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TWO-WEIGHTED ESTIMATES FOR GENERALIZED FRACTIONAL MAXIMAL OPERATORS ON NON-HOMOGENEOUS SPACES

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Abstract. Let μ be a nonnegative Borel measure on $\mathbb{R}^d$ satisfying that $\mu(Q) \leqslant l(Q)^n$ for every cube $Q \subset \mathbb{R}^n$, where $l(Q)$ is the side length of the cube Q and $0 < n \leqslant d$.

We study the class of pairs of weights related to the boundedness of radial maximal operators of fractional type associated to a Young function B in the context of non-homogeneous spaces related to the measure μ. Our results include two-weighted norm and weak type inequalities and pointwise estimates. Particularly, we give an improvement of a two-weighted result for certain fractional maximal operator proved in W. Wang, C. Tan, Z. Lou (2012).

Keywords: non-homogeneous space; generalized fractional operator; weight

MSC 2010: 42B25

1. Introduction

Let μ be a nonnegative upper Ahlfors n-dimensional measure on $\mathbb{R}^d$, that is, a Borel measure satisfying

$$\mu(Q) \leqslant l(Q)^n \tag{1.1}$$

for any cube $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes, where $l(Q)$ stands for the side length of Q and n is a fixed real number such that $0 < n \leqslant d$.

In the last decades, this measure have proved to be adequate for the development of many results in harmonic analysis which were known that hold in the context of doubling measures, that is, Borel measures ν for which there exists a positive constant D such that $\nu(2Q) \leqslant D\nu(Q)$ for every cube $Q \subset \mathbb{R}^d$. For example, many interesting results related to different operators and spaces of functions with non-doubling measures can be found in [16], [17], [14], [26], [8] and [15] among a vast bibliography on this topic.

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Particularly, in [8] the authors considered the radial maximal operator of fractional type associated to an upper Ahlfors n-dimensional measure μ which is defined for $0 \leqslant \alpha < n$ by

$$\mathcal{M}_\alpha f(x) = \sup_{Q \ni x} \frac{1}{l(Q)^{n-\alpha}} \int_Q |f(y)| \, \mathrm{d}\mu(y). \tag{1.2}$$

In the same article they study weighted boundedness properties for $\mathcal{M}_\alpha$ on non-homogeneous spaces. Concretely they characterize the pairs of weights for which these maximal operators satisfy weighted strong and weak type inequalities, obtaining Sawyer type conditions that involve the operators themselves, and Muckenhoupt type conditions, respectively. Furthermore, by strengthen Muckenhoupt type conditions by adding a "power-bump" to the right-hand side weight or even, by introducing certain Orlicz norm, strong type inequalities can be achieved.

A typical example of an upper Ahlfors 1-dimensional measure in $\mathbb{R}$ that satisfies condition (1.1) is given by $\mathrm{d}\mu(x) = \gamma(x) \, \mathrm{d}x$, where $\gamma(x) = \mathrm{e}^{-x^2}$. A similar version can be defined in $\mathbb{R}^n$. These examples show that the upper Ahlfors n-dimensional measures are not necessarily doubling.

Let $1 < p < \infty$. If g is a positive continuous function which is integrable with respect to the Lebesgue measure, then $\gamma(x) = (g(x)/Mg(x))^{(p-1)/p}$ defines an upper Ahlfors 1-dimensional measure, where M denotes the classical Hardy-Littlewood maximal operator (see [8]). Other examples of measures satisfying condition (1.1) can be found in [27].

In this paper we introduce a generalized version of the radial maximal operator of fractional type defined in (1.2), which is associated to a Young function B and will be denoted by $\mathcal{M}_{\alpha,B}$. We prove two-weighted norm inequalities for this operator in non-homogeneous spaces involving power bumps or Orlicz norms in the conditions on the weights. We also give weak type inequalities as well as a pointwise estimate between $\mathcal{M}_{\alpha,B}$ and the maximal operator $\mathcal{M}_\psi = \mathcal{M}_{0,\psi}$ for certain Young function ψ (for the definitions involved see below).

This type of maximal operators is not only a generalization but also they have proved to be the adequate operators related to commutators of singular and fractional integral operators in different settings (see for example [1], [2], [3], [6], [12], [13], [19], [20], [22] and [23]). Moreover, for certain Young functions, they are equivalent to the composition of some known operators such as the Hardy-Littlewood maximal operator or the fractional maximal operator (see [2], [4] and [20]). For example, when μ is the Lebesgue measure, if $k \in \mathbb{N}$ and $0 < \alpha < n$, it is known that

$$M_\alpha(M^k) \approx M_{\alpha, L(\log L)^k},$$

where M_α is the fractional maximal operator defined by

$$M_\alpha f(x) = \sup_{Q \ni x} |Q|^{\alpha/n-1} \int_Q |f(x)| \, \mathrm{d}x,$$

and M^k is the composition of the Hardy-Littlewood maximal operator $M = M_0$, k times (for more information see [4]).

We also give an example of an $A_1(\mu)$-weight that involves the radial fractional maximal operator defined in (1.2).

2. Statements and the proof of the main results

In order to state the main results we first introduce some preliminaries. A function $B\colon [0,\infty) \to [0,\infty)$ is a Young function if it is convex and increasing, if $B(0) = 0$, and if $B(t) \to \infty$ as $t \to \infty$. We also deal with submultiplicative Young functions, which means that $B(st) \leqslant B(s)B(t)$ for every s, $t > 0$. If B is a submultiplicative Young function, it follows that $B'(t) \simeq B(t)/t$ for every $t > 0$.

By a weight we understand a locally integrable function w which is positive almost everywhere. If w is a weight and $1 < p < \infty$, we define $L^p_w(\mathbb{R}^d)$ as the set of all measurable functions f for which

$$\int_{\mathbb{R}^d} |f(x)|^p w(x) \, \mathrm{d}\mu(x) < \infty.$$

Particularly, when $w = 1$, we simply denote $L^p_\mu(\mathbb{R}^d)$.

The radial maximal operator of fractional type associated to a Young function B is defined by

$$\mathcal{M}_{\alpha,B}(f)(x) = \sup_{Q \ni x} l(Q)^\alpha \|f\|_{B,Q}, \quad 0 \leqslant \alpha < n,$$

where

$$\|f\|_{B,Q} = \inf\Big\{\lambda > 0\colon \frac{1}{l(Q)^n} \int_Q B\Big(\frac{|f(x)|}{\lambda}\Big) \, \mathrm{d}\mu(x) \leqslant 1\Big\} \tag{2.1}$$

is the radial Luxemburg average. When $B(t) = t^r$, $1 \leqslant r < \infty$, then

$$\|f\|_{B,Q} = \Big(\frac{1}{l(Q)^n} \int_Q |f|^r \, \mathrm{d}\mu\Big)^{1/r}.$$

When $\alpha = 0$, we write $\mathcal{M}_{0,B} = \mathcal{M}_B$. If in addition $B(t) = t$, we denote $\mathcal{M}_{0,B} = \mathcal{M}$.

Before stating the next result we give the following definition. A Young function B satisfies the well known B_p condition, $1 < p < \infty$, if there is a positive constant c such that

$$\int_c^\infty \frac{B(t)}{t^p}\frac{\mathrm{d}t}{t} < \infty.$$

If $1 \leqslant r < \infty$ and $p > r$, it is not difficult to prove that both functions $B(t) = t^r$ and $D(t) = t^r \log(\mathrm{e}+t)^\delta$, $\delta > 0$, belong to B_p. Other example of a B_p function is given by $B(t) = t^p/\log(\mathrm{e}+t)^{1+\delta}$, $\delta > 0$. For more information related to this condition see for example [21], [18] or [25].

The following theorem gives sufficient conditions on a pair of weights in order to obtain weighted strong type inequalities for $\mathcal{M}_{\alpha,B}$ on non-homogeneous spaces. If $r > 0$, rQ denotes the cube with the same center as Q and with $l(rQ) = rl(Q)$.

Theorem 2.1. *Let $1 < p < q < \infty$, $0 \leqslant \alpha < n$ and let μ be an upper Ahlfors n-dimensional measure in $\mathbb{R}^d$. Let B be a submultiplicative Young function such that $B^{q_0/p_0} \in B_{q_0}$ for some $1 < p_0 \leqslant n/\alpha$ and $1/q_0 = 1/p_0 - \alpha/n$, and let φ be a Young function such that $C_1\varphi^{-1}(t)t^{\alpha/n} \leqslant B^{-1}(t) \leqslant C_2\varphi^{-1}(t)t^{\alpha/n}$ for some positive constants C_1 and C_2. If A and C are two Young functions such that $A^{-1}C^{-1} \preceq B^{-1}$ with $C \in B_p$, and (u, v) is a pair of weights such that for every cube Q,*

$$l(Q)^{\alpha-n/p}u(3Q)^{1/q}\|v^{-1/p}\|_{A,Q} \leqslant K, \tag{2.2}$$

then for all $f \in L^p_v(\mathbb{R}^d)$,

$$\|\mathcal{M}_{\alpha,B}(f)\|_{L^q_u(\mathbb{R}^d)} \leqslant C\|f\|_{L^p_v(\mathbb{R}^d)}.$$

It is important to note that Theorem 5.1 in [8] is a special case of the previous theorem considering $A(t) = t^{rp'}$, $C(t) = t^{(rp')'}$ and $B(t) = t$. In the classical setting of the Lebesgue measure, the theorem above was proved in [18] for $B(t) = t$.

Example 2.2. When $B(t) = t\log(\mathrm{e}+t)^k$, $k \geqslant 0$, it can be easily seen that B is submultiplicative, $B^{q_0/p_0} \in B_{q_0}$ for every $p_0, q_0 > 1$ and

$$B^{-1}(t) \approx t^{\alpha/n}\frac{t^{1-\alpha/n}}{\log(\mathrm{e}+t)^k} \approx t^{\alpha/n}\varphi^{-1}(t),$$

when $\varphi(t) = (t\log(\mathrm{e}+t)^k)^{n/(n-\alpha)}$. Moreover, the functions $A(t) = t^{rp'}$ and $C(t) = (t\log(\mathrm{e}+t)^k)^{(rp')'}$ satisfy

$$A^{-1}C^{-1} \preceq B^{-1}.$$

For $\delta > 0$, other examples are given by $A(t) = t^{p'}\log(\mathrm{e}+t)^{(k+1)p'-1+\delta}$ and $C(t) = t^p\log(\mathrm{e}+t)^{-(1+\delta(p-1))}$ (see [7]).

Example 2.3. When μ is the Lebesgue measure and $u = v = 1$, it is easy to note that condition (2.2) holds if and only if $1/q = 1/p - \alpha/n$ for any Young function A as in the hypotheses of Theorem 2.1. On the other hand, if we consider an upper Ahlfors n-dimensional measure μ, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$ and $u = v = 1$, then it is not difficult to check that condition (2.2) is satisfied. Thus, from this theorem we obtain that $\mathcal{M}_{\alpha,B}\colon L^p_\mu(\mathbb{R}^d) \hookrightarrow L^p_\mu(\mathbb{R}^d)$, that is, the unweighted boundedness holds for any upper Ahlfors n-dimensional measure μ.

Example 2.4. Let μ be an upper Ahlfors n-dimensional measure and let u be a weight. If A is a Young function satisfying the hypotheses of Theorem 2.1, then the pair of weights $(u, (\mathcal{M}u)^{p/q})$ satisfies condition (2.2) when $1/q = 1/p - \alpha/n$. In fact,

$$l(Q)^{\alpha-n/p} u(3Q)^{1/q} \|\mathcal{M}u^{(p/q)(-1/p)}\|_{A,Q} \leqslant C l(Q)^{\alpha-n/p} u(3Q)^{1/q} \frac{(3l(Q))^{n/q}}{u(3Q)^{1/q}} \leqslant C.$$

Therefore we obtain

$$\|\mathcal{M}_{\alpha,B}(f)\|_{L^q_u(\mathbb{R}^d)} \leqslant C\|f\|_{L^p_{(\mathcal{M}u)^{p/q}}(\mathbb{R}^d)}.$$

The same estimate holds if we replace $\mathcal{M}$ by M, where

$$Mu(x) = \sup_{Q \ni x} \frac{1}{\mu(Q)} \int_Q |u(y)| \, \mathrm{d}\mu(y). \tag{2.3}$$

Remark 2.5. In [28] the authors studied two-weighted norm inequalities for a fractional maximal operator associated to a measure μ satisfying condition (1.1). Concretely, they considered the following version of the fractional maximal operator defined for $0 \leqslant \alpha < 1$ by

$$M_\alpha f(x) = \sup_{Q \ni x} \frac{1}{\mu(5Q)^{1-\alpha}} \int_Q |f(y)| \, \mathrm{d}\mu(y),$$

and proved the following result.

Theorem 2.6 ([28])**.** *Let μ be an upper Ahlfors n-dimensional measure. Let $1 < p < q < \infty$ and $0 \leqslant \alpha < 1$. Let (u, v) be a pair of weights such that for every cube Q,*

$$l(Q)^{n(1-1/p)} \mu(Q)^{\alpha-1} u(3Q)^{1/q} \|v^{-1/p}\|_{\Phi,Q} \leqslant C, \tag{2.4}$$

where Φ is a Young function whose complementary function $\bar{\Phi} \in B_p$. Then

$$\|\mathcal{M}_\alpha(f)\|_{L^q_u(\mathbb{R}^d)} \leqslant C\|f\|_{L^p_v(\mathbb{R}^d)}$$

for every $f \in L^p_v(\mathbb{R}^d)$ bounded with compact support.

Let us make some comments about Theorem 2.6. When μ is the Lebesgue measure and $u = v = 1$, it is easy to note that condition (2.4) holds if and only if $1/q = 1/p - \alpha$ for any Φ as in the hypothesis. On the other hand, if we consider an upper Ahlfors n-dimensional measure μ and if we take $\Phi(t) = t^{rp'}$ for $1 < r < \infty$, $1/q = 1/p - \alpha$ and $u = v = 1$ in condition (2.4), we have that if the inequality

$$l(Q)^{n(1-1/p)} \mu(Q)^{\alpha-1} \mu(3Q)^{1/p-\alpha} \Big(\frac{\mu(Q)}{l(Q)^n}\Big)^{1/(rp')} \leqslant C$$

holds, then

$$\Big(\frac{l(Q)^n}{\mu(Q)}\Big)^{1/(p'r')} \leqslant C,$$

which implies that the measure μ satisfying the growth condition (1.1) also satisfies the "lower" case, that is $\mu(Q) \geqslant C l(Q)^n$ with a constant independent of Q. So, the weights $u = v = 1$ are not allowed in this case unless the measure is Ahlfors (and so, doubling), that is $\mu(Q) \simeq l(Q)^n$ for every cube Q. By taking into account Example 2.3, which shows that our result provides the boundedness with any upper Ahlfors n-dimensional measure μ, our theorem is an improvement of that given in [28] in this case.

Moreover, let M be the maximal operator defined in (2.3). When μ is the Lebesgue measure and $\Phi(t) = t^{rp'}$, a typical example of pair of weights satisfying condition (2.4) is $(u, (Mu)^{p/q})$ with $1/q = 1/p - \alpha$. On the other hand, suppose that this pair satisfies the same condition for a measure satisfying (1.1) and let $u \in A_1(\mu)$. Thus, the following chain of inequalities holds:

$$\begin{aligned}
C &\geqslant \frac{l(Q)^{n/p'}}{\mu(Q)^{1-\alpha-1/q}} \Big(\frac{1}{\mu(Q)} \int_Q u \, \mathrm{d}\mu\Big)^{1/q} \Big(\frac{1}{l(Q)^n} \int_Q ((Mu)^{p/q})^{-rp'/p}\Big)^{1/(rp')} \\
&\geqslant \frac{l(Q)^{n/p'-n/(rp')}}{\mu(Q)^{1/p'-1/(rp')}} \Big(\frac{1}{\mu(Q)} \int_Q u^{p/q} \, \mathrm{d}\mu\Big)^{1/p} \Big(\frac{1}{\mu(Q)} \int_Q (u^{p/q})^{-rp'/p}\Big)^{1/(rp')} \\
&\geqslant \frac{l(Q)^{n/(r'p')}}{\mu(Q)^{1/(r'p')}} \Big(\frac{1}{\mu(Q)} \int_Q u^{p/q} \, \mathrm{d}\mu\Big)^{1/p} \Big(\frac{1}{\mu(Q)} \int_Q (u^{p/q})^{-p'/p}\Big)^{1/p'} \\
&\geqslant \frac{l(Q)^{n/(r'p')}}{\mu(Q)^{1/(r'p')}}.
\end{aligned}$$

This implies again that μ must be an Ahlfors measure. Again, in this case, our result is an improvement of that given in [28] (see Example 2.4).

Let us make some comments about the upper Ahlfors n-dimensional measure μ satisfying (1.1). It is well known that for such measures the Lebesgue differentiation theorem holds; that is for every $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and a.e. x,

$$\frac{1}{\mu(Q)} \int_Q f(y) \, \mathrm{d}\mu(y) \to f(x)$$

when Q decreases to x (see [27]). However, if we take radial averages like those defined in (2.1), this is no longer true. In fact, let us consider μ defined by $\mathrm{d}\mu(t)=\mathrm{e}^{-t^2}\,\mathrm{d}t$, which is an upper Ahlfors 1-dimensional measure, and $f(t)=\mathrm{e}^{\theta t^2}$, $\theta\in\mathbb{R}$. Let $x\in\mathbb{R}$. Then

$$\lim_{r\to 0}\frac{1}{2r}\int_{x-r}^{x+r} f(t)\,\mathrm{d}\mu(t)=\lim_{r\to 0}\frac{1}{2r}\int_{x-r}^{x+r}\mathrm{e}^{(\theta-1)t^2}\,\mathrm{d}t=\mathrm{e}^{(\theta-1)x^2},$$

which differs from f in a.e. x.

Given a Young function B, let h_B be the function defined by

$$h_B(s)=\sup_{t>0}\frac{B(st)}{B(t)},\quad 0\leqslant s<\infty.$$

If B is submultiplicative, then $h_B\approx B$. More generally, given any B for every $s,t\geqslant 0$, $B(st)\leqslant h_B(s)B(t)$, it is easy to prove (see [5], Lemma 3.11) that if B is a Young function, then h_B is nonnegative, submultiplicative, increasing in $[0,\infty)$, strictly increasing in $[0,1]$ and $h_B(1)=1$.

The following theorem gives an modular endpoint estimate for $\mathcal{M}_{\alpha,B}$ on non-homogeneous spaces. This result was proved in [6] for μ being the Lebesgue measure.

Theorem 2.7. *Let $0\leqslant\alpha<n$ and let μ be an upper Ahlfors n-dimensional measure on $\mathbb{R}^d$. Let B be a Young function and suppose that if $\alpha>0$, $B(t)/t^{n/\alpha}$ is decreasing for all $t>0$. Then there exists a constant C depending only on B such that for all $t>0$, $\mathcal{M}_{\alpha,B}$ satisfies the modular weak-type inequality*

$$\varphi[\mu(\{x\in\mathbb{R}^d\colon\ \mathcal{M}_{\alpha,B}(f)(x)>t\})]\leqslant C\int_{\mathbb{R}^n}B\Big(\frac{|f(y)|}{t}\Big)\,\mathrm{d}\mu(y)$$

for all $f\in L^B_\mu(\mathbb{R}^d)$, where φ is any function such that

$$\varphi(s)\leqslant C_1\varphi_1(s)=\begin{cases}0 & \text{if } s=0,\\ \dfrac{s}{h_B(s^{\alpha/n})} & \text{if } s>0.\end{cases}$$

Remark 2.8. It is easy to see that the function $B(t)=t\log(\mathrm{e}+t)$ satisfies the hypothesis of the theorem above and thus

$$\mu(\{x\in\mathbb{R}^n\colon\ \mathcal{M}_{\alpha,B}(f)(x)>t\})\leqslant C\psi\Big[\int_{\mathbb{R}^n}B\Big(\frac{|f(y)|}{t}\Big)\,\mathrm{d}\mu(y)\Big]$$

for all $f\in L^B_\mu(\mathbb{R}^d)$, where $\psi=[t\log(\mathrm{e}+t^{\alpha/n})]^{n/(n-\alpha)}$. This last result was proved in [9] for μ being a doubling measure.

Remark 2.9. If $B(t) = t$, then $h_B(s) = s$ and $\varphi(s) = s^{1-\alpha/n}$. In this case, the theorem above provides the weak type $(1, n/(n-\alpha))$ proved in [8].

The proof of Theorem 2.7 requires some lemmas. The first of them was proved in [6] and the second in [11]. So, we only give the proof of the third.

Lemma 2.10. *Given $0 \leqslant \alpha < n$, let B be a Young function such that for $\alpha > 0$, $B(t)/t^{n/\alpha}$ is decreasing for all $t > 0$. Then the function φ_1 from Theorem 2.7 is increasing and $\varphi_1(s)/s$ is decreasing. Moreover, there exists φ such that $\varphi(s) \leqslant C_1\varphi_1(s)$ and φ is invertible.*

Lemma 2.11. *If $\varphi(t)/t$ is decreasing, then for any positive sequence $\{x_j\}$,*

$$\varphi\Big(\sum_j x_j\Big) \leqslant \sum_j \varphi(x_j).$$

The third lemma is a generalization of Lemma 3.2 in [8] for the radial Luxemburg type averages defined in (2.1). It was proved in [6] for μ being the Lebesgue measure.

Lemma 2.12. *Suppose that $0 \leqslant \alpha < n$, B is a Young function and f is a nonnegative bounded function with compact support. If for $t > 0$ and any cube Q*

$$l(Q)^\alpha \|f\|_{B,Q} > t,$$

then there exists a dyadic cube P such that $Q \subset 3P$ and satisfying

$$l(P)^\alpha \|f\|_{B,P} > \beta t,$$

where β is a nonnegative constant.

Proof. Let Q be a cube with

$$l(Q)^\alpha \|f\|_{B,Q} > t. \tag{2.5}$$

Let k be the unique integer such that $2^{-(k+1)} < l(Q) \leqslant 2^{-k}$. There are some dyadic cubes with side length 2^{-k}, and at most 2^d, let us denote them by $\{J_i\colon\ i = 1, \dots, N\}$, $N \leqslant 2^d$, meeting the interior of Q. It is easy to see that for one of these cubes, say J_1,

$$\frac{t}{2^d} < l(Q)^\alpha \|\chi_{J_1} f\|_{B,Q}.$$

This can be seen as follows. If for each $i = 1, 2, \ldots, N$ we have

$$l(Q)^{\alpha} \|\chi_{J_i} f\|_{B,Q} \leqslant \frac{t}{2^d},$$

since $Q \subset \bigcup_{i=1}^{N} J_i$, we obtain that

$$\begin{aligned} l(Q)^{\alpha} \|f\|_{B,Q} &= l(Q)^{\alpha} \|\chi_{\bigcup_{i=1}^{N} J_i} f\|_{B,Q} \\ &\leqslant l(Q)^{\alpha} \sum_{i=1}^{N} \|\chi_{J_i} f\|_{B,Q} \leqslant N \frac{t}{2^d} \leqslant t, \end{aligned}$$

contradicting (2.5). Using that $l(Q) \leqslant l(J_1) < 2l(Q)$ we can also show that

$$\frac{t}{2^d} < l(Q)^{\alpha} \|\chi_{J_1} f\|_{B,Q} \leqslant 2^n l(J_1)^{\alpha} \|f\|_{B,J_1}$$

and $Q \subset 3J_1$. □

Proof of Theorem 2.7. Fix a nonnegative function $f \in L^B_{\mu}(\mathbb{R}^d)$, fix $t > 0$ and define

$$E_t = \{x \in \mathbb{R}^d : \mathcal{M}_{\alpha,B} f(x) > t\}.$$

If t is such that the set E_t is empty, we have nothing to prove. Otherwise, for each $x \in E_t$ there exists a cube Q_x containing x such that

$$l(Q_x)^{\alpha} \|f\|_{B,Q_x} > t.$$

By Lemma 2.12, there exists a constant β and a dyadic cube P_x with $Q_x \subset 3P_x$ such that

$$l(P_x)^{\alpha} \|f\|_{B,P_x} > \beta t. \tag{2.6}$$

Since $f \in L^B_{\mu}(\mathbb{R}^d)$, it is not hard to prove that we can replace the collection $\{P_x\}$ with a maximal disjoint subcollection $\{P_j\}$. Each P_j satisfies (2.6) and, by our choice of the Q_x's, $E_t \subset \bigcup_j 3P_j$. By Lemmas 2.10 and 2.11,

$$\varphi_1(\mu(E_t)) \leqslant \sum_j \varphi_1(\mu(3P_j)).$$

On the other hand, inequality (2.6) implies that for each j,

$$\frac{1}{l(P_j)^n} \int_{P_j} B\Big(\frac{l(P_j)^{\alpha}|f|}{\beta t}\Big) \, \mathrm{d}\mu > 1,$$

and then by the definition and properties of h_B,

$$\begin{aligned}1 &< \frac{1}{l(P_j)^n}\int_{P_j} B\Big(\frac{3^\alpha l(P_j)^\alpha|f(x)|}{3^\alpha\beta t}\Big)\,\mathrm{d}\mu(x)\\&\leqslant \frac{3^n h_B(3^{-\alpha}\beta^{-1})h_B(l(3P_j)^\alpha)}{l(3P_j)^n}\int_{P_j} B\Big(\frac{|f(x)|}{t}\Big)\,\mathrm{d}\mu(x)\\&\leqslant \frac{C}{\varphi_1(l(3P_j)^n)}\int_{P_j} B\Big(\frac{|f(x)|}{t}\Big)\,\mathrm{d}\mu(x).\end{aligned}$$

Hence, since the P_j's are disjoint,

$$\begin{aligned}\sum_j \varphi_1(\mu(3P_j)) &\leqslant \sum_j \varphi_1(l(3P_j)^n)\\&\leqslant C\sum_j\int_{P_j} B\Big(\frac{|f(x)|}{t}\Big)\,\mathrm{d}\mu(x)\\&\leqslant C\int_{\mathbb{R}^d} B\Big(\frac{|f(x)|}{t}\Big)\,\mathrm{d}\mu(x).\end{aligned}$$

□

The radial Luxemburg average (2.1) has two rescaling properties which we will use repeatedly. Given any Young function A and $r > 0$,

$$\|f^r\|_{A,Q} = \|f\|_{B,Q}^r,$$

where $B(t) = A(t^r)$.

Given a Young function B, the complementary Young function $\widetilde{B}$ is defined by

$$\widetilde{B}(t) = \sup_{s>0}\{st - B(s)\}, \quad t > 0.$$

It is well known that B and $\widetilde{B}$ satisfy the following inequality:

$$t \leqslant B^{-1}(t)\widetilde{B}^{-1} \leqslant 2t, \quad t > 0.$$

It is also easy to check that the following version of the Hölder's inequality holds:

$$\frac{1}{l(Q)^n}\int_Q |f(x)g(x)|\,\mathrm{d}\mu(x) \leqslant 2\|f\|_{B,Q}\|g\|_{\widetilde{B},Q}.$$

Moreover, there is a further generalization of the inequality above. If A, B and C are Young functions such that for every $t \geqslant t_0 > 0$,

$$B^{-1}(t)C^{-1}(t) \leqslant cA^{-1}(t),$$

then the inequality

$$\|fg\|_{A,Q} \leqslant K\|f\|_{B,Q}\|g\|_{C,Q} \tag{2.7}$$

holds.

The following result is a pointwise estimate between the radial maximal operator fractional type associated with a Young function B and the radial maximal operator associated with a Young ψ related to B on non-homogeneous spaces.

Theorem 2.13. *Let $0 \leqslant \alpha < n$ and $1 < p < n/\alpha$. Let μ be an upper Ahlfors n-dimensional measure. Let q and s be defined by $1/q = 1/p - \alpha/n$ and $s = 1 + q/p'$, respectively. Let B and φ be Young functions such that $\varphi^{-1}(t)t^{\alpha/n} \geqslant CB^{-1}(t)$ and $\psi(t) = \varphi(t^{1-\alpha/n})$. Then for every measurable function f, the inequality*

$$\mathcal{M}_{\alpha,B}(f)(x) \leqslant C\mathcal{M}_{\psi}(|f|^{p/s})(x)^{1-\alpha/n}\biggl(\int_{\mathbb{R}^d} |f(y)|^p \,\mathrm{d}\mu(y)\biggr)^{\alpha/n}$$

holds for a.e. $x \in \mathbb{R}^d$.

When μ is the Lebesgue measure, the result above was proved in [1] (see also [24] for similar multilinear versions and [10] for the case $B(t) = t$, both in the euclidean context).

Proof. Let $g(x) = |f(x)|^{p/s}$. Then

$$|f(x)| = g(x)^{s/p+\alpha/n-1}g(x)^{1-\alpha/n}.$$

Let $x \in \mathbb{R}^d$ and Q be a fixed cube containing x. By the generalized Hölder's inequality (2.7) and the fact that

$$g(x)^{(s/p+\alpha/n-1)n/\alpha} = |f|^p$$

we get

$$\begin{aligned} l(Q)^{\alpha}\|f\|_{B,Q} &\leqslant Cl(Q)^{\alpha}\|g^{1-\alpha/n}\|_{\varphi,Q}\|g^{s/p+\alpha/n-1}\|_{n/\alpha,Q} \\ &= Cl(Q)^{\alpha}\|g\|_{\psi,Q}^{1-\alpha/n}\biggl(\frac{1}{l(Q)^n}\int_Q |f(y)|^p \,\mathrm{d}\mu(y)\biggr)^{\alpha/n} \\ &\leqslant C[\mathcal{M}_{\psi}(g)(x)]^{1-\alpha/n}\|f\|_{L^p(\mu)}^{p\alpha/n}. \end{aligned}$$

□

The next theorem gives sufficient conditions on the function B in order to obtain the boundedness of $\mathcal{M}_B$ on $L^p_\mu(\mathbb{R}^d)$. In the euclidean context, this result was proved in [21] and in [25] for the measure μ being doubling, that is $\mu(2Q) \leqslant D\mu(Q)$ for every $Q \in \mathbb{R}^d$.

Theorem 2.14. *Let μ be an upper Ahlfors n-dimensional measure. Let B be a Young function such that $B \in B_p$. Then*

$$\mathcal{M}_B\colon\ L^p_\mu(\mathbb{R}^d) \to L^p_\mu(\mathbb{R}^d).$$

P r o o f. From Theorem 2.7 applied to the case $\alpha = 0$ it is easy to check that

$$\mu(\{y \in \mathbb{R}^d\colon\ \mathcal{M}_B f(y) > 2t\}) \leqslant C \int_{\{|f|>t\}} B(|f(x)|/t)\,\mathrm{d}\mu(x).$$

Thus, by changing variables and using the inequality above we obtain that

$$\begin{aligned}\int_{\mathbb{R}^d} \mathcal{M}_B f(y)^p\,\mathrm{d}\mu(y) &= C\int_0^\infty t^p \mu(\{y \in \mathbb{R}^d\colon\ \mathcal{M}_B f(y) > 2t\})\frac{\mathrm{d}t}{t}\\ &\leqslant C\int_{\mathbb{R}^d}\int_0^{|f(y)|} t^p B\Big(\frac{|f(y)|}{t}\Big)\frac{\mathrm{d}t}{t}\,\mathrm{d}\mu(y)\\ &= C\Big(\int_{\mathbb{R}^d}|f(y)|^p\,\mathrm{d}\mu(y)\Big)\Big(\int_1^\infty \frac{B(s)}{s^p}\frac{\mathrm{d}s}{s}\Big).\end{aligned}$$

Thus, condition B_p allows us to obtain the desired result. $\square$

Proposition 2.15. *Let B be a submultiplicative Young function and let φ be a Young function such that $C_1\varphi^{-1}(t)t^{\alpha/n} \leqslant B^{-1}(t) \leqslant C_2\varphi^{-1}(t)t^{\alpha/n}$ for some positive constants C_1 and C_2. Let $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$ and $s = q(1-\alpha/n)$. If $B^{q/p} \in B_q$, then the function ψ defined by $\psi(t) = \varphi(t^{1-\alpha/n})$ belongs to B_s.*

Remark 2.16. It is easy to see that if $\delta > 0$, the function $B(t) = t\log(\mathrm{e}+t)^\delta$, $t > 0$ satisfies the hypothesis of Proposition 2.15. In fact,

$$B^{-1}(t) \approx \frac{t}{\log(\mathrm{e}+t)^\delta}.$$

P r o o f. From the definition of ψ and by changing variables we obtain that

$$\int_1^\infty \frac{\psi(t)}{t^s}\frac{\mathrm{d}t}{t} = \int_1^\infty \frac{\varphi(t^{1-\alpha/n})}{t^s}\frac{\mathrm{d}t}{t} = \frac{n}{n-\alpha}\int_1^\infty \frac{\varphi(r)}{r^{ns/(n-\alpha)}}\frac{\mathrm{d}r}{r}.$$

From the relation between B and φ it is easy to see that φ is a submultiplicative function. In fact, it is not hard to prove that a Young function B is submultiplicative if and only if its inverse function B^{-1} satisfies $B^{-1}(st) \geqslant B^{-1}(s)B^{-1}(t)$ for every $s, t > 0$.

Thus, noting that $q = ns/(n-\alpha)$, we obtain

$$\begin{aligned}\int_1^\infty \frac{\varphi(r)}{r^{ns/(n-\alpha)}}\frac{\mathrm{d}r}{r} &= \int_1^\infty \frac{\varphi(r)}{r^q}\frac{\mathrm{d}r}{r} \\ &\leqslant c\int_c^\infty \frac{u^{1+q\alpha/n}}{(\varphi^{-1}(u)u^{\alpha/n})^q}\frac{\mathrm{d}u}{u} \\ &\leqslant C\int_c^\infty \frac{u^{q/p}}{B^{-1}(u)^q}\frac{\mathrm{d}u}{u} = C\int_c^\infty \frac{B(t)^{q/p}}{t^q}\frac{\mathrm{d}t}{t} < \infty.\end{aligned}$$

□

The following result is a fractional version of Theorem 2.14 and gives a sufficient condition on the function B that guarantees the continuity of the radial maximal operator fractional type $\mathcal{M}_{\alpha,B}$ between Lebesgue spaces with not necessarily doubling measures.

Theorem 2.17. *Let μ be an upper Ahlfors n-dimensional measure. Let $0 < \alpha < n$ and $1 < p \leqslant n/\alpha$. Let q and s be defined by $1/q = 1/p - \alpha/n$ and $s = 1 + q/p'$, respectively. Let B be a submultiplicative Young function such that $B^{q/p} \in B_q$ and let φ be a Young function such that $C_1\varphi^{-1}(t)t^{\alpha/n} \leqslant B^{-1}(t) \leqslant C_2\varphi^{-1}(t)t^{\alpha/n}$ for some positive constants C_1 and C_2. Then*

$$\mathcal{M}_{\alpha,B}\colon\ L^p_\mu(\mathbb{R}^d) \to L^p_\mu(\mathbb{R}^d).$$

P r o o f. By Theorem 2.13, if $1 < p < n/\alpha$, we have

$$\begin{aligned}\Big(\int_{\mathbb{R}^d}(\mathcal{M}_{\alpha,B}(f))^q\,\mathrm{d}\mu\Big)^{1/q} &\leqslant C\Big(\int_{\mathbb{R}^d}(\mathcal{M}_\psi(|f|^{p/s})^{1-\alpha/n}\|f\|^{p\alpha/n}_{L^p(\mu)})^q\,\mathrm{d}\mu\Big)^{1/q} \\ &= C\|f\|^{p\alpha/n}_{L^p_\mu(\mathbb{R}^d)}\Big(\int_{\mathbb{R}^d}\mathcal{M}_\psi(|f|^{p/s})^s\,\mathrm{d}\mu\Big)^{1/q}.\end{aligned}$$

From Proposition 2.15 we have that the function $\psi \in B_s$. Thus, Theorem 2.14 implies that $\mathcal{M}_\psi\colon\ L^s_\mu(\mathbb{R}^d) \to L^s_\mu(\mathbb{R}^d)$, and thus,

$$\Big(\int_{\mathbb{R}^d}(\mathcal{M}_{\alpha,B}(f))^q\,\mathrm{d}\mu\Big)^{1/q} \leqslant C\|f\|^{p\alpha/n}_{L^p_\mu(\mathbb{R}^d)}\Big(\int_{\mathbb{R}^d}(|f|^{p/s})^s\,\mathrm{d}\mu\Big)^{1/q} = C\|f\|_{L^p_\mu(\mathbb{R}^d)}.$$

On the other hand, if $p=n/\alpha$ and Q is a cube such that $x\in Q$, we obtain that

$$l(Q)^{\alpha}\|f\|_{\eta,Q}\leqslant Cl(Q)^{\alpha}\|\chi_Q\|_{\varphi,Q}\|f\|_{n/\alpha,Q}\leqslant C\|f\|_{n/\alpha},$$

and thus

$$\mathcal{M}_{\alpha,B}(f)(x)\leqslant C\|f\|_{n/\alpha}$$

for a.e. x, which leads us to the desired result. □

The next theorem allows to find examples of A_1-weights on non-homogeneous spaces.

Theorem 2.18. *Given $0<\alpha<n$, and a nonnegative function f, there exists a constant C such that*

$$\mathcal{M}(\mathcal{M}_{\alpha}f)(x)\leqslant C\mathcal{M}_{\alpha}f(x).$$

P r o o f. Fix a cube Q. We will see that

$$\frac{1}{l(Q)^n}\int_Q\mathcal{M}_{\alpha}f(y)\,\mathrm{d}\mu(y)\leqslant C\mathcal{M}_{\alpha}f(x)\quad\text{for a.e. } x\in Q$$

with C independent of Q. Let $\widetilde{Q}=3Q$. We write $f=f_1+f_2$ with $f_1=f\chi_{\widetilde{Q}}$. Then

$$\mathcal{M}_{\alpha}f(x)\leqslant\mathcal{M}_{\alpha}f_1(x)+\mathcal{M}_{\alpha}f_2(x)$$

and

$$\begin{aligned}\frac{1}{l(Q)^n}\int_Q\mathcal{M}_{\alpha}f_1(y)\,\mathrm{d}\mu(y)&=\frac{1}{l(Q)^n}\int_0^{\infty}\mu\{x\in Q\colon\ \mathcal{M}_{\alpha}f_1(x)>t\}\,\mathrm{d}t\\&\leqslant\frac{1}{l(Q)^n}\Big(\mu(Q)R+\int_R^{\infty}\mu\{x\in Q\colon\ \mathcal{M}_{\alpha}f_1(x)>t\}\,\mathrm{d}t\Big).\end{aligned}$$

By [8], Proposition 2.1, we know that $\|\mathcal{M}_{\alpha}f\|_{L_{\mu}^{n/(n-\alpha),\infty}(\mathbb{R}^d)}\leqslant\|f\|_{L_{\mu}^1(\mathbb{R}^d)}$. Then since $\mu(Q)\leqslant l(Q)^n$,

$$\frac{1}{l(Q)^n}\int_Q\mathcal{M}_{\alpha}f_1(y)\,\mathrm{d}\mu(y)\leqslant R+\frac{c}{l(Q)^n}\|f_1\|_{L_{\mu}^1(\mathbb{R}^d)}^{n/(n-\alpha)}\int_R^{\infty}\frac{\mathrm{d}t}{t^{n/(n-\alpha)}}.$$

By taking $R=\|f_1\|_{L_{\mu}^1(\mathbb{R}^d)}/l(Q)^{n-\alpha}$, we get

$$\begin{aligned}\frac{1}{l(Q)^n}\int_Q\mathcal{M}_{\alpha}f_1(y)\,\mathrm{d}\mu(y)\leqslant C_{\alpha,n}\frac{\|f_1\|_{L_{\mu}^1(\mathbb{R}^d)}}{l(Q)^{n-\alpha}}&=\frac{C_{\alpha,n}}{l(\widetilde{Q})^{n-\alpha}}\int_{\widetilde{Q}}f(y)\,\mathrm{d}\mu(y)\\&\leqslant C_{\alpha,n}\mathcal{M}_{\alpha}f(x)\end{aligned}$$

for every $x\in Q$.

Dealing with $\mathcal{M}_\alpha f_2$ is even simpler. It is enough to see that because of the fact that f_2 lives far from Q (outside $\widetilde{Q}$). For any two points $x, y \in Q$ we have $\mathcal{M}_\alpha f_2(x) \leqslant C\mathcal{M}_\alpha f_2(y)$ with C an absolute constant. Indeed, if Q_0 is a cube containing x and meeting $\mathbb{R}^n \setminus \widetilde{Q}$, then $Q \subset Q_0^3$, so

$$\frac{1}{l(Q_0)^{n-\alpha}} \int_{Q_0} f_2(t)\,\mathrm{d}\mu(t) \leqslant \frac{3^{n-\alpha}}{l(Q_0^3)^{n-\alpha}} \int_{Q_0^3} f_2(t)\,\mathrm{d}\mu(t) \leqslant 3^{n-\alpha}\mathcal{M}_\alpha f_2(y).$$

Thus

$$\frac{1}{l(Q)^n} \int_Q \mathcal{M}_\alpha f_2(y)\,\mathrm{d}\mu(y) \leqslant C\frac{\mu(Q)}{l(Q)^n}\mathcal{M}_\alpha f(x) \leqslant C\mathcal{M}_\alpha f(x)$$

for every $x \in Q$. □

We finally give the proof of the two weighted estimate for $\mathcal{M}_{\alpha,B}$.

Proof of Theorem 2.1. Without loss of generality we assume that f is a nonnegative bounded function with compact support. This guarantees that $\mathcal{M}_{\alpha,B}f$ is finite μ-almost everywhere. In fact, $f \in L^{p_0}_\mu(\mathbb{R}^d)$, where p_0 is the exponent of the hypotheses. From Theorem 2.17 we get that $\mathcal{M}_{\alpha,B}f \in L^{q_0}_\mu(\mathbb{R}^d)$ and thus

$$\mathcal{M}_{\alpha,B}f(x) < \infty \quad \text{a.e. } x \in \mathbb{R}^d.$$

For each $k \in \mathbb{Z}$ let $\Omega_k = \{x \in \mathbb{R}^d \colon\ 2^k < \mathcal{M}_{\alpha,B}f(x) \leqslant 2^{k+1}\}$. Thus

$$\mathbb{R}^d = \bigcup_{k\in\mathbb{Z}} \Omega_k.$$

Then for every k and every $x \in \Omega_k$, by the definition of $\mathcal{M}_{\alpha,B}f$, there is a cube Q_x^k containing x such that

$$l(Q_x^k)^\alpha \|f\|_{B,Q_x^k} > 2^k,$$

and from Lemma 2.12 there exist a constant β and a dyadic cube P_x^k with $Q_x^k \subset 3P_x^k$ such that

$$l(P_x^k)^\alpha \|f\|_{B,P_x^k} > \beta 2^k. \tag{2.8}$$

From the fact that B is submultiplicative and $\operatorname{supp}(f)$ is a compact set, the inequality above allows us to obtain

$$\frac{l(P_x^k)^n}{B(l(P_x^k)^\alpha)} < \int_{P_x^k} B\Big(\frac{|f|}{2^k\beta}\Big)\,\mathrm{d}\mu \leqslant C\mu(\operatorname{supp}(f)) \leqslant C.$$

From the hypotheses on B it is easy to check that

$$C_1\varphi^{-1}\Big(\frac{l(P_x^k)^n}{C}\Big)\Big(\frac{l(P_x^k)^n}{C}\Big)^{\alpha/n} \leqslant B^{-1}\Big(\frac{l(P_x^k)^n}{C}\Big) \leqslant l(P_x^k)^\alpha,$$

which allows us to conclude that for each k, $l(P_x^k)$ is bounded by a constant independent of x. Then there is a subcollection of maximal cubes (and so disjoint) $\{P_j^k\}_j$ such that every Q_x^k is contained in $3P_j^k$ for some j and, as a consequence, $\Omega_k \subset_j 3P_j^k$. Next, decompose Ω_k into the sets

$$E_1^k = 3P_1^k \cap \Omega_k,\ E_2^k = (3P_2^k \setminus 3P_1^k) \cap \Omega_k,\ \ldots,\ E_j^k = \left(3P_j^k \setminus \bigcup_{r=1}^{j-1} 3P_r^k\right) \cap \Omega_k,\ \ldots$$

Then

$$\mathbb{R}^d = \bigcup_{k\in\mathbb{Z}} \Omega_k = \bigcup_{j,k} E_j^k$$

and these sets are pairwise disjoint. Let K be a fixed positive integer which will go to infinity later, and let $\Lambda_K = \{(j,k) \in \mathbb{N} \times \mathbb{Z} \colon\ |k| \leqslant K\}$. Using that $E_j^k \subset \Omega_k$ and that the cubes P_j^k satisfy (2.8) we obtain that

$$\begin{aligned}
\mathcal{I}_k &= \int_{\bigcup_{-K}^{K} \Omega_k} (\mathcal{M}_{\alpha,B} f(x))^q u(x)\,\mathrm{d}\mu(x) \\
&= \sum_{(j,k)\in\Lambda_k} \int_{E_j^k} (\mathcal{M}_{\alpha,B} f(x))^q u(x)\,\mathrm{d}\mu(x) \\
&\leqslant \sum_{(j,k)\in\Lambda_k} u(E_j^k) 2^{(k+1)q} \\
&\leqslant C 2^q \sum_{(j,k)\in\Lambda_k} u(E_j^k)(l(P_j^k)^\alpha \|f\|_{B,P_j^k})^q \\
&\leqslant C 2^q \sum_{(j,k)\in\Lambda_k} u(3P_j^k)(l(P_j^k)^\alpha \|fv^{1/p}\|_{C,P_j^k} \|v^{-1/p}\|_{A,P_j^k})^q,
\end{aligned}$$

where in the last inequality we have used the generalized Hölder's inequality (2.7) and the hypothesis on the functions A, B and C. Now, applying the hypothesis on the weights we obtain that

$$\mathcal{I}_k \leqslant C \sum_{(j,k)\in\Lambda_k} l(3P_j^k)^{nq/p} \|fv^{1/p}\|_{C,P_j^k}^q = C \int_{\mathcal{Y}} T_k (fv^{1/p})^q\,\mathrm{d}\nu,$$

where $\mathcal{Y} = \mathbb{N} \times \mathbb{Z}$, ν is a measure in $\mathcal{Y}$ given by $\nu(j,k) = l(3P_j^k)^{nq/p}$ and for every measurable function h, the operator T_k is defined by the expression

$$T_k h(j,k) = \|\varphi\|_{C,P_j^k} \chi_{\Lambda_k}(j,k).$$

Then, if we prove that $T_k\colon L^p(\mathbb{R}^d,\mu)\to L^q(\mathcal{Y},\nu)$ is bounded independently of K, we shall obtain that

$$\mathcal{I}_k\leqslant C\int_{\mathcal{Y}}T_k(fv^{1/p})^q\,\mathrm{d}\nu\leqslant C\biggl(\int_{\mathbb{R}^d}(fv^{1/p})^p\,\mathrm{d}\mu\biggr)^{q/p}=C\biggl(\int_{\mathbb{R}^d}f^pv\,\mathrm{d}\mu\biggr)^{q/p},$$

and we shall get the desired inequality by doing $K\to\infty$. But the proof of the boundedness of T_k follows the same arguments as in Theorem 5.3 in [8], using now that the function $C\in B_p$, so we omit it. □

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