$, $G(x) = xb$ and $H(x) = xab$ for all $x \in R$;

(2) there exist $a, b \in U$ such that $F(x) = xa$, $G(x) = bx$ and $H(x) = abx$ for all $x \in R$, with $ab \in C$;

(3) there exist $b \in C$ and $a \in U$ such that $F(x) = ax$, $G(x) = bx$ and $H(x) = abx$ for all $x \in R$;

(4) $f(x_1, \ldots, x_n)^2$ is central valued on R and one of the following conditions holds:
   - (a) there exist $a, b, p, p' \in U$ such that $F(x) = ax$, $G(x) = xb$ and $H(x) = px + xp'$ for all $x \in R$, with $ab = p + p'$;
   - (b) there exist $a, b, p, p' \in U$ such that $F(x) = xa$, $G(x) = bx$ and $H(x) = px + xp'$ for all $x \in R$, with $p + p' = ab \in C$.

Keywords: prime ring; derivation; generalized derivation; extended centroid; Utumi quotient ring

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1. Introduction

Throughout this paper R always denotes an associative prime ring with center $Z(R)$, extended centroid C, and U its Utumi quotient ring. The Lie commutator

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of x and y is denoted by $[x,y]$ and defined by $[x,y] = xy - yx$ for $x, y \in R$. An additive mapping $d\colon R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. An additive subgroup L of R is said to be a Lie ideal of R if $[L,R] \subseteq L$. An additive mapping $F\colon R \to R$ is called a generalized derivation if there exists a derivation $d\colon R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. Evidently, any derivation is a generalized derivation. Thus, the generalized derivation covers both the concepts of derivation and left multiplier mapping. The left multiplier mapping means an additive mapping $F\colon R \to R$ such that $F(xy) = F(x)y$ holds for all $x, y \in R$. We denote by s_4 the standard polynomial in four variables, which is $s_4(x_1, x_2, x_3, x_4) = \sum_{\sigma \in S_4} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)}$ where $(-1)^\sigma$ is $+1$ or -1 according to σ being an even or odd permutation in symmetric group S_4.

Let S be a nonempty subset of R and $F\colon R \to R$ an additive mapping. Then we say that F acts as a homomorphism or anti-homomorphism on S if $F(xy) = F(x)F(y)$ or $F(xy) = F(y)F(x)$ holds for all $x, y \in S$, respectively. The additive mapping F acts as a Jordan homomorphism on S if $F(x^2) = F(x)^2$ holds for all $x \in S$.

A series of papers in literature studied the homomorphism or anti-homomorphism of some specific type of additive mappings in prime and semiprime rings under certain conditions (see [1], [2], [4], [5], [10], [17], [14], [19], [30], [31]).

In [10], De Filippis studied the following cases: (i) when the generalized derivation F acts as a Jordan homomorphism on a noncentral Lie ideal L of R, that is $F(x)F(x) = F(x^2)$ for all $x \in L$, and (ii) $F(x)F(x) = F(x^2)$ for all $x \in [I, I]$, where I is a nonzero right ideal of a prime ring R.

It is natural to ask what happens, if we consider three generalized derivations $F, G, H\colon R \to R$ such that $F(x)G(x) = H(x^2)$ for all x in a suitable subset of R.

Recently, Dhara, Rehman and Raza in [16] proved that if R is a prime ring of characteristic not 2, L a nonzero square closed Lie ideal of R and F, G, H three generalized derivations associated with derivations $d(\neq 0)$, $\delta(\neq 0)$, h such that $F(u)G(v) \pm H(uv) \in Z(R)$ for all $u, v \in L$ or $F(u)G(v) \pm H(vu) \in Z(R)$ for all $u, v \in L$, then $L \subseteq Z(R)$.

In the present paper, our motive is to investigate the situation $F(x)G(x) = H(x^2)$ for all $x \in \{f(x_1, \ldots, x_n)\colon x_1, \ldots, x_n \in I\}$, where I is a nonzero ideal of R and $f(x_1, \ldots, x_n)$ is a multilinear polynomial over C. Note that in case $F = G = H$, Dhara, Huang and Pattanayak studied a more general situation in [15], that is, $F(x)^n = F(x^n)$ for all $x \in \{f(x_1, \ldots, x_n)\colon x_1, \ldots, x_n \in I\}$, where I is a nonzero right ideal of R and $f(x_1, \ldots, x_n)$ is a multilinear polynomial over C.

More precisely, we prove the following theorem:

Main theorem. *Let R be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, let F, G and H be three generalized derivations of R, I an ideal of R and $f(x_1, \dots, x_n)$ a multilinear polynomial over C which is not central valued on R. If*

$$F(f(r))G(f(r)) = H(f(r)^2)$$

for all $r = (r_1, \dots, r_n) \in I^n$, then one of the following conditions holds:

(1) *there exist $a \in C$ and $b \in U$ such that $F(x) = ax$, $G(x) = xb$ and $H(x) = xab$ for all $x \in R$;*

(2) *there exist $a, b \in U$ such that $F(x) = xa$, $G(x) = bx$ and $H(x) = abx$ for all $x \in R$, with $ab \in C$;*

(3) *there exist $b \in C$ and $a \in U$ such that $F(x) = ax$, $G(x) = bx$ and $H(x) = abx$ for all $x \in R$;*

(4) *$f(x_1, \dots, x_n)^2$ is central valued on R and one of the following conditions holds:*
  - (a) *there exist $a, b, p, p' \in U$ such that $F(x) = ax$, $G(x) = xb$ and $H(x) = px + xp'$ for all $x \in R$, with $ab = p + p'$;*
  - (b) *there exist $a, b, p, p' \in U$ such that $F(x) = xa$, $G(x) = bx$ and $H(x) = px + xp'$ for all $x \in R$, with $p + p' = ab \in C$.*

Example 1.1. Let Z be the set of all integers. Consider a ring $R = \left\{\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} : x, y \in Z\right\}$ and a multilinear polynomial $f(x, y) = xy$ which is not central valued on R. We define maps $F, G, d, g \colon R \to R$ by $G\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x & 2y \\ 0 & 0 \end{pmatrix}$, $g\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}$, $F\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x & 3y \\ 0 & 0 \end{pmatrix}$ and $d\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2y \\ 0 & 0 \end{pmatrix}$. Then F and G are generalized derivations of R associated with derivations d and g, respectively. We see that

$$G(f(x, y))F(f(x, y)) = F(f(x, y)^2)$$

for all $x, y \in R$.

As an immediate application of the main theorem, in particular, when $H = 0$, we obtain the result of Carini, De Filippis and Scudo in [7]:

Corollary 1.2. *Let R be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, let F, G be two nonzero generalized derivations of R, I an ideal of R and $f(x_1, \dots, x_n)$ a multilinear polynomial over C which is not central valued on R. If*

$$F(f(r))G(f(r)) = 0$$

for all $r = (r_1, \dots, r_n) \in I^n$, then one of the following conditions holds:

(1) *there exist* $a, b \in U$ *such that* $F(x) = xa$, $G(x) = bx$ *for all* $x \in R$, *with* $ab = 0$;
(2) $f(x_1, \ldots, x_n)^2$ *is central valued on* R *and there exist* $a, b \in U$ *such that* $F(x) = ax$, $G(x) = xb$ *for all* $x \in R$, *with* $ab = 0$.

In particular, when $F = G$ in our Main theorem, we obtain Theorem 1 of De Filippis and Scudo in [12] as a special case.

Corollary 1.3. *Let* R *be a noncommutative prime ring of characteristic different from* 2 *with Utumi quotient ring* U *and extended centroid* C, *let* F *and* H *be two generalized derivations of* R, I *an ideal of* R *and* $f(x_1, \ldots, x_n)$ *a multilinear polynomial over* C *which is not central valued on* R. *If*

$$F(f(r))^2 = H(f(r)^2)$$

for all $r = (r_1, \ldots, r_n) \in I^n$, *then one of the following conditions holds:*
(1) *there exists* $a \in C$ *such that* $F(x) = ax$, *and* $H(x) = a^2 x$ *for all* $x \in R$;
(2) $f(x_1, \ldots, x_n)^2$ *is central valued on* R *and there exist* $a \in C$, $p, p' \in U$ *such that* $F(x) = ax$, *and* $H(x) = px + xp'$ *for all* $x \in R$, *with* $p + p' = a^2$.

In particular, when $F = G = H$, our Main theorem yields the following corollary which is Corollary 2.3 in [15].

Corollary 1.4. *Let* R *be a noncommutative prime ring of characteristic different from* 2 *with Utumi quotient ring* U *and extended centroid* C, *let* F *be a generalized derivation of* R, I *an ideal of* R *and* $f(x_1, \ldots, x_n)$ *a multilinear polynomial over* C *which is not central valued on* R. *If*

$$F(f(r))^2 = F(f(r)^2)$$

for all $r = (r_1, \ldots, r_n) \in I^n$, *then* $F(x) = x$ *for all* $x \in R$.

Another immediate corollary is obtained by taking $F(x) = x$ for all $x \in R$, $G = 2d$ and $H = d$, where d is a derivation in our Main theorem, which gives the particular case of the main result of Lee and Lee in [26]. Moreover, replacing multilinear polynomial $f(x_1, \ldots, x_n)$ by x, the corollary gives the famous result of Posner in [29].

Corollary 1.5. *Let* R *be a prime ring of characteristic different from* 2 *with extended centroid* C, *let* d *be a nonzero derivation of* R, I *an ideal of* R *and* $f(x_1, \ldots, x_n)$ *a multilinear polynomial over* C. *If* $[d(f(r)), f(r)] = 0$ *for all* $r = (r_1, \ldots, r_n) \in I^n$, *then* $f(x_1, \ldots, x_n)$ *is central valued on* R.

2. Main results

First we consider the inner generalized derivation cases. Let $F(x) = ax + xc$, $G(x) = bx + xq$ and $H(x) = px + xp'$ for all $x \in R$, for some $a, b, c, p, q, p' \in U$. Then $F(f(r))G(f(r)) = H(f(r)^2)$ for all $x \in f(R)$ yields

$$(af(r) + f(r)c)(bf(r) + f(r)q) = pf(r)^2 + f(r)^2p',$$

which gives

$$af(r)bf(r) + af(r)^2q + f(r)c'f(r) + f(r)cf(r)q = pf(r)^2 + f(r)^2p'$$

for all $r = (r_1, \ldots, r_n) \in R^n$, where $c' = cb$. We investigate this generalized polynomial identity in the prime ring.

We need the following known results:

Lemma 2.1 ([3], Lemma 1). *Let R be a noncommutative prime ring, $a, b \in U$, let $p(x_1, \ldots, x_n)$ be any polynomial over C which is not an identity for R. If $ap(r) - p(r)b = 0$ for all $r = (r_1, \ldots, r_n) \in R^n$, then one of the following conditions holds:*
(1) $a = b \in C$,
(2) $a = b$ *and* $p(x_1, \ldots, x_n)$ *is central valued on* R,
(3) $\mathrm{char}(R) = 2$ *and* R *satisfies* s_4.

Lemma 2.2 ([3], Lemma 3). *Let R be a noncommutative prime ring with Utumi quotient ring U and extended centroid C, and let $f(x_1, \ldots, x_n)$ be a multilinear polynomial over C which is not central valued on R. Suppose that there exist $a, b, c, q \in U$ such that $(af(r) + f(r)b)f(r) - f(r)(cf(r) + f(r)q) = 0$ for all $r = (r_1, \ldots, r_n) \in R^n$. Then one of the following conditions holds:*
(1) $a, q \in C$ *and* $q - a = b - c = \alpha \in C$;
(2) $f(x_1, \ldots, x_n)^2$ *is central valued on* R *and there exists* $\alpha \in C$ *such that* $q - a = b - c = \alpha$;
(3) $\mathrm{char}(R) = 2$ *and* R *satisfies* s_4.

In particular, from the above lemma, we have the following result:

Lemma 2.3. *Let R be a noncommutative prime ring with Utumi quotient ring U and extended centroid C, and let $f(x_1, \ldots, x_n)$ be a multilinear polynomial over C which is not central valued on R. Suppose that there exist $a, b, c \in U$ such that $f(r)af(r) + f(r)^2b - cf(r)^2 = 0$ for all $r = (r_1, \ldots, r_n) \in R^n$. Then one of the following conditions holds:*

(1) $b, c \in C$ and $c - b = a = \alpha \in C$;
(2) $f(x_1, \ldots, x_n)^2$ *is central valued on* R *and there exists* $\alpha \in C$ *such that* $c - b = a = \alpha$;
(3) $\operatorname{char}(R) = 2$ *and* R *satisfies* s_4.

Lemma 2.4. *Let* R *be a noncommutative prime ring with Utumi quotient ring* U *and extended centroid* C, *and let* $f(x_1, \ldots, x_n)$ *be a multilinear polynomial over* C *which is not central valued on* R. *Suppose that there exist* $a, b \in U$ *such that* $(af(r) + f(r)b)f(r) = 0$ *for all* $r = (r_1, \ldots, r_n) \in R^n$. *Then one of the following conditions holds:*
(1) $a, b \in C$ and $a + b = 0$;
(2) $\operatorname{char}(R) = 2$ *and* R *satisfies* s_4.

Lemma 2.5. *Let* R *be a noncommutative prime ring with Utumi quotient ring* U *and extended centroid* C, *and let* $f(x_1, \ldots, x_n)$ *be a multilinear polynomial over* C *which is not central valued on* R. *Suppose that there exist* $c, q \in U$ *such that* $f(r)(cf(r) + f(r)q) = 0$ *for all* $r = (r_1, \ldots, r_n) \in R^n$. *Then one of the following conditions holds:*
(1) $c, q \in C$ and $q + c = 0$;
(2) $\operatorname{char}(R) = 2$ *and* R *satisfies* s_4.

Lemma 2.6 ([11], Lemma 1)**.** *Let* C *be an infinite field and* $m \geqslant 2$. *If* $A_1, \ldots, A_k$ *are not scalar matrices in* $M_m(C)$ *then there exists an invertible matrix* $P \in M_m(C)$ *such that all matrices* $PA_1P^{-1}, \ldots, PA_kP^{-1}$ *have entries different from zero.*

Proposition 2.7. *Let* $R = M_m(C)$, $m \geqslant 2$, *be the ring of all* $m \times m$ *matrices over the infinite field* C, $f(x_1, \ldots, x_n)$ *a noncentral multilinear polynomial over* C *and* $a, b, c, p, q, c', p' \in R$. *If*

$$af(r)bf(r) + af(r)^2q + f(r)c'f(r) + f(r)cf(r)q = pf(r)^2 + f(r)^2p'$$

for all $r = (r_1, \ldots, r_n) \in R^n$, *then either* a *or* b *and either* c *or* q *are central.*

P r o o f. By our assumption R satisfies the generalized identity

$$\begin{aligned}(2.1)\qquad & af(x_1, \ldots, x_n)bf(x_1, \ldots, x_n) + af(x_1, \ldots, x_n)^2q \\ &+ f(x_1, \ldots, x_n)c'f(x_1, \ldots, x_n) + f(x_1, \ldots, x_n)cf(x_1, \ldots, x_n)q \\ &= pf(x_1, \ldots, x_n)^2 + f(x_1, \ldots, x_n)^2p'.\end{aligned}$$

We assume first that $a \notin Z(R)$ and $b \notin Z(R)$. Now we shall show that this case leads to a contradiction.

Since $a \notin Z(R)$ and $b \notin Z(R)$, by Lemma 2.6 there exists a C-automorphism φ of $M_m(C)$ such that $a_1 = \varphi(a)$, $b_1 = \varphi(b)$ have all nonzero entries. Clearly a_1, b_1, $c_1 = \varphi(c)$, $c'_1 = \varphi(c')$, $q_1 = \varphi(q)$, $p_1 = \varphi(p)$ and $p'_1 = \varphi(p')$ must satisfy the condition

$$\begin{aligned}(2.2)\qquad & a_1 f(x_1, \ldots, x_n) b_1 f(x_1, \ldots, x_n) + a_1 f(x_1, \ldots, x_n)^2 q_1 \\ & \quad + f(x_1, \ldots, x_n) c'_1 f(x_1, \ldots, x_n) + f(x_1, \ldots, x_n) c_1 f(x_1, \ldots, x_n) q_1 \\ & = p_1 f(x_1, \ldots, x_n)^2 + f(x_1, \ldots, x_n)^2 p'_1\end{aligned}$$

for all $x_1, \ldots, x_n \in R$.

Here e_{kl} denotes the usual matrix unit with 1 in (k, l)-entry and zero elsewhere. Since $f(x_1, \ldots, x_n)$ is not central, by [24] (see also [27]) there exist $u_1, \ldots, u_n \in M_m(C)$ and $0 \neq \gamma \in C$ such that $f(u_1, \ldots, u_n) = \gamma e_{kl}$, with $k \neq l$. Moreover, since the set $\{f(r_1, \ldots, r_n)\colon\ r_1, \ldots, r_n \in M_m(C)\}$ is invariant under the action of all C-automorphisms of $M_m(C)$ for any $i \neq j$ there exist $r_1, \ldots, r_n \in M_m(C)$ such that $f(r_1, \ldots, r_n) = \gamma e_{ij}$, where $0 \neq \gamma \in C$. Hence by (2.2) we have

$$(2.3)\qquad a_1 e_{ij} b_1 e_{ij} + e_{ij} c'_1 e_{ij} + e_{ij} c_1 e_{ij} q_1 = 0$$

and then left multiplying by e_{ij} implies $e_{ij} a_1 e_{ij} b_1 e_{ij} = 0$, which is a contradiction, since a_1 and b_1 have all nonzero entries. Thus we conclude that either a or b are central.

Similarly we can prove that c or q are central. □

Proposition 2.8. *Let $R = M_m(C)$, $m \geqslant 2$, be the ring of all matrices over the field C with* $\operatorname{char}(R) \neq 2$, *$f(x_1, \ldots, x_n)$ a noncentral multilinear polynomial over C and $a, b, c, p, q, c', p' \in R$. If*

$$af(r)bf(r) + af(r)^2 q + f(r)c'f(r) + f(r)cf(r)q = pf(r)^2 + f(r)^2 p'$$

for all $r = (r_1, \ldots, r_n) \in R^n$, then either a or b and either c or q are central.

Proof. If one assumes that C is infinite, then the conclusions follow by Proposition 2.7.

Now let C be finite and let K be an infinite field which is an extension of the field C. Let $\overline{R} = M_m(K) \cong R \otimes_C K$. Notice that the multilinear polynomial $f(x_1, \ldots, x_n)$ is central valued on R if and only if it is central valued on $\overline{R}$. Consider the generalized polynomial

$$\begin{aligned}(2.4)\qquad P(x_1, \ldots, x_n) &= af(x_1, \ldots, x_n) bf(x_1, \ldots, x_n) + af(x_1, \ldots, x_n)^2 q \\ & \quad + f(x_1, \ldots, x_n) c' f(x_1, \ldots, x_n) + f(x_1, \ldots, x_n) cf(x_1, \ldots, x_n) q \\ & \quad - (pf(x_1, \ldots, x_n)^2 + f(x_1, \ldots, x_n)^2 p') = 0\end{aligned}$$

which is a generalized polynomial identity for R.

Moreover, it is multi-homogeneous of multi-degree $(2, \dots, 2)$ in the indeterminates $x_1, \dots, x_n$.

Hence the complete linearization of $P(x_1, \dots, x_n)$ is a multilinear generalized polynomial $\Theta(x_1, \dots, x_n, y_1, \dots, y_n)$ in $2n$ indeterminates, moreover,

$$\Theta(x_1, \dots, x_n, x_1, \dots, x_n) = 2^n P(x_1, \dots, x_n).$$

Clearly the multilinear polynomial $\Theta(x_1, \dots, x_n, y_1, \dots, y_n)$ is a generalized polynomial identity for R and $\overline{R}$ too. Since $\mathrm{char}(C) \neq 2$ we obtain $P(r_1, \dots, r_n) = 0$ for all $r_1, \dots, r_n \in \overline{R}$ and then the conclusion follows from Proposition 2.7. □

Lemma 2.9. *Let R be a noncommutative prime ring of $\mathrm{char}(R) \neq 2$, $a, b, c, c' \in U$, let $p(x_1, \dots, x_n)$ be any polynomial over C which is not an identity for R. If $ap(r) + p(r)b + cp(r)c' = 0$ for all $r = (r_1, \dots, r_n) \in R^n$, then one of the following conditions holds:*

(1) *$b, c' \in C$ and $a + b + cc' = 0$,*

(2) *$a, c \in C$ and $a + b + cc' = 0$,*

(3) *$a + b + cc' = 0$ and $p(x_1, \dots, x_n)$ is central valued on R.*

Proof. If $p(x_1, \dots, x_n)$ is central valued on R, then our assumption $ap(r) + p(r)b + cp(r)c' = 0$ yields $(a + b + cc')p(r) = 0$ for all $r = (r_1, \dots, r_n) \in R^n$. Since $p(r_1, \dots, r_n)$ is nonzero valued on R, $a + b + cc' = 0$ and hence we obtain our conclusion (3).

If $c' \in C$, then by assumption we have $(a + cc')p(r) + p(r)b = 0$ for all $r = (r_1, \dots, r_n) \in R^n$. By Lemma 2.1, we have one of the following conditions: (1) $a + cc' = -b \in C$, which is our conclusion (1); (2) $a + cc' = -b$ and $p(r_1, \dots, r_n)$ is central valued on R, which is our conclusion (3).

If $c \in C$, then by assumption we have $ap(r) + p(r)(b + cc') = 0$ for all $r = (r_1, \dots, r_n) \in R^n$. By Lemma 2.1, we have one of the following conditions: (1) $b + cc' = -a \in C$, which is our conclusion (2); (2) $b + cc' = -a$ and $p(r_1, \dots, r_n)$ is central valued on R, which is our conclusion (3).

Next, we assume that $p(x_1, \dots, x_n)$ is not central valued on R and $c, c' \notin C$. Let G be the additive subgroup of R generated by the set $S = \{p(x_1, \dots, x_n) \colon x_1, \dots, x_n \in R\}$. Then $S \neq \{0\}$, since $p(x_1, \dots, x_n)$ is nonzero valued on R. By our assumption we get $ax + xb + cxc' = 0$ for any $x \in G$. By [8], either $G \subseteq Z(R)$ or $\mathrm{char}(R) = 2$ and R satisfies s_4, except when G contains a noncentral Lie ideal L of R. Since $p(x_1, \dots, x_n)$ is not central valued on R, the first case cannot occur. Moreover, since $\mathrm{char}(R) \neq 2$, we have only the case that G contains a noncentral Lie ideal L of R. By [6], Lemma 1, there exists a noncentral two sided ideal I of R such that $[I, R] \subseteq L$. In particular, $a[x_1, x_2] + [x_1, x_2]b + c[x_1, x_2]c' = 0$ for all $x_1, x_2 \in I$.

By [9], $a[x_1, x_2] + [x_1, x_2]b + c[x_1, x_2]c' = 0$ is a generalized polynomial identity for R and for U.

Since c and c' are not in C, the generalized polynomial identity (GPI) $a[x_1, x_2] + [x_1, x_2]b + c[x_1, x_2]c' = 0$ is nontrivial GPI for U and $U \otimes_C \overline{C}$. Since both U and $U \otimes_C \overline{C}$ are centrally closed (see [18]), we may replace R by U or $U \otimes_C \overline{C}$ according as C is finite or infinite. Thus we may assume that R is centrally closed over C which is either finite or algebraically closed. By Martindale's theorem in [28], R is a primitive ring having a nonzero socle $\mathrm{Soc}(R)$ with C as the associated division ring. In light of Jacobson's theorem in [20], page 75, R is isomorphic to a dense ring of linear transformations on some vector space V over C. Since R is not commutative, $\dim_C V \geqslant 2$. If $\dim_C V = n$, then by density of R we have $R \cong M_n(C)$, $n \geqslant 2$. Replacing $[x_1, x_2] = [e_{ii}, e_{ij}] = e_{ij}$, we have $0 = ae_{ij} + e_{ij}b + ce_{ij}c'$. Left and right multiplying by e_{ij}, we have $0 = c_{ji}c'_{ji}e_{ij}$. This implies $c_{ji}c'_{ji} = 0$. Then by the same argument as before Proposition 2.7 and Proposition 2.8, we conclude that either $c \in C$ or $c' \in C$, a contradiction. Assume now that V is infinite dimensional over C. Then for any $e = e^2 \in \mathrm{Soc}(R)$ we have $eRe \cong M_k(C)$ with $k = \dim_C Ve$. Since $c \notin C$ and $c' \notin C$, c and c' do not centralize the nonzero ideal $\mathrm{Soc}(R)$ of R, so $ch_0 \neq h_0c$ and $c'h_1 \neq h_1c'$ for some $h_0, h_1 \in \mathrm{Soc}(R)$. By Litoff's theorem in [22], page 280, there exists an idempotent $e \in \mathrm{Soc}(R)$ such that h_0, h_1, h_0c, ch_0, h_1c', $c'h_1$ are all in eRe. We have $eRe \cong M_k(C)$ where $k = \dim_C Ve$. Since R satisfies GPI $e(a[ex_1e, ex_2e] + [ex_1e, ex_2e]b + c[ex_1e, ex_2e]c')e = 0$, the subring eRe satisfies the GPI $eae[x_1, x_2] + [x_1, x_2]ebe + ece[x_1, x_2]ec'e = 0$. Then by the above finite dimensional case, we conclude that either $ece \in Z(eRe)$ or $ec'e \in Z(eRe)$. Then

$$ch_0 = ech_0 = eceh_0 = h_0ece = h_0ce = h_0c$$

and

$$c'h_1 = ec'h_1 = ec'eh_1 = h_1ec'e = h_1c'e = h_1c'.$$

Both the cases lead to contradiction. □

Lemma 2.10. *Let R be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, and let $f(x_1, \ldots, x_n)$ be a multilinear polynomial over C which is not central valued on R. If F, G and H are three inner generalized derivations of R such that*

$$F(f(r))G(f(r)) = H(f(r)^2)$$

for all $r = (r_1, \ldots, r_n) \in R^n$, then one of the following conditions holds:

(1) *there exist $a \in C$ and $b \in U$ such that $F(x) = ax$, $G(x) = xb$ and $H(x) = xab$ for all $x \in R$;*

(2) *there exist* $a, b \in U$ *such that* $F(x) = xa$, $G(x) = bx$ *and* $H(x) = abx$ *for all* $x \in R$, *with* $ab \in C$;

(3) *there exist* $b \in C$ *and* $a \in U$ *such that* $F(x) = ax$, $G(x) = bx$ *and* $H(x) = abx$ *for all* $x \in R$;

(4) $f(x_1, \ldots, x_n)^2$ *is central valued on* R *and one of the following conditions holds:*

   (a) *there exist* $a, b, p, p' \in U$ *such that* $F(x) = ax$, $G(x) = xb$ *and* $H(x) = px + xp'$ *for all* $x \in R$, *with* $ab = p + p'$;

   (b) *there exist* $a, b, p, p' \in U$ *such that* $F(x) = xa$, $G(x) = bx$ *and* $H(x) = px + xp'$ *for all* $x \in R$, *with* $p + p' = ab \in C$.

Proof. Since F, G and H are three inner generalized derivations of R, we assume that $F(x) = ax + xc$, $G(x) = bx + xq$ and $H(x) = px + xp'$ for all $x \in R$ for some $a, b, c, p, q, p' \in U$. Then by hypothesis we have

$$\begin{aligned}(2.5)\qquad \Psi(x_1, \ldots, x_n) = {} & af(x_1, \ldots, x_n)bf(x_1, \ldots, x_n) + af(x_1, \ldots, x_n)^2q \\ & + f(x_1, \ldots, x_n)cbf(x_1, \ldots, x_n) + f(x_1, \ldots, x_n)cf(x_1, \ldots, x_n)q \\ & - (pf(x_1, \ldots, x_n)^2 + f(x_1, \ldots, x_n)^2p') = 0\end{aligned}$$

for all $x_1, \ldots, x_n \in R$. Since R and U satisfy the same generalized polynomial identities (see [9]), U satisfies $\Psi(x_1, \ldots, x_n) = 0$. Suppose that $\Psi(x_1, \ldots, x_n)$ is a trivial GPI for U. Let $T = U *_C C\{x_1, x_2, \ldots, x_n\}$, the free product of U and $C\{x_1, \ldots, x_n\}$, be the free C-algebra in noncommuting indeterminates $x_1, x_2, \ldots, x_n$. Then, $\Psi(x_1, \ldots, x_n)$ is the zero element in $T = U *_C C\{x_1, \ldots, x_n\}$. This implies that $\{p, a, 1\}$ is linearly dependent over C. Let $\alpha p + \beta a + \gamma = 0$. If $\alpha = 0$, then $\beta \neq 0$, and hence $a \in C$. If $\alpha \neq 0$, then $p = \lambda a + \mu$ for some $\lambda, \mu \in C$. In this case our identity reduces to

$$\begin{aligned}(2.6)\qquad & af(x_1, \ldots, x_n)bf(x_1, \ldots, x_n) + af(x_1, \ldots, x_n)^2q \\ & + f(x_1, \ldots, x_n)cbf(x_1, \ldots, x_n) + f(x_1, \ldots, x_n)cf(x_1, \ldots, x_n)q \\ & - ((\lambda a + \mu)f(x_1, \ldots, x_n)^2 + f(x_1, \ldots, x_n)^2p') = 0.\end{aligned}$$

If $a \notin C$, then

$$(2.7)\qquad af(x_1, \ldots, x_n)bf(x_1, \ldots, x_n) + af(x_1, \ldots, x_n)^2q - \lambda af(x_1, \ldots, x_n)^2 = 0,$$

that is

$$(2.8)\qquad af(x_1, \ldots, x_n)(bf(x_1, \ldots, x_n) + f(x_1, \ldots, x_n)q - \lambda f(x_1, \ldots, x_n)) = 0.$$

This implies $b \in C$. Thus we conclude that either $a \in C$ or $b \in C$.

Similarly, we can prove that either $c \in C$ or $q \in C$.

Next suppose that $\Psi(x_1, \ldots, x_n)$ is a nontrivial GPI for U. In case C is infinite, we have $\Psi(x_1, \ldots, x_n) = 0$ for all $x_1, \ldots, x_n \in U \otimes_C \overline{C}$, where $\overline{C}$ is the algebraic closure of C. Since both U and $U \otimes_C \overline{C}$ are prime and centrally closed [18], (see Theorems 2.5 and 3.5), we may replace R by U or $U \otimes_C \overline{C}$ according to C being finite or infinite. Then R is centrally closed over C and $\Psi(x_1, \ldots, x_n) = 0$ for all $x_1, \ldots, x_n \in R$. By Martindale's theorem in [28], R is then a primitive ring with a nonzero socle $\mathrm{soc}(R)$ and with C as its associated division ring. Then, by Jacobson's theorem (see [20], page 75), R is isomorphic to a dense ring of linear transformations of a vector space V over C. Assume first that V is finite dimensional over C, that is, $\dim_C V = m$. By density of R, we have $R \cong M_m(C)$. Since $f(r_1, \ldots, r_n)$ is not central valued on R, R must be noncommutative and so $m \geqslant 2$. In this case, by Proposition 2.8, we get that a or b and c or q are in C. If V is infinite dimensional over C, then for any $e^2 = e \in \mathrm{soc}(R)$ we have $eRe \cong M_t(C)$ with $t = \dim_C Ve$. We want to show that in this case also a or b and c or q are in C. To prove this, let none of a and b and none of c and q be in C. Then a, b, c and q do not centralize the nonzero ideal $\mathrm{soc}(R)$. Hence there exist $h_1, h_2, h_3, h_4 \in \mathrm{soc}(R)$ such that $[a, h_1] \neq 0$, $[b, h_2] \neq 0$, $[c, h_3] \neq 0$ and $[q, h_4] \neq 0$. By Litoff's theorem [22], page 280, there exists an idempotent $e \in \mathrm{soc}(R)$ such that $ah_1, h_1a, bh_2, h_2b, ch_3, h_3c, qh_4, h_4q, h_1, h_2, h_3, h_4 \in eRe$. We have $eRe \cong M_k(C)$ with $k = \dim_C Ve$. Since R satisfies the generalized identity

$$
\begin{aligned}
(2.9) \qquad & e\{af(ex_1e, \ldots, ex_ne)bf(ex_1e, \ldots, ex_ne) + af(ex_1e, \ldots, ex_ne)^2q \\
& + f(ex_1e, \ldots, ex_ne)cbf(ex_1e, \ldots, ex_ne) \\
& + f(ex_1e, \ldots, ex_ne)cf(ex_1e, \ldots, ex_ne)q \\
& - (pf(ex_1e, \ldots, ex_ne)^2 + f(ex_1e, \ldots, ex_ne)^2p')\}e = 0
\end{aligned}
$$

the subring eRe satisfies

$$
\begin{aligned}
(2.10) \quad & eaef(x_1, \ldots, x_n)ebef(x_1, \ldots, x_n) + eaef(x_1, \ldots, x_n)^2eqe \\
& + f(x_1, \ldots, x_n)ecbef(x_1, \ldots, x_n) + f(x_1, \ldots, x_n)ecef(x_1, \ldots, x_n)eqe \\
& - (epef(x_1, \ldots, x_n)^2 + f(x_1, \ldots, x_n)^2ep'e) = 0.
\end{aligned}
$$

Then by Proposition 2.8, either eae or ebe and either ece or eqe are central elements of eRe. Thus $ah_1 = (eae)h_1 = h_1eae = h_1a$ or $bh_2 = (ebe)h_2 = h_2(ebe) = h_2b$ and $ch_3 = (ece)h_3 = h_3(ece) = h_3c$ or $qh_4 = (eqe)h_4 = h_4eqe = h_4q$, a contradiction.

Thus up to now, we have proved that a or b and c or q are in C. Thus we have the following four cases:

Case I: $a, c \in C$. In this case, (2.5) reduces to

$$
(2.11) \qquad f(r)abf(r) + f(r)^2aq + f(r)cbf(r) + f(r)^2cq - (pf(r)^2 + f(r)^2p') = 0
$$

that is

$$f(r)(ab+cb)f(r)+f(r)^2(aq+cq-p')-pf(r)^2=0 \tag{2.12}$$

for all $r=(r_1,\dots,r_n)\in R^n$. Then by Lemma 2.3, we have any one of the following cases:

▷ $aq+cq-p', p\in C$ and $p-(aq+cq-p')=ab+cb=\alpha\in C$. Thus in this case we have $a,c,p\in C$, $(a+c)b\in C$ and $p+p'=(a+c)(q+b)$. Since $F\neq 0$, we have $0\neq a+c\in C$. Hence $(a+c)b\in C$ implies $b\in C$. Thus we have $F(x)=(a+c)x$, $G(x)=x(b+q)$ and $H(x)=x(p+p')=x(a+c)(q+b)$ for all $x\in R$, which is our conclusion (1).

▷ $f(x_1,\dots,x_n)^2$ is central valued on R and there exists $\alpha\in C$ such that $p-(aq+cq-p')=ab+cb=\alpha$. In this case we have $a,c\in C$, $(a+c)b\in C$ and $p+p'=(a+c)(q+b)$. Since $F\neq 0$, we have $0\neq a+c\in C$. Hence $(a+c)b\in C$ implies $b\in C$. Hence $F(x)=(a+c)x$, $G(x)=x(b+q)$ and $H(x)=px+xp'$ for all $x\in R$, which is our conclusion 4 (a).

Case II: $a,q\in C$. In this case, (2.5) reduces to

$$f(r)(ab+cb+cq+aq)f(r)-(pf(r)^2+f(r)^2p')=0 \tag{2.13}$$

for all $r=(r_1,\dots,r_n)\in R^n$. Then by Lemma 2.3, we have any one of the following cases:

▷ $p,p'\in C$ and $p+p'=ab+cb+cq+aq=\alpha\in C$. Thus in this case we have $a,q,p,p'\in C$, with $p+p'=(a+c)(b+q)\in C$. Hence $F(x)=x(a+c)$, $G(x)=(b+q)x$ and $H(x)=(p+p')x=(a+c)(b+q)x$ for all $x\in R$, which is our conclusion (2).

▷ $f(x_1,\dots,x_n)^2$ is central valued on R and there exists $\alpha\in C$ such that $p+p'=ab+cb+cq+aq=\alpha\in C$. In this case we have $a,q\in C$, with $p+p'=(a+c)(b+q)\in C$. Hence $F(x)=x(a+c)$, $G(x)=(b+q)x$ and $H(x)=px+xp'$ for all $x\in R$, which is our conclusion 4 (b).

Case III: $b,c\in C$. In this case, (2.5) reduces to

$$(ab+bc-p)f(r)^2+af(r)^2q+f(r)^2(cq-p')=0 \tag{2.14}$$

for all $r=(r_1,\dots,r_n)\in R^n$. Then by Lemma 2.9, we have any one of the following three cases:

▷ $q, cq-p'\in C$ and $ab+bc-p+aq+cq-p'=0$. Thus in this case we have $b,c,q,p'\in C$ and $(a+c)(b+q)=p+p'$. Hence $F(x)=(a+c)x$, $G(x)=(b+q)x$ and $H(x)=(p+p')x=(a+c)(b+q)x$ for all $x\in R$, which gives conclusion (3).

▷ $a, ab+bc-p \in C$ and $ab+bc-p+aq+cq-p' = 0$. In this case we have $a, b, c, p \in C$ and $(a+c)(b+q) = p+p'$. In this case $F(x) = (a+c)x$, $G(x) = x(b+q)$ and $H(x) = x(p+p') = x(a+c)(b+q)$ for all $x \in R$. This gives conclusion (1).

▷ $f(x_1, \dots, x_n)^2$ is central valued on R and $ab+bc-p+aq+cq-p' = 0$. Thus in this case we have $b, c \in C$ and $(a+c)(b+q) = p+p'$. Hence $F(x) = (a+c)x$, $G(x) = x(b+q)$ and $H(x) = px + xp'$ for all $x \in R$. This gives conclusion 4 (a).

Case IV: $b, q \in C$. In this case, (2.5) reduces to

$$(ab+aq-p)f(r)^2 + f(r)(cb+cq)f(r) - f(r)^2p' = 0 \tag{2.15}$$

for all $r = (r_1, \dots, r_n) \in R^n$. Then by Lemma 2.3, we have any one of the following cases:

▷ $ab+aq-p, p' \in C$ with $p' - (ab+aq-p) = cb+cq \in C$. In this case we have $b, q, p' \in C$ and $p+p' = (a+c)(b+q)$. Since $G \neq 0$, we have $0 \neq b+q \in C$. Hence $cb+cq = c(b+q) \in C$ implies $c \in C$. Thus $F(x) = (a+c)x$, $G(x) = (b+q)x$ and $H(x) = (p+p')x = (a+c)(b+q)x$ for all $x \in R$, which is our conclusion (3).

▷ $f(x_1, \dots, x_n)^2$ is central valued on R and there exists $\alpha \in C$ such that $p' - (ab+aq-p) = cb+cq = \alpha$. In this case, we have $b, q, (b+q)c \in C$ and $p+p' = (a+c)(b+q)$. Since $G \neq 0$, we have $0 \neq b+q \in C$. Hence $(b+q)c \in C$ implies $c \in C$. Thus $F(x) = (a+c)x$, $G(x) = x(b+q)$ and $H(x) = px+xp'$ for all $x \in R$, which is our conclusion 4 (a). □

Lemma 2.11. *Let R be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C. Let F, G be two generalized derivations of R, H an inner generalized derivation of R, I an ideal of R and $f(x_1, \dots, x_n)$ a multilinear polynomial over C which is not central valued on R. If*

$$F(f(r))G(f(r)) = H(f(r)^2)$$

for all $r = (r_1, \dots, r_n) \in I^n$, then one of the following conditions holds:

(1) *there exist $a \in C$ and $b \in U$ such that $F(x) = ax$, $G(x) = xb$ and $H(x) = xab$ for all $x \in R$;*
(2) *there exist $a, b \in U$ such that $F(x) = xa$, $G(x) = bx$ and $H(x) = abx$ for all $x \in R$, with $ab \in C$;*
(3) *there exist $b \in C$ and $a \in U$ such that $F(x) = ax$, $G(x) = bx$ and $H(x) = abx$ for all $x \in R$;*
(4) *$f(x_1, \dots, x_n)^2$ is central valued on R and one of the following conditions holds:*
    (a) *there exist $a, b, p, p' \in U$ such that $F(x) = ax$, $G(x) = xb$ and $H(x) = px + xp'$ for all $x \in R$, with and $ab = p + p'$;*
    (b) *there exist $a, b, p, p' \in U$ such that $F(x) = xa$, $G(x) = bx$ and $H(x) = px + xp'$ for all $x \in R$, with $p + p' = ab \in C$.*

Proof. Since H is an inner generalized derivation of R, let $H(x) = cx + xc'$ for all $x \in R$ and for some $c, c' \in U$. In view of [25], Theorem 3, we may assume that there exist $a, b \in U$ and derivations d, δ of U such that $F(x) = ax + d(x)$ and $G(x) = bx + \delta(x)$. Since R and U satisfy the same generalized polynomial identities (see [9]) as well as the same differential identities (see [24]), we may assume that

$$(2.16) \qquad (af(r) + d(f(r)))(bf(r) + \delta(f(r))) = cf(r)^2 + f(r)^2c'$$

for all $r = (r_1, \dots, r_n) \in U^n$, where d, δ are two derivations on U.

If both F and G are inner generalized derivations of R, then by Lemma 2.10, we obtain our conclusions. Thus we assume that not both of F and G are inner. Then d and δ cannot be both inner derivations of U. Now we consider the following two cases:

Case I: Assume that d and δ are C-dependent modulo inner derivations of U, say $\alpha d + \beta\delta = ad_q$, where $\alpha, \beta \in C$, $q \in U$ and $ad_q(x) = [q, x]$ for all $x \in U$.

Subcase i: Let $\alpha \neq 0$.

Then $d(x) = \lambda\delta(x) + [p, x]$ for all $x \in U$, where $\lambda = -\beta\alpha^{-1}$ and $p = \alpha^{-1}q$.

Then δ cannot be inner derivation of U. From (2.16), we obtain

$$(2.17) \qquad (af(r) + \lambda\delta(f(r)) + [p, f(r)])(bf(r) + \delta(f(r))) = cf(r)^2 + f(r)^2c'$$

for all $r = (r_1, \dots, r_n) \in U^n$, that is, U satisfies

$$(2.18) \quad \begin{aligned} &\Big(af(r_1, \dots, r_n) + \lambda f^{\delta}(r_1, \dots, r_n) \\ &\qquad + \lambda\sum_i f(r_1, \dots, \delta(r_i), \dots, r_n) + [p, f(r_1, \dots, r_n)]\Big) \\ &\qquad \times \Big(bf(r_1, \dots, r_n) + f^{\delta}(r_1, \dots, r_n) + \sum_i f(r_1, \dots, \delta(r_i), \dots, r_n)\Big) \\ &\quad = cf(r_1, \dots, r_n)^2 + f(r_1, \dots, r_n)^2c', \end{aligned}$$

where $f^{\delta}(r_1, \dots, r_n)$ is the polynomial obtained from $f(r_1, \dots, r_n)$ by replacing each of the coefficients α_σ by $\delta(\alpha_\sigma)$ and then we have $\delta(f(r_1, \dots, r_n)) = f^{\delta}(r_1, \dots, r_n) + \sum_i f(r_1, \dots, \delta(r_i), \dots, r_n)$. By Kharchenko's theorem, see [21], we have that U satisfies

$$\begin{aligned}(2.19)\qquad &\Big(af(r_1,\ldots,r_n)+\lambda f^{\delta}(r_1,\ldots,r_n)\\ &\quad+\lambda\sum_i f(r_1,\ldots,y_i,\ldots,r_n)+[p,f(r_1,\ldots,r_n)]\Big)\\ &\quad\times\Big(bf(r_1,\ldots,r_n)+f^{\delta}(r_1,\ldots,r_n)+\sum_i f(r_1,\ldots,y_i,\ldots,r_n)\Big)\\ &=cf(r_1,\ldots,r_n)^2+f(r_1,\ldots,r_n)^2c'.\end{aligned}$$

In particular, for $r_1=0$ we have that U satisfies

$$\lambda f(y_1,\ldots,r_n)^2=0. \tag{2.20}$$

This implies $\lambda=0$ or U satisfies $f(r_1,\ldots,r_n)^2=0$. In the latter case U satisfies the polynomial identity $f(r_1,\ldots,r_n)^2=0$ and hence there exists a field E such that $U\subseteq M_k(E)$ and U and $M_k(E)$ satisfy the same polynomial identities [23], Lemma 1. Then again by [27], Corollary 5, $f(r_1,\ldots,r_n)$ is an identity for $M_k(E)$ and so for U, a contradiction. Hence we conclude that $\lambda=0$. Thus from (2.19), U satisfies the blended component

$$(af(r_1,\ldots,r_n)+[p,f(r_1,\ldots,r_n)])\sum_i f(r_1,\ldots,y_i,\ldots,r_n)=0. \tag{2.21}$$

In particular, for $y_1=r_1$ and $y_2=\ldots=y_n=0$ we have that U satisfies

$$(af(r_1,\ldots,r_n)+[p,f(r_1,\ldots,r_n)])f(r_1,\ldots,r_n)=0. \tag{2.22}$$

By Lemma 2.4, this yields that $p\in C$ and $a=0$, implying $F=0$, a contradiction.

Subcase ii: *Let* $\alpha=0$.

Then $\delta(x)=[q',x]$ for all $x\in U$, where $q'=\beta^{-1}q$. Since δ is inner, d cannot be an inner derivation. From (2.16), we obtain

$$(af(r)+d(f(r)))(bf(r)+[q',f(r)])=cf(r)^2+f(r)^2c' \tag{2.23}$$

for all $r=(r_1,\ldots,r_n)\in U^n$.

Since $d(f(r_1,\ldots,r_n))=f^d(r_1,\ldots,r_n)+\sum_i f(r_1,\ldots,d(r_i),\ldots,r_n)$, by Kharchenko's theorem, see [21], we can replace $d(f(r_1,\ldots,r_n))$ by $f^d(r_1,\ldots,r_n)+\sum_i f(r_1,\ldots,y_i,\ldots,r_n)$ in (2.23) and then U satisfies the blended component

$$\sum_i f(r_1,\ldots,y_i,\ldots,r_n)(bf(r_1,\ldots,r_n)+[q',f(r_1,\ldots,r_n)])=0 \tag{2.24}$$

and so in particular

$$f(r_1,\ldots,r_n)(bf(r_1,\ldots,r_n)+[q',f(r_1,\ldots,r_n)])=0. \tag{2.25}$$

By Lemma 2.5, this yields $q'\in C$ and $b=0$, implying $G=0$, a contradiction.

Case II: Assume next that d and δ are C-independent modulo inner derivations of U.

Then applying Kharchenko's theorem from [21], we have from (2.16) that U satisfies the blended component

$$\sum_i f(r_1,\ldots,y_i,\ldots,r_n)\sum_i f(r_1,\ldots,t_i,\ldots,r_n)=0. \tag{2.26}$$

This gives $f(r_1,\ldots,r_n)^2=0$, implying $f(r_1,\ldots,r_n)=0$ as above, a contradiction. □

Lemma 2.12. *Let R be a prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, let F, G, H be three generalized derivations of R, I an ideal of R and $f(x_1,\ldots,x_n)$ a multilinear polynomial over C which is not central valued on R. If F is the inner generalized derivation of R such that*

$$F(f(r))G(f(r))=H(f(r)^2)$$

for all $r=(r_1,\ldots,r_n)\in I^n$, then one of the following conditions holds:

1. *there exist $a\in C$ and $b\in U$ such that $F(x)=ax$, $G(x)=xb$ and $H(x)=xab$ for all $x\in R$;*
2. *there exist $a,b\in U$ such that $F(x)=xa$, $G(x)=bx$ and $H(x)=abx$ for all $x\in R$, with $ab\in C$;*
3. *there exist $b\in C$ and $a\in U$ such that $F(x)=ax$, $G(x)=bx$ and $H(x)=abx$ for all $x\in R$;*
4. *$f(x_1,\ldots,x_n)^2$ is central valued on R and one of the following conditions holds:*
   - (a) *there exist $a,b,p,p'\in U$ such that $F(x)=ax$, $G(x)=xb$ and $H(x)=px+xp'$ for all $x\in R$, with $ab=p+p'$;*
   - (b) *there exist $a,b,p,p'\in U$ such that $F(x)=xa$, $G(x)=bx$ and $H(x)=px+xp'$ for all $x\in R$, with $p+p'=ab\in C$.*

Proof. Since F is inner, let $F(x)=ax+xa'$ for all $x\in R$ for some $a,a'\in U$. In view of [25], Theorem 3, we may assume that there exist $b,c\in U$ and derivations δ, h of U such that $G(x)=bx+\delta(x)$ and $H(x)=cx+h(x)$. Since R and U satisfy the same generalized polynomial identities (see [9]) as well as the same differential identities (see [24]), we may assume that

$$(af(r)+f(r)a')(bf(r)+\delta(f(r))) = cf(r)^2 + f(r)h(f(r)) + h(f(r))f(r) \tag{2.27}$$

for all $r=(r_1,\dots,r_n)\in U^n$, where d,δ are two derivations on U.

If H is inner, then the result follows by Lemma 2.11. So we assume that H is not the inner generalized derivation of U. Now we consider the following two cases:

Case I: Assume that h and δ are C-dependent modulo inner derivations of U, say $\alpha\delta+\beta h = ad_q$, where $\alpha,\beta\in C$, $q\in U$ and $ad_q(x)=[q,x]$ for all $x\in U$. If $\alpha=0$, then β cannot be equal to zero, implying that h is the inner derivation, a contradiction. Thus $\alpha\neq 0$.

Then $\delta(x)=\lambda h(x)+[p,x]$ for all $x\in U$, where $\lambda=-\beta\alpha^{-1}$ and $p=\alpha^{-1}q$.

From (2.27) we obtain

$$\begin{aligned}(af(r)+f(r)a')&(bf(r)+\lambda h(f(r))+[p,f(r)])\\ &= cf(r)^2+f(r)h(f(r))+h(f(r))f(r)\end{aligned} \tag{2.28}$$

for all $r=(r_1,\dots,r_n)\in U^n$, that is, U satisfies

$$\begin{aligned}(af(r_1,\dots,r_n)&+f(r_1,\dots,r_n)a')\Big(bf(r_1,\dots,r_n)+\lambda f^h(r_1,\dots,r_n)\\ &+\lambda\sum_i f(r_1,\dots,h(r_i),\dots,r_n)+[p,f(r_1,\dots,r_n)]\Big)\\ =\ & cf(r_1,\dots,r_n)^2\\ &+f(r_1,\dots,r_n)\Big(f^h(r_1,\dots,r_n)+\sum_i f(r_1,\dots,h(r_i),\dots,r_n)\Big)\\ &+\Big(f^h(r_1,\dots,r_n)+\sum_i f(r_1,\dots,h(r_i),\dots,r_n)\Big)f(r_1,\dots,r_n),\end{aligned} \tag{2.29}$$

where $f^h(r_1,\dots,r_n)$ is the polynomial obtained from $f(r_1,\dots,r_n)$ by replacing each of the coefficients α_σ by $h(\alpha_\sigma)$ and then we have $h(f(r_1,\dots,r_n))=f^h(r_1,\dots,r_n)+\sum_i f(r_1,\dots,h(r_i),\dots,r_n)$. By Kharchenko's theorem, see [21], we have that U satisfies

$$(2.30)\qquad (af(r_1,\ldots,r_n)+f(r_1,\ldots,r_n)a')\Big(bf(r_1,\ldots,r_n)+\lambda f^h(r_1,\ldots,r_n) + \lambda\sum_i f(r_1,\ldots,y_i,\ldots,r_n)+[p,f(r_1,\ldots,r_n)]\Big)$$
$$= cf(r_1,\ldots,r_n)^2 + f(r_1,\ldots,r_n)\Big(f^h(r_1,\ldots,r_n)+\sum_i f(r_1,\ldots,y_i,\ldots,r_n)\Big) + \Big(f^h(r_1,\ldots,r_n)+\sum_i f(r_1,\ldots,y_i,\ldots,r_n)\Big)f(r_1,\ldots,r_n).$$

In particular, U satisfies the blended component

$$(2.31)\qquad (af(r_1,\ldots,r_n)+f(r_1,\ldots,r_n)a')\lambda\sum_i f(r_1,\ldots,y_i,\ldots,r_n) = f(r_1,\ldots,r_n)\sum_i f(r_1,\ldots,y_i,\ldots,r_n) + \sum_i f(r_1,\ldots,y_i,\ldots,r_n)f(r_1,\ldots,r_n).$$

In particular, for $y_1=r_1$ and $y_2=\ldots=y_n=0$ we have

$$(2.32)\qquad \lambda(af(r)+f(r)a')f(r)=2f(r)^2,$$

that is,

$$(2.33)\qquad \big((\lambda a-2)f(r)+f(r)\lambda a'\big)f(r)=0$$

for all $r=(r_1,\ldots,r_n)\in U^n$. By Lemma 2.4, this gives $\lambda a'\in C$ and $\lambda a+\lambda a'-2=0$. Then (2.31) gives

$$(2.34)\qquad 2f(r_1,\ldots,r_n)\sum_i f(r_1,\ldots,y_i,\ldots,r_n) = f(r_1,\ldots,r_n)\sum_i f(r_1,\ldots,y_i,\ldots,r_n) + \sum_i f(r_1,\ldots,y_i,\ldots,r_n)f(r_1,\ldots,r_n),$$

that is

$$(2.35)\qquad \Big[\sum_i f(r_1,\ldots,y_i,\ldots,r_n),f(r_1,\ldots,r_n)\Big]=0.$$

Then by [13], Lemma 1.2, $f(x_1,\ldots,x_n)$ is central valued, a contradiction.

Case II: Assume now that h and δ are C-independent modulo inner derivations of U.

Then applying Kharchenko's theorem [21], we have from (2.27) that U satisfies

$$\begin{aligned}(2.36)\quad &(af(r_1,\dots,r_n)+f(r_1,\dots,r_n)a')\Big(bf(r_1,\dots,r_n)\\&\quad+f^{\delta}(r_1,\dots,r_n)+\sum_i f(r_1,\dots,y_i,\dots,r_n)\Big)\\&=cf(r_1,\dots,r_n)^2\\&\quad+f(r_1,\dots,r_n)\Big(f^h(r_1,\dots,r_n)+\sum_i f(r_1,\dots,t_i,\dots,r_n)\Big)\\&\quad+\Big(f^h(r_1,\dots,r_n)+\sum_i f(r_1,\dots,t_i,\dots,r_n)\Big)f(r_1,\dots,r_n).\end{aligned}$$

In particular, U satisfies the blended component

$$\begin{aligned}(2.37)\quad 0&=f(r_1,\dots,r_n)\sum_i f(r_1,\dots,t_i,\dots,r_n)\\&\quad+\sum_i f(r_1,\dots,t_i,\dots,r_n)f(r_1,\dots,r_n).\end{aligned}$$

This gives $2f(r_1,\dots,r_n)^2=0$, implying $f(r_1,\dots,r_n)=0$ as before, a contradiction. □

P r o o f of Main theorem. If $F=0$ or $G=0$, then by hypothesis $H(f(r)^2)=0$, which yields $H(f(r))f(r)+f(r)d(f(r))=0$ for all $r=(r_1,\dots,r_n)\in I^n$, where d is a derivation associated with H. Then by [3], Theorem 1, we have $f(x_1,\dots,x_n)^2$ is central valued on R and H is an inner derivation of R, which is our conclusion (4). So, we assume that $F\neq 0$ and $G\neq 0$.

In [25], Theorem 3, Lee proved that every generalized derivation g on a dense right ideal of R can be uniquely extended to a generalized derivation of U and thus can be assumed to be defined on the whole U in the form $g(x)=ax+d(x)$ for some $a\in U$ where d is a derivation of U. In light of this, we may assume that there exist $a,b,c\in U$ and derivations d,δ,h of U such that $F(x)=ax+d(x)$, $G(x)=bx+\delta(x)$ and $H(x)=cx+h(x)$. Since I, R and U satisfy the same generalized polynomial identities (see [9]) as well as the same differential identities (see [24]), without loss of generality, to prove our results, we may assume $(af(r)+d(f(r)))(bf(r)+\delta(f(r)))=cf(r)^2+h(f(r)^2)$ for all $r=(r_1,\dots,r_n)\in U^n$, where d,δ,h are three derivations on U.

If F or H is an inner generalized derivation of R, then by Lemma 2.11 and Lemma 2.12 we obtain our conclusions. Thus we assume that F and H are not inner. Hence

$$\{af(r)+d(f(r))\}\{bf(r)+\delta(f(r))\}=cf(r)^2+f(r)h(f(r))+h(f(r))f(r) \tag{2.38}$$

for all $r=(r_1,\dots,r_n)\in U^n$. Then neither d nor h can be inner derivations of U.

Now we consider the following two cases:

Case 1: Let d and δ be C-dependent modulo inner derivations of U, i.e., $\alpha d+\beta\delta = ad_{p'}$. Then $\beta\neq 0$, otherwise d is inner, a contradiction. Hence $\delta=\lambda d+ad_q$, where $\lambda=-\beta^{-1}\alpha$ and $q=\beta^{-1}p'$. Hence (2.38) becomes

$$\begin{aligned}\{af(r)+d(f(r))\}\{bf(r)+\lambda d(f(r))+[q,f(r)]\}\\ =cf(r)^2+f(r)h(f(r))+h(f(r))f(r)\end{aligned} \tag{2.39}$$

for all $r=(r_1,\dots,r_n)\in U^n$. Now we have the following two subcases:

Subcase i: Let d and h be C-dependent modulo inner derivations of U.

Then there exist $\alpha_1,\alpha_2\in C$ such that $\alpha_1 d+\alpha_2 h=ad_{q'}$. Since both d and h are outer derivations of U, $\alpha_1\neq 0$, $\alpha_2\neq 0$. Then $d=\mu h+ad_{c'}$, where $\mu=-\alpha_2\alpha_1^{-1}$ and $c'=q'\alpha_1^{-1}$. Then (2.39) gives

$$\begin{aligned}\{af(r)+\mu h(f(r))+[c',f(r)]\}\{bf(r)+\lambda\mu h(f(r))+[\lambda c'+q,f(r)]\}\\ =cf(r)^2+f(r)h(f(r))+h(f(r))f(r)\end{aligned} \tag{2.40}$$

for all $r=(r_1,\dots,r_n)\in U^n$. Since h is an outer derivation, by Kharchenko's theorem, see [21], we can replace $h(f(r_1,\dots,r_n))$ by $f^h(r_1,\dots,r_n)+\sum_i f(r_1,\dots,y_i,\dots,r_n)$ in (2.40) and then in particular for $r_1=0$, U satisfies

$$\lambda\mu^2 f(y_1,\dots,r_n)^2=0. \tag{2.41}$$

This implies that either $\lambda=0$ or $\mu=0$, since $f(r_1,\dots,r_n)\neq 0$ for all $r_1,\dots,r_n\in U$. Now $\mu=0$ gives d is inner, a contradiction. Hence $\lambda=0$ and thus (2.40) gives

$$\begin{aligned}\{af(r)+\mu h(f(r))+[c',f(r)]\}\{bf(r)+[q,f(r)]\}\\ =cf(r)^2+f(r)h(f(r))+h(f(r))f(r)\end{aligned} \tag{2.42}$$

for all $r = (r_1, \ldots, r_n) \in U^n$. Then again by Kharchenko's theorem, see [21], U satisfies the blended component

$$\left\{\mu \sum_i f(r_1, \ldots, y_i, \ldots, r_n)\right\}\{bf(r_1, \ldots, r_n) + [q, f(r_1, \ldots, r_n)]\} \tag{2.43}$$
$$= f(r_1, \ldots, r_n) \sum_i f(r_1, \ldots, y_i, \ldots, r_n)$$
$$+ \sum_i f(r_1, \ldots, y_i, \ldots, r_n) f(r_1, \ldots, r_n).$$

In particular, for $y_1 = r_1$ and $y_2 = \ldots = y_n = 0$, we have that U satisfies

$$\mu f(r_1, \ldots, r_n)\{bf(r_1, \ldots, r_n) + [q, f(r_1, \ldots, r_n)]\} = 2f(r_1, \ldots, r_n)^2, \tag{2.44}$$

that is

$$f(r_1, \ldots, r_n)(\mu(b+q)f(r_1, \ldots, r_n) - f(r_1, \ldots, r_n)(2 + \mu q)) = 0. \tag{2.45}$$

Then by Lemma 2.5, $2 + \mu q \in C$ and $\mu(b+q) - (2+\mu q) = 0$, that is, $\mu b, \mu q \in C$ and $\mu b = 2$. Then (2.43) gives

$$\left[\sum_i f(r_1, \ldots, y_i, \ldots, r_n), f(r_1, \ldots, r_n)\right] = 0. \tag{2.46}$$

Then by [13], Lemma 1.2, $f(x_1, \ldots, x_n)$ is central valued, a contradiction.

Subcase ii: Let d and h be C-independent modulo inner derivations of U.

Then applying Khrachenko's theorem, see [21], to (2.39), we can replace $d(f(r_1, \ldots, r_n))$ by $f^d(r_1, \ldots, r_n) + \sum_i f(r_1, \ldots, y_i, \ldots, r_n)$ and $h(f(r_1, \ldots, r_n))$ by $f^h(r_1, \ldots, r_n) + \sum_i f(r_1, \ldots, t_i, \ldots, r_n)$ and then U satisfies blended components

$$0 = f(r_1, \ldots, r_n) \sum_i f(r_1, \ldots, t_i, \ldots, r_n) + \sum_i f(r_1, \ldots, t_i, \ldots, r_n) f(r_1, \ldots, r_n).$$

In particular, this yields $0 = 2f(r_1, \ldots, r_n)^2$, which implies $f(r_1, \ldots, r_n) = 0$ for all $r_1, \ldots, r_n \in U$, a contradiction.

Case 2: Let d and δ be C-independent modulo inner derivations of U.

Subcase i: Let d, δ and h be C-dependent modulo inner derivations of U.

In this case there exist $\alpha_1, \alpha_2, \alpha_3 \in C$ such that $\alpha_1 d + \alpha_2 \delta + \alpha_3 h = ad_{a'}$. Then $\alpha_3 \neq 0$, otherwise d and δ would be C-dependent modulo inner derivation of U, a contradiction. Then we can write $h = \beta_1 d + \beta_2 \delta + ad_{a''}$ for some $\beta_1, \beta_2 \in C$ and $a'' \in U$. Then (2.38) becomes

$$\begin{aligned}(2.47)\quad & \{af(r_1,\ldots,r_n) + d(f(r_1,\ldots,r_n))\}\{bf(r_1,\ldots,r_n) + \delta(f(r_1,\ldots,r_n))\} \\ & = cf(r_1,\ldots,r_n)^2 + f(r_1,\ldots,r_n)\{\beta_1 d(f(r_1,\ldots,r_n)) + \beta_2\delta(f(r_1,\ldots,r_n)) \\ & \quad + [a'', f(r_1,\ldots,r_n)]\} + \{\beta_1 d(f(r_1,\ldots,r_n)) \\ & \quad + \beta_2\delta(f(r_1,\ldots,r_n)) + [a'', f(r_1,\ldots,r_n)]\}f(r_1,\ldots,r_n).\end{aligned}$$

Since d and δ are C-independent modulo inner derivations of U, by Kharchenko's theorem, see [21], U satisfies

$$\begin{aligned}(2.48)\quad & \Big\{af(r_1,\ldots,r_n) + f^d(r_1,\ldots,r_n) + \sum_i f(r_1,\ldots,y_i,\ldots,r_n)\Big\} \\ & \quad \times \Big\{bf(r_1,\ldots,r_n) + f^\delta(r_1,\ldots,r_n) + \sum_i f(r_1,\ldots,t_i,\ldots,r_n)\Big\} \\ & = cf(r_1,\ldots,r_n)^2 + f(r_1,\ldots,r_n)\Big\{\beta_1 f^d(r_1,\ldots,r_n) \\ & \quad + \beta_1 \sum_i f(r_1,\ldots,y_i,\ldots,r_n) + \beta_2 f^\delta(r_1,\ldots,r_n) \\ & \quad + \beta_2 \sum_i f(r_1,\ldots,t_i,\ldots,r_n) + [a'', f(r_1,\ldots,r_n)]\Big\} \\ & \quad + \Big\{\beta_1 f^d(r_1,\ldots,r_n) + \beta_1 \sum_i f(r_1,\ldots,y_i,\ldots,r_n) \\ & \quad + \beta_2 f^\delta(r_1,\ldots,r_n) + \beta_2 \sum_i f(r_1,\ldots,t_i,\ldots,r_n) \\ & \quad + [a'', f(r_1,\ldots,r_n)]\Big\}f(r_1,\ldots,r_n).\end{aligned}$$

In particular, for $r_1 = 0$, U satisfies

$$(2.49)\qquad f(y_1,\ldots,r_n)f(t_1,\ldots,r_n) = 0.$$

This gives $f(r_1,\ldots,r_n)^2 = 0$, implying $f(r_1,\ldots,r_n) = 0$, a contradiction.

Subcase ii: Let d, δ and h be C-independent modulo inner derivations of U.

Then from (2.38), by Kharchenko's theorem [21], U satisfies

$$\tag{2.50} \begin{aligned} &\Big\{af(r_1,\dots,r_n)+f^d(r_1,\dots,r_n)+\sum_i f(r_1,\dots,y_i,\dots,r_n)\Big\}\\ &\qquad\times\Big\{bf(r_1,\dots,r_n)+f^\delta(r_1,\dots,r_n)+\sum_i f(r_1,\dots,t_i,\dots,r_n)\Big\}\\ &= cf(r_1,\dots,r_n)^2\\ &\quad + f(r_1,\dots,r_n)\Big\{f^h(r_1,\dots,r_n)+\sum_i f(r_1,\dots,z_i,\dots,r_n)\Big\}\\ &\quad + \Big\{f^h(r_1,\dots,r_n)+\sum_i f(r_1,\dots,z_i,\dots,r_n)\Big\}f(r_1,\dots,r_n). \end{aligned}$$

In particular, U satisfies the blended component

$$\tag{2.51} f(y_1,\dots,r_n)f(t_1,\dots,r_n)=0,$$

implying $f(r_1,\dots,r_n)^2=0$ and so $f(r_1,\dots,r_n)=0$ as before, a contradiction. □

In particular, when F,G and H all are derivations, we have the following result:

Corollary 2.13. *Let R be a noncommutative prime ring of characteristic different from 2 with extended centroid C, let D_1, D_2 and D_3 be three derivations of R, I an ideal of R and $f(x_1,\dots,x_n)$ a multilinear polynomial over C which is not central valued on R. If*

$$D_1(f(r))D_2(f(r))=D_3(f(r)^2)$$

for all $r=(r_1,\dots,r_n)\in I^n$, then $D_1=D_2=0$, $f(r_1,\dots,r_n)^2$ is central valued on R and there exists $p\in U$ such that $D_3(x)=[p,x]$ for all $x\in R$.

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