Applications of Mathematics

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Applications of Mathematics, Vol. 63 (2018), No. 1, 7–35

Persistent URL: http://dml.cz/dmlcz/147112

STOCHASTIC AFFINE EVOLUTION EQUATIONS WITH MULTIPLICATIVE FRACTIONAL NOISE

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Received February 13, 2017. First published January 8, 2018.

Abstract. A stochastic affine evolution equation with bilinear noise term is studied, where the driving process is a real-valued fractional Brownian motion with Hurst parameter greater than 1/2. Stochastic integration is understood in the Skorokhod sense. The existence and uniqueness of weak solution is proved and some results on the large time dynamics are obtained.

Keywords: geometric fractional Brownian motion; stochastic differential equations in Hilbert space; stochastic bilinear equation

MSC 2010: 60H15, 60G22

1. Introduction

In the paper the formula for stochastic evolution system generated by an equation with bilinear stochastic term and affine drift term is studied. The existence and uniqueness of solutions is proved and the relation between weak and "mild" form of solutions is investigated. Some peculiarities of large time behaviour are also demonstrated. The results obtained for the equation in the general infinite-dimensional form are applied to linear stochastic PDE of second order.

Stochastic differential equations in Hilbert spaces with multiplicative white noise have been studied in numerous papers, e.g. Da Prato, Iannelli, Tubaro [5], [4], Flandoli [9], and in Chapter 6 of the monograph by Da Prato and Zabczyk [6]. The solution to such equations may be viewed as a generalization of the geometric Brownian motion, which has a wide range of applications. In all these cases the driving process is the Brownian motion. Later, Bonaccorsi in [3] studied mild solutions of

This work was partially supported by the GAČR grant no. 15-08819S.

DOI: 10.21136/AM.2018.0036-17

equations with additional nonlinear terms in the drift and diffusion parts, defined by means of the stochastic evolution system induced by the bilinear equation.

Analogous results have been obtained for bilinear evolution equations of the same type with fractional Gaussian noise by Duncan, Maslowski, Pasik-Duncan [7] (for $H > 1/2$, where H denotes the Hurst parameter of the driving fractional Brownian motion) and Šnupárková [20] (for $H < 1/2$). In these papers the stochastic integral is understood in the Skorokhod sense, i.e. as the adjoint operator to the Malliavin derivative, the existence (not uniqueness) of the solution is proved and large time behaviour is studied. On the other hand, semilinear evolution equations with bilinear Stratonovich noise have been studied in [10]. It was shown that the equation defines a random dynamical system (which is not true in the case of Skorokhod integration) and the long time behaviour was dealt with.

The present paper is organized as follows. In Section 2 the notion of Skorokhod integral with respect to fractional Brownian motion with Hurst parameter $H > 1/2$ and its basic properties are recalled. Section 3 is devoted to an extension of a result from [7] on the existence of a weak solution to the bilinear equation. In Section 4 the existence and uniqueness of the mild solution is proved (Theorem 4.1). If the perturbation F does not depend on the solution process, the mild solution of the corresponding affine equation is the weak one (Theorem 4.7). Unlike in the case of the standard Brownian motion, this is no longer true if the equation is semilinear (the perturbation F depends on the solution) as shown by a simple counterexample. Note that it is not completely clear how to define the candidate on the "random evolution system", as explained in Example 4.5. Following the ideas from [3], this system is used to define the mild solution of an equation with additional nonlinearity in the drift part for $H > 1/2$. While the mild formulation implies the weak one in the Wiener case it need not be true in the case of fractional Brownian motion unless the perturbation is independent of the solution process. It is shown that this "mild" solution satisfies a certain different equation in the weak sense.

Some large time behaviour results are also proved. Sections 5 and 6 are devoted to the proof of uniqueness of solutions to the affine equation. This problem is nontrivial, because the Gronwall lemma is not applicable as in the case of standard Brownian motion. Instead, we prove the uniqueness of the mild solution to the bilinear equation inductively, showing uniqueness of the coefficients in the Wiener chaos expansions, and using this result we prove the uniqueness of weak solutions to the affine equation.

2. Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. A stochastic process $B^H = \{B_t^H, t \in [0,T]\}$ is a *fractional Brownian motion* with Hurst parameter $H \in (0,1)$ if it is a real-valued centered Gaussian process with the covariance function given by

$$\mathbb{E}[B_t^H B_s^H] = \tfrac{1}{2}(s^{2H} + t^{2H} - |s-t|^{2H}), \quad s, t \geqslant 0.$$

In what follows

(i) Hurst parameter $H > 1/2$ is assumed,
(ii) a version of B^H with continuous path is considered (it always exists by Kolmogorov continuity theorem).

Let $\mathcal{E}$ be the set of step functions of the form

$$f = \sum_{k=0}^{N-1} a_k I_{(t_k, t_{k+1}]},$$

where $N \in \mathbb{N}$, $0 = t_0 < t_1 < \ldots < t_N = T$, $a_k \in \mathbb{R}$, $k = 0, \ldots, N$. Define the linear operator $\mathcal{K}_H^*\colon \mathcal{E} \to L^2([0,T])$ as

$$(\mathcal{K}_H^* f)(t) = C_H \Gamma(H - 1/2) t^{1/2-H} (I_{T-}^{H-1/2} f_{H-1/2})(t),$$

where $f \in \mathcal{E}$, $f_{H-1/2}(t) = t^{H-1/2} f(t)$, $t \in [0,T]$, $I_{T-}^{H-1/2}$ is a Riemann-Liouville fractional right-sided integral defined as

$$(I_{T-}^{H-1/2} f)(t) = \frac{1}{\Gamma(H-1/2)} \int_t^T \frac{f(s)}{(s-t)^{3/2-H}}\, \mathrm{d}s \quad \text{for a.e. } t \in [0,T],$$

and

$$C_H = \sqrt{\frac{H(2H-1)}{\mathrm{B}(2-2H, H-1/2)}}$$

(see [19] for a detailed treatment on fractional calculus).

Using the operator $\mathcal{K}_H^*$, define the scalar product on $\mathcal{E}$ as

$$\langle \varphi, \psi \rangle_{\mathcal{H}} := \langle \mathcal{K}_H^*(\varphi), \mathcal{K}_H^*(\psi) \rangle_{L^2([0,T])}, \quad \varphi, \psi \in \mathcal{E}.$$

Denote by $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ the Hilbert space obtained as the completion of $\mathcal{E}$ with respect to the scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and let $\|\cdot\|_{\mathcal{H}}$ be the norm induced by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$.

For $\varphi \in \mathcal{E}$ define the stochastic integral with respect to the fractional Brownian motion

$$I(\varphi) \equiv \int_0^T \varphi(s)\,\mathrm{d}B_s^H := \sum_{k=0}^{N-1} a_k(B^H(t_{k+1}) - B^H(t_k)).$$

Since

$$\mathbb{E}\left[\int_0^T \varphi(s)\,\mathrm{d}B_s^H \int_0^T \psi(s)\,\mathrm{d}B_s^H\right] = \langle\varphi,\psi\rangle_{\mathcal{H}}, \quad \varphi,\psi \in \mathcal{E},$$

(see [1]), the integral can be uniquely extended to $\mathcal{H}$ (the standard notation $I(\varphi) = B^H(\varphi) = \int_0^T \varphi(r)\,\mathrm{d}B_r^H$ is also used) and the operator $\mathcal{K}_H^*$ provides an isometry between spaces $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ and $L^2(\Omega)$.

Let $\mathcal{S}$ be a set of smooth cylindrical random variables of the form

$$F = f(B^H(\varphi_1),\dots,B^H(\varphi_n)), \tag{2.1}$$

where $n \geqslant 1$, $f \in \mathcal{C}_b^\infty(\mathbb{R}^n)$ (f and all its partial derivatives are bounded) and $\varphi_i \in \mathcal{H}$, $i = 1,\dots,n$. The *derivative operator (Malliavin derivative)* of a smooth cylindrical random variable F of the form (2.1) is an $\mathcal{H}$-valued random variable

$$D^H F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(B^H(\varphi_1),\dots,B^H(\varphi_n))\varphi_i.$$

The derivative operator D^H is closable from $L^p(\Omega)$ into $L^p(\Omega;\mathcal{H})$ for any $p \in [1,\infty)$. Let $\mathbb{D}_H^{1,p}$ be the Sobolev space obtained as the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,p} := (\mathbb{E}[|F|^p] + \mathbb{E}[\|D^H F\|_{\mathcal{H}}^p])^{1/p}$$

for any $p \in [1,\infty)$. Similarly, given a Hilbert space $\widetilde{V} \subset \mathcal{H}$, set $\mathbb{D}_H^{1,p}(\widetilde{V})$ for the corresponding Sobolev space of $\widetilde{V}$-valued random variables.

Definition 2.1. The *divergence operator* (*Skorokhod integral*) $\delta_H\colon \operatorname{Dom}\delta_H \to L^2(\Omega)$ is defined as the adjoint operator of the derivative operator $D^H\colon L^2(\Omega) \to L^2(\Omega;\mathcal{H})$, i.e. for any $u \in \operatorname{Dom}\delta_H$ the duality relationship

$$\mathbb{E}[F\delta_H(u)] = \mathbb{E}[\langle D^H F, u\rangle_{\mathcal{H}}]$$

holds for any $F \in \mathbb{D}_H^{1,2}$.

A random variable $u \in L^2(\Omega;\mathcal{H})$ belongs to the domain $\operatorname{Dom}\delta_H$ if there exists a constant $0 < c_u < \infty$ depending only on u such that

$$|\mathbb{E}[\langle D^H F, u\rangle_{\mathcal{H}}]| \leqslant c_u\|F\|_{L^2(\Omega)}$$

for any $F \in \mathcal{S}$.

The useful facts listed below can be found e.g. in [16]. Let $|\mathcal{H}| \subset \mathcal{H}$ be the linear space of measurable functions φ on $[0,T]$ such that

$$\|\varphi\|_{|\mathcal{H}|}^2 = \alpha_H \int_0^T \int_0^T |\varphi(r)||\varphi(s)||r-s|^{2H-2} \,\mathrm{d}r \,\mathrm{d}s < \infty,$$

where $\alpha_H = H(2H-1)$. Then $\mathcal{E}$ is dense in $|\mathcal{H}|$ and $(|\mathcal{H}|, \|\cdot\|_{|\mathcal{H}|})$ is a Banach space. Moreover,

$$L^2([0,T]) \subset L^{1/H}([0,T]) \subset |\mathcal{H}| \subset \mathcal{H},$$

thus there exists a constant $K_e < \infty$ such that

$$\|\mathcal{K}_H^*(\varphi)\|_{L^2([0,T])} = \|\varphi\|_{\mathcal{H}} \leqslant K_e \|\varphi\|_{L^2([0,T])} \tag{2.2}$$

for any $\varphi \in \mathcal{H}$. Note that

$$\mathbb{D}_H^{1,2}(|\mathcal{H}|) \subset \mathbb{D}_H^{1,2}(\mathcal{H}) \subset \operatorname{Dom} \delta_H \tag{2.3}$$

and for some constant $0 < \widetilde{C}_{H,2} < \infty$

$$\mathbb{E}[\delta_H^2(u)] \leqslant \widetilde{C}_{H,2}(\mathbb{E}[\|u\|_{|\mathcal{H}|}^2] + \mathbb{E}[\|D^H u\|_{|\mathcal{H}|\otimes|\mathcal{H}|}^2]), \ u \in \mathbb{D}_H^{1,2}(|\mathcal{H}|),$$

holds, where $\mathbb{D}_H^{1,p}(|\mathcal{H}|)$ $(p \in (1,\infty))$ contains processes $u \in \mathbb{D}_H^{1,p}(\mathcal{H})$ such that $u \in |\mathcal{H}|$, $D^H u \in |\mathcal{H}| \otimes |\mathcal{H}|$ $\mathbb{P}$-a.s. and

$$\mathbb{E}[\|u\|_{|\mathcal{H}|}^p] + \mathbb{E}[\|D^H u\|_{|\mathcal{H}|\otimes|\mathcal{H}|}^p] < \infty.$$

The normed linear space $(|\mathcal{H}| \otimes |\mathcal{H}|, \|\cdot\|_{|\mathcal{H}|\otimes|\mathcal{H}|})$ is defined in a similar way as $(|\mathcal{H}|, \|\cdot\|_{|\mathcal{H}|})$ (for a precise definition see e.g. [16]). Hence, for some constant $0 < C_{H,2} < \infty$,

$$\mathbb{E}[\delta_H^2(u)] \leqslant C_{H,2}(\mathbb{E}[\|u\|_{L^{1/H}([0,T])}^2] + \mathbb{E}[\|D^H u\|_{L^{1/H}([0,T]^2)}^2]), \quad u \in \mathbb{D}_H^{1,2}(|\mathcal{H}|). \tag{2.4}$$

Since the process B^H has an integral representation (see e.g. [16])

$$B_t^H = \int_0^t (\mathcal{K}_H^* I_{(0,t]})(s) \,\mathrm{d}W_s, \quad t \geqslant 0, \tag{2.5}$$

where $W = \{W_t, t \geqslant 0\}$ is a Wiener process on $(\Omega, \mathcal{F}, \mathbb{P})$, similar relations are valid for derivatives and divergence operators, i.e.

(i) for any $F \in \mathbb{D}_W^{1,2}$

$$\mathcal{K}_H^*(D^H F) = D^W F,$$

where D^W denotes the derivative operator with respect to W and $\mathbb{D}_W^{1,2}$ the corresponding Sobolev space,

(ii) $\operatorname{Dom}\delta_W = \mathcal{K}_H^*(\operatorname{Dom}\delta_H)$ and

$$\delta_H(u) = \delta_W(\mathcal{K}_H^* u) \tag{2.6}$$

for any $u \in \operatorname{Dom}\delta_H$, where δ_W denotes the divergence operator with respect to W.

Remark 2.2. The construction (and the properties) of Malliavin derivative and Skorokhod integral for Hilbert space-valued random variables are completely analogous.

3. Random evolution system

In this short overview section, a result from [7] is slightly extended to obtain a random two-parameter evolution system representing the solution to the equation

$$\begin{aligned} \mathrm{d}Y_t &= AY_t\,\mathrm{d}t + BY_t\,\mathrm{d}B_t^H, \quad t > s, \\ Y_s &= x, \end{aligned} \tag{3.1}$$

in a separable Hilbert space V on a finite interval $[0,T]$ with general initial time $s \in [0,T]$ and deterministic initial value $x \in V$. The driving process $\{B_t^H,\ t \geqslant 0\}$ is a one-dimensional fractional Brownian motion with Hurst parameter $H > 1/2$ on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the stochastic integral is understood in the Skorokhod sense (see [1] for more details).

The linear operators A and B on V satisfy

(A1) the operator A is closed and densely defined with the domain $D := \operatorname{Dom}(A)$,

(A2) the resolvent set contains all $\lambda \in \mathbb{C}$ such that $\operatorname{Re}(\lambda) \geqslant \omega$ for some fixed $\omega \in \mathbb{R}$ and for some constant $M > 0$ the resolvent $R(\lambda, A)$ satisfies

$$\|R(\lambda, A)\|_{\mathcal{L}(V)} \leqslant \frac{M}{|\lambda - \omega| + 1}$$

for all $\lambda \in \mathbb{C}$, $\operatorname{Re}(\lambda) \geqslant \omega$, where $\mathcal{L}(V)$ stands for the space of all linear bounded operators on V,

(B2) the operator B is closed, densely defined and generates a strongly continuous group $\{S_B(u),\ u \in \mathbb{R}\}$ on V.

The conditions (A1) and (A2) imply that the operator A generates an analytic semigroup $\{S_A(t),\ 0 \leqslant t \leqslant T\}$ on V. The condition (B2) ensures the existence of constants $M_B \geqslant 1$, $\omega_B \geqslant 0$ such that the inequality

$$\|S_B(u)\|_{\mathcal{L}(V)} \leqslant M_B \exp\{\omega_B |u|\} \tag{3.2}$$

holds for each $u \in \mathbb{R}$.

For simplicity assume that $\omega < 0$ (cf. (A2)). Note that since the operator $-A$ is sectorial, the fractional powers $(-A)^\alpha$ for $\alpha \in (0,1]$ are well-defined (see e.g. [17]), so the following condition makes sense. Suppose that

(B3) B^2 is closed and

$$\mathrm{Dom}(B^2) \supset \mathrm{Dom}((-A)^\alpha) \tag{3.3}$$

for some $\alpha \in (0,1)$.

Define the operators $\bar{A}(t)\colon\ D \to V$ as

$$\bar{A}(t) = A - Ht^{2H-1}B^2$$

for any $t \in [0,T]$.

Lemma 3.1. *Under the assumptions* (A1), (A2), (B2), *and* (B3) *the system* $\{\bar{A}(t),\ t \in [0,T]\}$ *generates a strongly continuous evolution system* $\{U(t,s),\ 0 \leqslant s \leqslant t \leqslant T\}$ *on* V.

Proof. See [7]. □

The system $\{U(t,s),\ 0 \leqslant s \leqslant t \leqslant T\}$ satisfies

$$\begin{gathered}
\mathrm{Im}(U(t,s)) \subset D,\\
\|U(t,s)\|_{\mathcal{L}(V)} \leqslant C_U,\\
\left\|\frac{\partial}{\partial t}U(t,s)\right\|_{\mathcal{L}(V)} = \|\bar{A}(t)U(t,s)\|_{\mathcal{L}(V)} \leqslant \frac{C_U}{t-s},\\
\|\bar{A}(t)U(t,s)(\bar{A}(s) - \varpi I)^{-1}\|_{\mathcal{L}(V)} \leqslant C_U
\end{gathered} \tag{3.4}$$

for some constants $C_U > 0$, $\varpi \in \mathbb{R}$ and any $0 \leqslant s < t \leqslant T$ (see e.g. [21], Theorem 5.2.1).

R e m a r k 3.2. Instead of (A2) and (B3) we may assume directly that $\{\bar{A}(t),\ t \in [0,T]\}$ generates a strongly continuous evolution system $\{U(t,s),\ 0 \leqslant s \leqslant t \leqslant T\}$ on V. Nevertheless, the condition (A2) can be useful in applications to stochastic partial differential equations (as shown in [7]).

Note that if $H < 1/2$ the system of operators $\{\bar{A}(t),\ t \in (0,T)\}$ is singular at $t = 0$. In this case the theory of strong evolution operators does not seem to give outcomes satisfactory for our purpose (in [20] some results have been achieved by means of approximative systems). We do not consider this case here.

Let A^* denote the adjoint operator to the operator A. Let $\mathrm{Dom}(A^*) = D^*$ be the domain of A^* and suppose that

(B1) $D^* \subset \mathrm{Dom}((B^*)^2)$.

Definition 3.3. A $(\mathcal{B}([s,T]) \otimes \mathcal{F})$-measurable stochastic process $\{Y_t,\ t \in [s,T]\}$ is said to be a *weak solution* to the equation (3.1) if for any $y \in D^*$

$$\langle Y_t, y\rangle_V = \langle x, y\rangle_V + \int_s^t \langle Y_r, A^* y\rangle_V \,\mathrm{d}r + \int_s^t \langle Y_r, B^* y\rangle_V \,\mathrm{d}B_r^H \quad \mathbb{P}\text{-a.s.}$$

for all $t \in [s,T]$, where the integrals have to be well-defined.

Theorem 3.4. *Let*

(AB) *the operators A and $\{S_B(u), u \in \mathbb{R}\}$ commute on the domain D, i.e.*

$$S_B(u)Ay = AS_B(u)y$$

for any $u \in \mathbb{R}$ and $y \in D$.

The process $\{U_Y(t,s)x,\ s \leqslant t \leqslant T\}$ defined as

$$U_Y(t,s)x = S_B(B_t^H - B_s^H)U(t-s,0)x, \quad s \leqslant t \leqslant T, \tag{3.5}$$

is a weak solution to the equation (3.1) *for any fixed $x \in V$ and $s \in [0,T]$ under the assumptions* (A1), (A2) *and* (B1), (B2), (B3).

P r o o f. The proof is completely analogous to the proof of Theorem 2.3 in [7].

□

R e m a r k 3.5. The system $\{U_Y(t,s),\ 0 \leqslant s \leqslant t \leqslant T\}$ is not a random continuous evolution system, because it does not possess the standard composition property.

4. Perturbed equation

In this section the equation with a perturbation in the drift part is studied.

Let $H > 1/2$ and let $\{U_Y(t,s),\ 0 \leqslant s \leqslant t \leqslant T\}$ be the system of operators defined as

$$U_Y(t,s)x := S_B(B_t^H - B_s^H)U(t-s,0)x, \quad x \in V,$$

where $\{U(t,s),\ 0 \leqslant s \leqslant t \leqslant T\}$ is a strongly continuous evolution system associated with operators $\{A - Ht^{2H-1}B^2,\ t \in [0,T]\}$, and $\{S_B(u),\ u \in \mathbb{R}\}$ is a strongly continuous group associated with an operator B satisfying the conditions from Theorem 3.4. Recall from the previous section that for any fixed $s \in [0,T]$ the process $\{U_Y(t,s)x,\ s \leqslant t \leqslant T\}$ is a weak solution to the equation

$$\begin{aligned} \mathrm{d}Y_t &= AY_t\,\mathrm{d}t + BY_t\,\mathrm{d}B_t^H, \quad t > s, \\ Y_s &= x \in V. \end{aligned} \tag{4.1}$$

Theorem 4.1. *Let $F\colon [0,T] \times V \to V$ be a measurable function satisfying*

$(\mathrm{i})_{\mathrm{F}}$ *there exists a function $\bar{L} \in L^1([0,T])$ such that*

$$\|F(t,x) - F(t,y)\|_V \leqslant \bar{L}(t)\|x-y\|_V, \quad x,y \in V,\ t \in [0,T],$$

$(\mathrm{ii})_{\mathrm{F}}$ *for some function $\overline{K} \in L^1([0,T])$*

$$\|F(t,0)\|_V \leqslant \overline{K}(t), \quad t \in [0,T].$$

Then the equation

$$y(t) = U_Y(t,0)x + \int_0^t U_Y(t,r)F(r,y(r))\,\mathrm{d}r \tag{4.2}$$

has a unique solution in the space $\mathcal{C}([0,T];V)$ for a.e. $\omega \in \Omega$ and any initial value $x \in V$.

R e m a r k 4.2. The conditions $(\mathrm{i})_{\mathrm{F}}$ and $(\mathrm{ii})_{\mathrm{F}}$ imply

$$\|F(t,x)\|_V \leqslant \overline{C}(t)(1 + \|x\|_V), \quad x \in V,\ t \in [0,T] \tag{4.3}$$

for a function $\overline{C} \in L^1([0,T])$.

P r o o f of Theorem 4.1. Fix $x \in V$ and show that the mapping

$$(\mathcal{K}(y))(t) = U_Y(t,0)x + \int_0^t U_Y(t,r)F(r,y(r))\,\mathrm{d}r$$

is continuous from $\mathcal{C}([0,T];V)$ into $\mathcal{C}([0,T];V)$ and that $\mathcal{K}$ is a contraction mapping.

Take $y \in \mathcal{C}([0,T];V)$ and $t,s \in [0,T]$. Then

$$\begin{aligned}\|(\mathcal{K}(y))(t) - (\mathcal{K}(y))(s)\|_V &\leqslant \|U_Y(t,0)x - U_Y(s,0)x\|_V \\ &+ \left\|\int_0^t U_Y(t,r)F(r,y(r))\,\mathrm{d}r - \int_0^s U_Y(s,r)F(r,y(r))\,\mathrm{d}r\right\|_V = I_1 + I_2.\end{aligned}$$

Note that due to (3.2) and by the continuity of trajectories of $\{B_t^H, t \in [0,T]\}$

$$\begin{aligned}&\sup_{t\in[0,T]} \|S_B(B_t^H(\omega))\|_{\mathcal{L}(V)} \leqslant M_B \exp\{\omega_B \|B^H(\omega)\|_{\mathcal{C}([0,T])}\} \leqslant C_B(\omega),\\ &\sup_{s,t\in[0,T]} \|S_B(B_t^H(\omega) - B_s^H(\omega))\|_{\mathcal{L}(V)} \\ &\qquad \leqslant M_B \exp\{2\omega_B \|B^H(\omega)\|_{\mathcal{C}([0,T])}\} \leqslant C_B(\omega)\end{aligned} \tag{4.4}$$

hold for some constant $0 < C_B(\omega) < \infty$ depending on $\omega \in \Omega$.

The strong continuity of S_B and $U(\cdot,0)$ on V yields

$$\begin{aligned}I_1 &= \|U_Y(t,0)x - U_Y(s,0)x\|_V \\ &\leqslant \|(S_B(B_t^H) - S_B(B_s^H))U(t,0)x\|_V + \|S_B(B_s^H)(U(t,0) - U(s,0))x\|_V \\ &\leqslant \|(S_B(B_t^H) - S_B(B_s^H))U(t,0)x\|_V + C_B(\omega)\|(U(t,0) - U(s,0))x\|_V \underset{s\to t}{\longrightarrow} 0.\end{aligned}$$

Now, let $t > s$. Then

$$\begin{aligned}I_2 &= \left\|\int_0^t U_Y(t,r)F(r,y(r))\,\mathrm{d}r - \int_0^s U_Y(s,r)F(r,y(r))\,\mathrm{d}r\right\|_V \\ &\leqslant \left\|\int_0^s (U_Y(t,r) - U_Y(s,r))F(r,y(r))\,\mathrm{d}r\right\|_V \\ &\quad + \left\|\int_s^t U_Y(t,r)F(r,y(r))\,\mathrm{d}r\right\|_V = J_1 + J_2.\end{aligned}$$

Using (4.4), (3.4), and (4.3), we obtain

$$\begin{aligned}J_2 &= \left\|\int_s^t U_Y(t,r)F(r,y(r))\,\mathrm{d}r\right\|_V \leqslant \int_s^t C_U \|S_B(B_t^H - B_r^H)\|_{\mathcal{L}(V)} \|F(r,y(r))\|_V\,\mathrm{d}r \\ &\leqslant C_U C_B(\omega)(1 + \|y\|_{\mathcal{C}([0,T];V)}) \int_s^t \overline{C}(r)\,\mathrm{d}r \to 0\end{aligned}$$

as $s \to t-$ or $t \to s+$.

Also

$$\begin{aligned}
J_1 &= \left\| \int_0^s (U_Y(t,r) - U_Y(s,r)) F(r, y(r)) \, \mathrm{d}r \right\|_V \\
&\leqslant \left\| \int_0^s (S_B(B_t^H - B_r^H) - S_B(B_s^H - B_r^H)) U(t-r,0) F(r,y(r)) \, \mathrm{d}r \right\|_V \\
&+ \left\| \int_0^s S_B(B_s^H - B_r^H)(U(t-r,0) - U(s-r,0)) F(r,y(r)) \, \mathrm{d}r \right\|_V = K_1 + K_2.
\end{aligned}$$

Since for any fixed $0 \leqslant r \leqslant s$

$$\|(U(t-r,0) - U(s-r,0)) F(r,y(r))\|_V \to 0$$

as $s \to t-$ or $t \to s+$ and by (3.4)

$$\begin{aligned}
\|(U(t-r,0) - U(s-r,0)) F(r,y(r))\|_V &\leqslant 2C_U \|F(r,y(r))\|_V \\
&\leqslant 2C_U (1 + \|y\|_{\mathcal{C}([0,T];V)}) \overline{C}(r) \in L^1([0,T]),
\end{aligned}$$

the convergence

$$\begin{aligned}
K_2 &= \left\| \int_0^s S_B(B_s^H - B_r^H)(U(t-r,0) - U(s-r,0)) F(r,y(r)) \, \mathrm{d}r \right\|_V \\
&\leqslant C_B(\omega) \int_0^s \|(U(t-r,0) - U(s-r,0)) F(r,y(r))\|_V \, \mathrm{d}r \to 0
\end{aligned}$$

is obtained as $s \to t-$ or $t \to s+$ by the Lebesgue dominated convergence theorem. Note that the set

$$K := \left\{ \overline{y} \in V; \ \exists\, 0 \leqslant s_1 \leqslant t_1 \leqslant T, \quad \overline{y} = \int_0^{s_1} S_B(-B_r^H) U(t_1 - r, 0) F(r, y(r)) \, \mathrm{d}r \right\}$$

is compact (being a continuous image of a compact set) and

$$\lim_{t \to s} \sup_{z \in K} \|(S_B(B_t^H) - S_B(B_s^H)) z\|_V = 0.$$

Therefore,

$$\begin{aligned}
K_1 &= \left\| \int_0^s (S_B(B_t^H - B_r^H) - S_B(B_s^H - B_r^H)) U(t-r,0) F(r,y(r)) \, \mathrm{d}r \right\|_V \\
&= \left\| (S_B(B_t^H) - S_B(B_s^H)) \int_0^s S_B(-B_r^H) U(t-r,0) F(r,y(r)) \, \mathrm{d}r \right\|_V \\
&\leqslant \sup_{z \in K} \|(S_B(B_t^H) - S_B(B_s^H)) z\|_V \to 0
\end{aligned}$$

as $s \to t-$ or $t \to s+$. Thus

$$\|(\mathcal{K}(y))(t) - (\mathcal{K}(y))(s)\|_V \to 0$$

as $s \to t-$ or $t \to s+$ and the function $t \mapsto (\mathcal{K}(y))(t)$ is continuous on the interval $[0,T]$ for any $y \in \mathcal{C}([0,T];V)$.

For any $y_1, y_2 \in \mathcal{C}([0,T];V)$, $t \in [0,T]$ and $T > 0$ small enough there exists a constant $0 < L_T(\omega) < 1$ such that

$$\begin{aligned}
&\|(\mathcal{K}(y_1))(t) - (\mathcal{K}(y_2))(t)\|_V = \left\| \int_0^t U_Y(t,r)(F(r,y_1(r)) - F(r,y_2(r)))\,\mathrm{d}r \right\|_V \\
&\leqslant C_B(\omega) C_U \int_0^t \|(F(r,y_1(r)) - F(r,y_2(r)))\|_V\,\mathrm{d}r \\
&\leqslant C_B(\omega) C_U \|y_1 - y_2\|_{\mathcal{C}([0,T];V)} \int_0^T \bar{L}(r)\,\mathrm{d}r \leqslant L_T(\omega)\|y_1 - y_2\|_{\mathcal{C}([0,T];V)}
\end{aligned}$$

holds so that $\mathcal{K}$ is a contraction mapping. Hence, by the Banach fixed-point theorem there exists a unique solution to the equation (4.2) for T small enough. Applying standard methods a unique continuous solution to (4.2) for any $T > 0$ can be obtained. □

Consider an equation with a nonlinear perturbation of the drift part

$$\begin{aligned}
\mathrm{d}X_t &= AX_t\,\mathrm{d}t + F(t,X_t)\,\mathrm{d}t + BX_t\,\mathrm{d}B_t^H, \\
X_0 &= x \in V.
\end{aligned} \tag{4.5}$$

Definition 4.3. A $(\mathcal{B}([0,T]) \otimes \mathcal{F})$-measurable process $\{X_t, t \in [0,T]\}$ is a *weak solution* to the equation (4.5) if for any $y \in D^*$

$$\begin{aligned}
\langle X_t, y\rangle_V = \langle x, y\rangle_V &+ \int_0^t \langle X_r, A^* y\rangle_V\,\mathrm{d}r \\
&+ \int_0^t \langle F(r,X_r), y\rangle_V\,\mathrm{d}r + \int_0^t \langle X_r, B^* y\rangle_V\,\mathrm{d}B_r^H \quad \mathbb{P}\text{-a.s.}
\end{aligned}$$

for all $t \in [0,T]$, where the integrals have to be well-defined.

Remark 4.4. In the Wiener case $H = 1/2$ the solution to the equation (4.2) is called the mild solution to the equation

$$\begin{aligned}
\mathrm{d}X_t &= AX_t\,\mathrm{d}t + F(t,X_t)\,\mathrm{d}t + BX_t\,\mathrm{d}W_t, \\
X_0 &= x \in V.
\end{aligned}$$

In this case, Bonaccorsi ([3]) has shown that the solution to the equation (4.2) is also a weak solution to the above equation. This in general is not true for the equation (4.5) as is shown in a simple counterexample below.

E x a m p l e 4.5. Consider a one-dimensional equation

$$\mathrm{d}X_t = aX_t\,\mathrm{d}t + bX_t\,\mathrm{d}B_t^H, \quad X_0 = 1, \tag{4.6}$$

where $a, b \in \mathbb{R}$ are nonzero constants. Note that the equation (4.6) takes the form (4.5) with $F(t,x) = ax$, $A = 0$, $B = bI$, and $x_0 = 1$.

The solution to the equation (4.6) is given by the formula

$$X_t = \exp\{bB_t^H - \tfrac{1}{2}b^2t^{2H} + at\}, \quad t \in [0,T],$$

(applying Theorem 3.4 with the choice $V = \mathbb{R}$, $A = aI$, $B = bI$, $s = 0$ and $x = 1$) and the random evolution system corresponding to the above choice of coefficients is

$$\begin{aligned} U_Y(t,s) &= S_B(B_t^H - B_s^H)U(t-s,0) \\ &= \exp\{b(B_t^H - B_s^H) - \tfrac{1}{2}b^2(t-s)^{2H}\}, \quad 0 \leqslant s \leqslant t \leqslant T. \end{aligned}$$

It is now easy to compute that the solution $\{X_t,\ t \in [0,T]\}$ does not satisfy the mild formula

$$y(t) = U_Y(t,0) + \int_0^t U_Y(t,r)F(r,y(r))\,\mathrm{d}r. \tag{4.7}$$

Note that if we define the system $\{\overline{U}_Y(t,s),\ 0 \leqslant s \leqslant t \leqslant T\}$ as

$$\begin{aligned} \overline{U}_Y(t,s) &= S_B(B_t^H - B_s^H)U(t,s) \\ &= \exp\{b(B_t^H - B_s^H) - \tfrac{1}{2}b^2(t^{2H} - s^{2H})\}, \quad 0 \leqslant s \leqslant t \leqslant T, \end{aligned}$$

the above mild formula holds if U_Y is replaced by $\overline{U}_Y$.

R e m a r k 4.6. Let the assumptions of Theorem 3.4 be satisfied. Then the system $\{\overline{U}_Y(t,s),\ 0 \leqslant s \leqslant t \leqslant T\}$ defined as

$$\overline{U}_Y(t,s)x = S_B(B_t^H - B_s^H)U(t,s)x, \quad x \in V,\ 0 \leqslant s \leqslant t \leqslant T, \tag{4.8}$$

is a weak solution to the equation

$$\begin{aligned} &\mathrm{d}Y_t = A(t)Y_t\,\mathrm{d}t + H((t-s)^{2H-1} - t^{2H-1})B^2Y_t\,\mathrm{d}t + BY_t\,\mathrm{d}B_t^H, \quad t > s, \\ &Y_s = x. \end{aligned}$$

This result can be obtained in the same way as Theorem 3.4. The system $\{\overline{U}_Y(t,s),\ 0 \leqslant s \leqslant t \leqslant T\}$ defined in Example 4.5 is a particular case of (4.8). Moreover, this system has the composition property.

It is easy to verify the fact that $\{U_Y(t,s),\ 0 \leqslant s \leqslant t \leqslant T\}$ does not possess the composition property, which means that the equation (3.1) does not define a cocycle in the usual way. On the other hand, in [2] it has been proved (for the case of stochastic equation with homogeneous right-hand side and bilinear fractional noise) that the cocycle property does hold in the case when stochastic integration in Stratonovich sense is considered.

Also, it is interesting to note that if the state space V is finite-dimensional (and under our commutativity assumption) the evolution system $\{U_Y(t,s),\ 0 \leqslant s \leqslant t \leqslant T\}$ may be expressed as the Wick exponential

$$U_Y(t,s) = \exp^{\diamond}\{A(t-s) + B(B_t^H - B_s^H)\}$$

and it is easy to verify that it satisfies the evolution property with respect to the Wick multiplication

$$U_Y(t,s) = U_Y(t,u) \diamond U_Y(u,s), \quad 0 \leqslant s \leqslant u \leqslant t.$$

In infinite dimensions (and in particular for unbounded B) the problem seems to be more complex, since the explicit form of $\{U(t,s),\ 0 \leqslant s \leqslant t \leqslant T\}$ is not available (in fact, to show the existence of strong evolution operators is much more complicated than for one-parameter semigroups) and working with Wick compositions of such systems may not be easy in general.

Similarly, to derive the “appropriate” mild formula for the semilinear equation (4.5) would have to take into account the Wick correction (for deeper insight to these questions in one-dimensional case see [12], [13]).

The natural question is whether there is a chance to obtain a weak solution as the unique solution to the equation (4.2). The affirmative answer is given by the next theorem but only under the restrictive assumption on F that it does not depend on the space variable.

Theorem 4.7. *Assume that the measurable function $F\colon [0,T] \to V$ is affine and that $\|F\|_V \in L^2([0,T])$. Then the unique continuous solution $\{X_t, t \in [0,T]\}$ to the equation*

$$X_t^M = U_Y(t,0)x + \int_0^t U_Y(t,r)F(r)\,\mathrm{d}r \tag{4.9}$$

stated in Theorem 4.1 is a weak solution to the equation

$$\begin{aligned} \mathrm{d}X_t &= AX_t\,\mathrm{d}t + F(t)\,\mathrm{d}t + BX_t\,\mathrm{d}B_t^H, \\ X_0 &= x \in V. \end{aligned} \tag{4.10}$$

The main idea of the proof is to use the standard and stochastic Fubini theorems for the Skorokhod integral stated in [11], Lemma 2.10, or [15], Exercise 3.2.8.

Lemma 4.8. *Consider a random field* $\{u(t,x),\ t \in [0,T], x \in G\}$, *where* $G \subset \mathbb{R}$ *is a bounded set, such that*

$(\mathrm{i})_{\mathrm{W}}$ $u \in L^2(\Omega \times [0,T] \times G)$,
$(\mathrm{ii})_{\mathrm{W}}$ $u(\cdot, x) \in \operatorname{Dom} \delta_W$ *for a.e.* $x \in G$,
$(\mathrm{iii})_{\mathrm{W}}$ $\mathbb{E}\big[\int_G \big(\int_0^T u(t,x)\,\mathrm{d}W_t\big)^2\,\mathrm{d}x\big] < \infty$.

Then the process $\big\{\int_G u(t,x)\,\mathrm{d}x, t \in [0,T]\big\} \in \operatorname{Dom} \delta_W$ *and*

$$\int_0^T \left(\int_G u(t,x)\,\mathrm{d}x\right)\mathrm{d}W_t = \int_G \left(\int_0^T u(t,x)\,\mathrm{d}W_t\right)\mathrm{d}x.$$

Due to the relationship between Skorokhod integral with respect to Wiener process and fractional Brownian motion (see (2.6) or [16] for more detailes) $(\mathrm{ii})_{\mathrm{W}}$, $(\mathrm{iii})_{\mathrm{W}}$ are equivalent to

$(\mathrm{ii})_{\mathrm{H}}$ $u_H(\cdot, x) \in \operatorname{Dom} \delta_H$ for a.e. $x \in G$,
$(\mathrm{iii})_{\mathrm{H}}$ $\mathbb{E}\big[\int_G \big(\int_0^T u_H(t,x)\,\mathrm{d}B_t^H\big)^2\,\mathrm{d}x\big] < \infty$,

respectively, where $u_H(t,x) = (\mathcal{K}_H^*)^{-1}(u(\cdot,x))(t)$, $t \in [0,T]$. The conclusion of Lemma 4.8 can be reformulated in the following way. The process

$$\left\{\int_G u_H(t,x)\,\mathrm{d}x,\ t \in [0,T]\right\} \in \operatorname{Dom} \delta_H$$

and

$$\int_0^T \left(\int_G u_H(t,x)\,\mathrm{d}x\right)\mathrm{d}B_t^H = \int_G \left(\int_0^T u_H(t,x)\,\mathrm{d}B_t^H\right)\mathrm{d}x.$$

The proof of Theorem 4.7 is based on the following lemma.

Lemma 4.9. *The equalities*

$$\int_0^t \int_0^r \langle U_Y(r,v)F(v), A^*\zeta\rangle_V\,\mathrm{d}v\,\mathrm{d}r = \int_0^t \int_v^t \langle U_Y(r,v)F(v), A^*\zeta\rangle_V\,\mathrm{d}r\,\mathrm{d}v \tag{4.11}$$

and

$$\int_0^t \int_0^r \langle U_Y(r,v)F(v), B^*\zeta\rangle_V\,\mathrm{d}v\,\mathrm{d}B_r^H = \int_0^t \int_v^t \langle U_Y(r,v)F(v), B^*\zeta\rangle_V\,\mathrm{d}B_r^H\,\mathrm{d}v$$

hold $\mathbb{P}$*-a.s. for any* $t \in [0,T]$ *and fixed* $\zeta \in D^*$.

P r o o f. It is necessary to verify the assumptions of the standard and stochastic Fubini theorems.

Notice that the Fernique theorem (see e.g. [8]) yields that there exists a random variable $C_{B^H}(\omega)$ such that $C_{B^H} \in L^q(\Omega)$ for any $q \in [1, \infty)$ and $l > 0$, and

$$M_B \exp\{l\omega_B \|B^H(\omega)\|_{\mathcal{C}([0,T])}\} \leqslant C_{B^H}(\omega), \ \omega \in \Omega. \tag{4.12}$$

Since by (4.12) and (3.4)

$$\begin{aligned}\int_0^t \int_0^r |\langle U_Y(r,v)F(v), A^*\zeta\rangle_V| \, \mathrm{d}v \, \mathrm{d}r &\leqslant \int_0^T \int_0^T C_{B^H}(\omega) C_U \|F(v)\|_V \|A^*\zeta\|_V \, \mathrm{d}v \, \mathrm{d}r \\ &\leqslant K(\omega) \int_0^T \|F(v)\|_V \, \mathrm{d}v < \infty\end{aligned}$$

for a.e. $\omega \in \Omega$, (4.11) follows by the standard Fubini theorem.

Denote

$$\begin{aligned} u_H(r,s) &= \langle U_Y(r,s)F(s), B^*\zeta\rangle_V, \quad 0 \leqslant s \leqslant r \leqslant t, \\ u(r,s) &= (\mathcal{K}_H^* u_H(\cdot, s))(r), \quad 0 \leqslant s \leqslant r \leqslant t, \end{aligned}$$

and verify that $(\mathrm{i})_{\mathrm{W}}$, $(\mathrm{ii})_{\mathrm{H}}$ and $(\mathrm{iii})_{\mathrm{H}}$ hold for the corresponding processes. First show that $u \in L^2([0,t]^2 \times \Omega)$. Using (2.2),

$$\begin{aligned}\mathbb{E}\left[\int_0^t \int_0^t u^2(r,s) \, \mathrm{d}r \, \mathrm{d}s\right] &\leqslant K_e \mathbb{E}\left[\int_0^t \int_0^t u_H^2(r,s) \, \mathrm{d}r \, \mathrm{d}s\right] \\ &\leqslant K_e \mathbb{E}\left[\int_0^t \int_0^t (C_{B^H}(\omega) C_U \|F(s)\|_V \|B^*\zeta\|_V)^2 \, \mathrm{d}r \, \mathrm{d}s\right] < \infty,\end{aligned}$$

and $(\mathrm{i})_{\mathrm{W}}$ follows. To prove $(\mathrm{ii})_{\mathrm{H}}$ it sufficies to show (in view of (2.3)) that $u_H(\cdot, s) \in \mathbb{D}_H^{1,2}(|\mathcal{H}|)$ for a.e. $s \in [0,t]$, which is true whenever

$$\max\Big\{\sup_{r\in[0,t]} \mathbb{E}[u_H^2(r,s)], \sup_{r\in[0,t]} \sup_{v\in[0,t]} \mathbb{E}[(D_v^H u_H(r,s))^2]\Big\} < \infty \tag{4.13}$$

for a.e. $s \in [0,t]$. Since

$$D_v^H u_H(r,s) = \langle U_Y(r,s)F(s), (B^*)^2\zeta\rangle_V I_{(s,r]}(v),$$

the inequalities

$$\begin{aligned}&\sup_{r\in[0,t]} \sup_{v\in[0,t]} \mathbb{E}[(D_v^H u_H(r,s))^2] \\ &\qquad \leqslant \sup_{r\in[0,t]} \mathbb{E}[(C_{B^H}(\omega) C_U \|F(s)\|_V \|(B^*)^2\zeta\|_V)^2] = K\|F(s)\|_V^2 < \infty\end{aligned}$$

and

$$\sup_{r\in[0,t]} \mathbb{E}[u_H^2(r,s)] \leqslant \mathbb{E}[(C_{B^H}(\omega)C_U\|F(s)\|_V\|B^*\zeta\|_V)^2] \leqslant K\|F(s)\|_V^2 < \infty$$

hold for a.e. $s \in [0,t]$, which completes the proof of (4.13).

Finally, applying the estimate on the Skorokhod integral (2.4) and the previous part of the proof of (4.13) we conclude that

$$\begin{aligned}\mathbb{E}\left[\int_0^t \left(\int_0^t u_H(r,s)\,\mathrm{d}B_r^H\right)^2 \mathrm{d}s\right] &= \int_0^t \mathbb{E}\left[\left(\int_0^t u_H(r,s)\,\mathrm{d}B_r^H\right)^2\right] \mathrm{d}s \\ &\leqslant C_{H,2}\int_0^t (\mathbb{E}[\|u_H(\cdot,s)\|^2_{L^2([0,t])}] + \mathbb{E}[\|D^H u_H(\cdot,s)\|^2_{L^2([0,t]^2)}])\,\mathrm{d}s \\ &\leqslant C_{H,2}\int_0^t (t+t^2)K\|F(s)\|_V^2\,\mathrm{d}s < \infty\end{aligned}$$

holds and $(\mathrm{iii})_{\mathrm{H}}$ follows. □

P r o o f of Theorem 4.7. Fix $\zeta \in D^*$. Since $\{X_t,\ t\in[0,T]\}$ satisfies (4.9) and $\{U_Y(t,s)x,\ s \leqslant t \leqslant T\}$ is a weak solution to the equation (4.1),

$$\begin{aligned}\int_0^t \langle X_r, A^*\zeta\rangle_V\,\mathrm{d}r + \int_0^t \langle X_r, B^*\zeta\rangle_V\,\mathrm{d}B_r^H &= \int_0^t \langle U_Y(r,0)x, A^*\zeta\rangle_V\,\mathrm{d}r \\ &\quad + \int_0^t\int_0^r \langle U_Y(r,v)F(v), A^*\zeta\rangle_V\,\mathrm{d}v\,\mathrm{d}r + \int_0^t \langle U_Y(r,0)x, B^*\zeta\rangle_V\,\mathrm{d}B_r^H \\ &\quad + \int_0^t\int_0^r \langle U_Y(r,v)F(v), B^*\zeta\rangle_V\,\mathrm{d}v\,\mathrm{d}B_r^H \\ &= \langle U_Y(t,0)x,\zeta\rangle_V - \langle x,\zeta\rangle_V + \int_0^t\int_v^t \langle U_Y(r,v)F(v), A^*\zeta\rangle_V\,\mathrm{d}r\,\mathrm{d}v \\ &\quad + \int_0^t\int_v^t \langle U_Y(r,v)F(v), B^*\zeta\rangle_V\,\mathrm{d}B_r^H\,\mathrm{d}v \quad \mathbb{P}\text{-a.s.}\end{aligned}$$

holds for any $t\in[0,T]$, where in the last equality Lemma 4.9 is used.

Applying again that $\{U_Y(t,s)x, s\leqslant t\leqslant T\}$ is a weak solution to the equation (4.1)

$$\begin{aligned}&\int_0^t \langle X_r, A^*\zeta\rangle_V\,\mathrm{d}r + \int_0^t \langle X_r, B^*\zeta\rangle_V\,\mathrm{d}B_r^H \\ &\quad = \langle U_Y(t,0)x,\zeta\rangle_V - \langle x,\zeta\rangle_V + \int_0^t \langle U_Y(t,v)F(v),\zeta\rangle_V\,\mathrm{d}v - \int_0^t \langle F(v),\zeta\rangle_V\,\mathrm{d}v \\ &\quad = \langle X_t,\zeta\rangle_V - \langle x,\zeta\rangle_V - \int_0^t \langle F(v),\zeta\rangle_V\,\mathrm{d}v \quad \mathbb{P}\text{-a.s.}\end{aligned}$$

is obtained for any $t\in[0,T]$ and the conclusion follows. □

Remark 4.10. In view of Example 4.5, one can ask whether the solution to the equation (4.2) is a weak one to some equation. A partial answer is given by the next theorem the proof of which is similar to that of Theorem 4.7 (but more technical) and is omitted.

Theorem 4.11. *Let the assumptions of Theorem* 4.1 *hold and let* $\{X_t,\ t\in[0,T]\}$ *be the solution to the equation* (4.2) *such that there exists a constant* $C_X<\infty$,

$$\max\Big\{\sup_{t\in[0,T]}\mathbb{E}\|X_t\|_V^4,\ \sup_{t\in[0,T]}\sup_{v\in[0,T]}\mathbb{E}\|D_v^H X_t\|_V^4\Big\}\leqslant C_X. \tag{4.14}$$

In addition, let F be Fréchet differentiable with respect to the space variable for any time $t\in[0,T]$. Suppose that there exists a function $C\in L^4([0,T])$ such that

$$\max\{\|F(t,x)\|_V,\|F'_x(t,x)\|\}\leqslant C(t),\quad t\in[0,T], \tag{4.15}$$

holds. Then $\{X_t,t\in[0,T]\}$ is a solution to the integral equation

$$\begin{aligned}X_t=x&+\int_0^t AX_r\,\mathrm{d}r+\int_0^t F(r,X_r)\,\mathrm{d}r+\int_0^t BX_r\,\mathrm{d}B_r^H\\&+\int_0^t\alpha_H\int_0^T\int_r^t|v-w|^{2H-2}BU_Y(v,r)F'_x(r,X_r)D_w^HX_r\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}r\end{aligned}$$

in a weak sense, i.e. for any $y\in D^*$

$$\begin{aligned}\langle X_t,y\rangle_V=\langle x,y\rangle_V&+\int_0^t\langle X_r,A^*y\rangle_V\,\mathrm{d}r+\int_0^t\langle F(r,X_r),y\rangle_V\,\mathrm{d}r+\int_0^t\langle X_r,B^*y\rangle_V\,\mathrm{d}B_r^H\\+\int_0^t\alpha_H\int_0^T\int_r^t&|v-w|^{2H-2}\langle U_Y(v,r)F'_x(r,X_r)D_w^HX_r,B^*y\rangle_V\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}r\quad\mathbb{P}\text{-a.s.}\end{aligned}$$

holds for all $t\in[0,T]$.

Remark 4.12. The condition (4.14) implies that $X\in\mathbb{D}_H^{1,4}(|\mathcal{H}|)$.

Example 4.13. Consider the stochastic parabolic equation of the second order with the additional affine term in a drift part

$$\begin{aligned}\frac{\partial u}{\partial t}(t,x)&=(Lu(t,\cdot))(x)+f(t,x)+bu(t,x)\frac{\mathrm{d}B^H}{\mathrm{d}t},\\u(0,x)&=x_0(x),\quad x\in\mathcal{O},\\u(t,x)&=0,\quad(t,x)\in[0,T]\times\partial\mathcal{O},\end{aligned}\tag{4.16}$$

where $\mathcal{O} \subset \mathbb{R}^d$ is a bounded domain with the boundary of class $\mathcal{C}^2$, $b \in \mathbb{R} \setminus \{0\}$, and

$$(Lu(t,\cdot))(x) = a_0(x)u(t,x) + \sum_{i=1}^{d} a_i(x)\frac{\partial u}{\partial x_i}(t,x) + \sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}(t,x)$$

is a strongly elliptic operator on $\mathcal{O}$.

Suppose that the functions $a_0, a_i, a_{ij} \in \mathcal{C}^\infty(\overline{\mathcal{O}})$ for $i, j = 1, \ldots, d$. Let $V = L^2(\mathcal{O})$. Assume that the mapping $F\colon [0,T] \to V$; $F(t) := f(t,\cdot)$, satisfies $F \in L^2([0,T]; V)$.

Equation (4.16) can be rewritten in the form (4.10), where

$$(Au(t,\cdot))(x) = (Lu(t,\cdot))(x),$$

with $\mathrm{Dom}(A) = D = H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$ and $B = bI \in \mathcal{L}(V)$. The adjoint operator A^* has the same form as the operator A (possibly, with different coefficients), hence $\mathrm{Dom}(A^*) = D$. In this case the assumptions of Theorem 4.7 (including those of Theorems 3.4 and 4.1) are satisfied, therefore the process $\{X_t, t \in [0,T]\}$ defined as

$$X_t = U_Y(t,0)x_0 + \int_0^t U_Y(t,r)F(r)\,\mathrm{d}r$$

is a weak solution to the equation (4.16). Note that the process $\{U_Y(t,s),\ 0 \leqslant s \leqslant t \leqslant T\}$ defined in Theorem 3.4 has the form

$$U_Y(t,s) = \exp\{b(B_t^H - B_s^H) - \tfrac{1}{2}b^2(t-s)^{2H}\}S_L(t-s), \quad 0 \leqslant s \leqslant t \leqslant T,$$

where $\{S_L(t),\ t \in [0,T]\}$ is the strongly continuous semigroup on V generated by operator A.

Theorem 4.7 may serve as a useful tool for an analysis of the behaviour of the weak solutions to (4.10). As an example a simple result on large time behaviour of the solution to the equation

$$\begin{aligned} \mathrm{d}X_t &= (AX_t + F(t))\,\mathrm{d}t + bX_t\,\mathrm{d}B_t^H, \quad t > 0, \\ X_0 &= x, \end{aligned} \tag{4.17}$$

is provided, where $A\colon \mathrm{Dom}(A) \subset V \to V$ is the generator of a strongly continuous semigroup $\{S_A(t),\ t \geqslant 0\}$ and $b \in \mathbb{R} \setminus \{0\}$.

It is easily seen that

$$U_Y(t,s) = \exp\{b(B_t^H - B_s^H) - \tfrac{1}{2}b^2(t-s)^{2H}\}S_A(t-s), \quad 0 \leqslant s \leqslant t < \infty,$$

and since there exist constants $M > 0$, $\omega \in \mathbb{R}$, such that

$$\|S_A(t)\|_{\mathcal{L}(V)} \leqslant M\mathrm{e}^{\omega t}, \quad t \geqslant 0,$$

the inequality

$$\begin{aligned} &\|U_Y(t,s)\|_{\mathcal{L}(V)} \\ &\quad \leqslant M \exp\{b(B_t^H - B_s^H) - \tfrac{1}{2}b^2(t-s)^{2H} + \omega(t-s)\}, \quad 0 \leqslant s \leqslant t < \infty, \end{aligned} \tag{4.18}$$

is obtained.

Proposition 4.14. *Assume that $F \in L^2([0,T];V)$. Then the solution $\{X_t, t \geqslant 0\}$ to the equation (4.17) satisfies*

$$\|X_t\|_V \leqslant y(t), \ t \geqslant 0, \quad \mathbb{P}\text{-a.s.},$$

where y is a solution to the one-dimensional equation

$$\begin{aligned} \mathrm{d}y(t) &= (\omega y(t) + \|F(t)\|_V)\,\mathrm{d}t + by(t)\,\mathrm{d}B_t^H, \quad t > 0, \\ y(0) &= M\|x\|_V. \end{aligned} \tag{4.19}$$

P r o o f. The proof easily follows from (4.9) and (4.18), because

$$\begin{aligned} \|X_t\|_V &\leqslant M \exp\{bB_t^H - \tfrac{1}{2}b^2 t^{2H} + \omega t\}\|x\|_V \\ &\quad + \int_0^t \exp\{b(B_t^H - B_s^H) - \tfrac{1}{2}b^2(t-s)^{2H} + \omega(t-s)\}M\|F(s)\|_V\,\mathrm{d}s \end{aligned} \tag{4.20}$$

and by Theorem 4.7 the right-hand side of (4.20) is exactly the formula for the solution to (4.19). □

Corollary 4.15. *For each $p \geqslant 1$ there exists a constant $c_p > 0$ depending only on p such that*

$$\begin{aligned} \mathbb{E}[\|X_t\|_V^p] &\leqslant c_p M \exp\{\tfrac{1}{2}(p^2-p)b^2 t^{2H} + p\omega t\}\|x\|_V^p \\ &\quad + Mt^{p-1}\int_0^t \exp\{\tfrac{1}{2}(p^2-p)b^2(t-s)^{2H} + p\omega(t-s)\}\|F(s)\|_V^p\,\mathrm{d}s, \ t \geqslant 0. \end{aligned} \tag{4.21}$$

In particular, if $F(t) \equiv F$ does not depend on $t \geqslant 0$, for each $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\mathbb{E}[\|X_t\|_V^p] \leqslant C_\varepsilon \exp\{(\hat{c} + \varepsilon)t^{2H}\}, \quad t \geqslant 0, \tag{4.22}$$

holds with $\hat{c} = \frac{1}{2}b^2(p^2 - p)$.

P r o o f. The inequality (4.21) easily follows from (4.20) if we take into account that

$$\mathbb{E}[\exp\{p(b(B_t^H - B_s^H) - \tfrac{1}{2}b^2(t-s)^{2H} + \omega(t-s))\}] = \exp\{\hat{c}(t-s)^{2H} + p\omega(t-s)\}$$

for all $0 \leqslant s \leqslant t$ and apply the Hölder inequality to the second term on the right-hand side of (4.20). The inequality (4.22) is an immediate consequence of (4.21). $\square$

R e m a r k 4.16. A simple one-dimensional example shows that the bound $\hat{c}$ in (4.22) is, in some sense, sharp. Take $V = \mathbb{R}$, $A = \omega$, $F = 0$, and $x \neq 0$. Then

$$|X_t|^p = |x|^p \exp\{p\omega t - \tfrac{1}{2}b^2 p t^{2H} + pbB_t^H\}, \quad t \geqslant 0,\ p > 1,$$

hence for each $\varepsilon > 0$ there exists $\widetilde{C}_\varepsilon > 0$ such that

$$\mathbb{E}[|X_t|_V^p] = |x|^p \exp\{\hat{c}t^{2H} + p\omega t\} \geqslant \widetilde{C}_\varepsilon \exp\{(\hat{c} - \varepsilon)t^{2H}\}, \quad t \geqslant 0.$$

It means that for $p > 1$ the pth moment of the solution to the linear equation may be destabilized by adding bilinear fractional noise of the form $bX_t\dot{B}_t^H, b \neq 0$, even if the original equation is stable (here $\omega < 0$). It may be interesting to note that from [4], Remark 3.7, applied to the same example it follows that the solution tends to zero pathwise exponentially fast as $t \to \infty$, even if the equation without noise is not stable (i.e. $\omega > 0$).

5. Uniqueness of mild solution

This section is devoted to the proof of the uniqueness of the mild solution to the equation

$$\mathrm{d}X_t = AX_t\,\mathrm{d}t + BX_t\,\mathrm{d}B_t^H, \quad X_0 = x, \tag{5.1}$$

on the interval $[0,T]$. This result is used later in Section 6 to prove the uniqueness of the weak solution to the perturbed equation. Let $H > 1/2$ and recall that S_B is a strongly continuous group generated by B and U is a strongly continuous evolution system associated with operators $A - Ht^{2H-1}B^2$, $t \in [0,T]$.

Theorem 5.1. *Let the conditions* (A1), (A2), (AB) *be satisfied and let* $B \in \mathcal{L}(V)$. *Then the process* $X = \{X_t,\ t \in [0,T]\}$ *given by*

$$X_t = S_B(B_t^H)U(t,0)x \tag{5.2}$$

is a mild solution to the equation (5.1), *i.e.*

$$X_t = S_A(t)x + \int_0^t S_A(t-r)BX_r \, \mathrm{d}B_r^H \quad \mathbb{P}\text{-a.s.}$$

for all $t \in [0,T]$, *where* $\{S_A(t), t \geqslant 0\}$ *is an analytic semigroup generated by* A.

Proof. See [7]. □

Our aim is to show that X defined by (5.2) is a unique mild solution to (5.1). The idea of the proof is to use the fractional Wiener chaos decomposition as in the paper [18], where the result is proved in a one-dimensional case.

The construction of multiple fractional integrals and fractional Wiener chaos decomposition that are used below are made only for real-valued random variables. However, all remains true for Hilbert space-valued random variables (see e.g. [14]). For simplicity the Hilbert space-valued notation is the same as the real-valued notation.

Let $\mathcal{H}^{\otimes n}$ denote the nth tensor product of $\mathcal{H}$ for any $n \geqslant 2$. Set $\mathcal{H}^{\otimes 1} \equiv \mathcal{H}$ and $\mathcal{H}^{\otimes 0} \equiv \mathbb{R}$ or V, respectively.

Definition 5.2. Let $n \in \mathbb{N}$. For $f \in \mathcal{H}^{\otimes n}$ symmetric the multiple fractional integral of order n of f is defined as

$$I_n^H(f) = \delta_H^n(f),$$

where δ_H^n is the multiple divergence operator (Skorokhod integral) of order n (for the definition see e.g. [14]).

Note that $\delta_H^1 \equiv \delta_H$.

As in the Wiener case the functions $F \in L^2(\Omega; \mathcal{G}, \mathbb{P})$ (where $\mathcal{G}$ denotes the σ-field generated by $\{B_t^H, t \in [0,T]\}$) admit the unique fractional Wiener chaos decomposition

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} I_n^H(f_n),$$

where $f_n \in \mathcal{H}^{\otimes n}$ are symmetric elements which are uniquely determined (see [14] or [15]). Let

$$\mathcal{H}_n = I_n^H(\mathcal{H}^{\otimes n})$$

be the fractional Wiener chaos of order n.

Theorem 5.3. *Under the assumptions of Theorem* 5.1 *the mild solution*

$$\{X_t = S_B(B_t^H)U(t,0)x, \; t \in [0,T]\},$$

to the equation (5.1) *is unique in* $\operatorname{Dom} \delta_H$.

P r o o f. Clearly, $X = \{X_t,\ t \in [0,T]\} \in \mathbb{D}_H^{1,2}(|\mathcal{H}|) \subset \operatorname{Dom}\delta_H$. Take another mild solution $Y = \{Y_t,\ t \in [0,T]\} \in \operatorname{Dom}\delta_H$ to the equation (5.1). Then the processes X and Y satisfy

$$X_t = S_A(t)x + \int_0^t S_A(t-r)BX_r \,\mathrm{d}B_r^H,$$
$$Y_t = S_A(t)x + \int_0^t S_A(t-r)BY_r \,\mathrm{d}B_r^H,$$

respectively. Define the process $Z = \{Z_t,\ t \in [0,T]\}$ as

$$Z_t = X_t - Y_t, \quad t \in [0,T].$$

Let

$$Z_t = \sum_{n=0}^{\infty} I_n(z_n(t,\cdot))$$

be the fractional Wiener chaos decomposition of process Z, where $z_n(t,\cdot) \in \mathcal{H}^{n+1}$ are the symmetric elements in the last n variables. Since

$$z_0(t) = I_0(z_0(t)) = \mathbb{E}[Z_t] = \mathbb{E}[X_t - Y_t] = S_A(t)x - S_A(t)x = 0$$

for all $t \in [0,T]$, we get

$$Z_t = \sum_{n=1}^{\infty} I_n(z_n(t,\cdot)).$$

The definition of Skorokhod integral via multiple integrals yields

$$\sum_{n=1}^{\infty} I_n(z_n(t,\cdot)) = Z_t = \int_0^t S_A(t-r)BZ_r \,\mathrm{d}B_r^H = \int_0^t \sum_{n=0}^{\infty} I_n(S_A(t-r)Bz_n(r,\cdot)) \,\mathrm{d}B_r^H$$
$$= \sum_{n=0}^{\infty} I_{n+1}(\operatorname{Sym}(S_A(t-\cdot)Bz_n(\cdot))) = \sum_{n=1}^{\infty} I_n(\operatorname{Sym}(S_A(t-\cdot)Bz_{n-1}(\cdot))),$$

where $\operatorname{Sym}(f)$ denotes the symmetrization of f in all variables. From the uniqueness of Wiener chaos expansion we obtain

$$z_n(t,\cdot) = \operatorname{Sym}(S_A(t-\cdot)Bz_{n-1}(\cdot)), \quad n \geqslant 1.$$

Since $z_0 \equiv 0$, we obtain by induction that

$$z_1 \equiv 0,\ z_2 \equiv 0,\ \ldots,$$

hence $Z \equiv 0$ and the proof is completed. $\square$

6. Uniqueness of weak solution

Let $\mathcal{M}$ be the space of $\mathcal{B}([0,T])\otimes\mathcal{F}$-measurable processes $Z\colon [0,T]\times\Omega\to V$ with continuous trajectories such that

$$Z\in\operatorname{Dom}\delta_H \quad \text{and} \quad \mathbb{E}\sup_{t\in[0,T]}\|Z_t\|_V^2<\infty.$$

The aim is to show that a weak solution to the equation

$$\mathrm{d}X_t=(AX_t+F(t))\,\mathrm{d}t+BX_t\,\mathrm{d}B_t^H,\quad X_0=x, \tag{6.1}$$

is unique in the space $\mathcal{M}$. To this purpose the following version of integration by parts formula is necessary.

Lemma 6.1. *Let $Y\in\mathcal{M}$ be a weak solution to the equation*

$$\mathrm{d}Y_t=AY_t\,\mathrm{d}t+BY_t\,\mathrm{d}B_t^H,\quad X_0=0.$$

Then

$$\langle Y_t,\zeta(t)\rangle_V=\int_0^t\langle Y_r,A^*\zeta(r)+\zeta'(r)\rangle_V\,\mathrm{d}r+\int_0^t\langle Y_r,B^*\zeta(r)\rangle_V\,\mathrm{d}B_r^H \tag{6.2}$$

for any $\zeta\in\mathcal{C}^1([0,T];D^*)$.

Proof. *1st step:* Let ζ have the form

$$\zeta(t)=\varphi(t)\xi,\quad \varphi\in\mathcal{C}^1([0,T]),\quad \xi\in D^*. \tag{6.3}$$

Let $\{t_k,k=0,\dots,n\}$ be the partition of interval $[0,t]$. Then

$$\begin{aligned}\langle Y_t,\zeta(t)\rangle_V=\varphi(t)\langle Y_t,\xi\rangle_V&=\sum_{k=0}^{n-1}(\varphi(t_{k+1})\langle Y_{t_{k+1}},\xi\rangle_V-\varphi(t_k)\langle Y_{t_k},\xi\rangle_V)\\&=\sum_{k=0}^{n-1}(\varphi(t_{k+1})-\varphi(t_k))\langle Y_{t_{k+1}},\xi\rangle_V\\&\quad+\sum_{k=0}^{n-1}\varphi(t_k)(\langle Y_{t_{k+1}},\xi\rangle_V-\langle Y_{t_k},\xi\rangle_V)=S_1+S_2.\end{aligned} \tag{6.4}$$

Since

$$\left|\sum_{k=0}^{n-1}(\varphi(t_{k+1})-\varphi(t_k))\langle Y_{t_{k+1}},\xi\rangle_V\right|\leqslant\|\varphi\|_{\mathcal{C}^1([0,T])}\sup_{r\in[0,T]}\|Y_r\|_V\left(\sum_{k=0}^{n-1}(t_{k+1}-t_k)\right)\|\xi\|_V$$

and

$$\mathbb{E}\Big[\sup_{r\in[0,T]}\|Y_r\|_V^2\Big]<\infty,$$

it follows that

$$S_1=\sum_{k=0}^{n-1}(\varphi(t_{k+1})-\varphi(t_k))\langle Y_{t_{k+1}},\xi\rangle_V \underset{n\to\infty}{\longrightarrow}\int_0^t \varphi'(r)\langle Y_r,\xi\rangle_V\,\mathrm{d}r \quad \text{in } L^2(\Omega)$$

in virtue of the Lebesgue dominated convergence theorem. The second sum S_2 can be split into two summands

$$S_2=\sum_{k=0}^{n-1}\varphi(t_k)\bigg(\int_{t_k}^{t_{k+1}}\langle Y_r,A^*\xi\rangle_V\,\mathrm{d}r+\int_{t_k}^{t_{k+1}}\langle Y_r,B^*\xi\rangle_V\,\mathrm{d}B_r^H\bigg)=S_{21}+S_{22}.$$

The first summand is

$$S_{21}=\sum_{k=0}^{n-1}\varphi(t_k)\int_{t_k}^{t_{k+1}}\langle Y_r,A^*\xi\rangle_V\,\mathrm{d}r \underset{n\to\infty}{\longrightarrow}\int_0^t \varphi(r)\langle Y_r,A^*\xi\rangle_V\,\mathrm{d}r \quad \text{in } L^2(\Omega)$$

by the Lebesgue dominated convergence theorem, because

$$\bigg|\sum_{k=0}^{n-1}\varphi(t_k)\int_{t_k}^{t_{k+1}}\langle Y_r,A^*\xi\rangle_V\,\mathrm{d}r\bigg|\leqslant\|\varphi\|_{\mathcal{C}^1([0,T])}\sup_{r\in[0,T]}\|Y_r\|_V\|A^*\xi\|_V T.$$

Since $Y_t\in L^2(\Omega)$ satisfies (6.4), we conclude that

$$S_{22}=\sum_{k=0}^{n-1}\varphi(t_k)\int_{t_k}^{t_{k+1}}\langle Y_r,B^*\xi\rangle_V\,\mathrm{d}B_r^H=\int_0^t\sum_{k=0}^{n-1}\varphi(t_k)I_{(t_k,t_{k+1}]}(r)\langle Y_r,B^*\xi\rangle_V\,\mathrm{d}B_r^H$$

converges to a random variable denoted by Y_t^1 in $L^2(\Omega)$ as $n\to\infty$. It remains to show that $Y_t^1=\int_0^t\varphi(r)\langle Y_r,B^*\xi\rangle_V\,\mathrm{d}B_r^H$ by using the closedness of Skorokhod integral. Denote

$$\Phi_n(r)=\sum_{k=0}^{n-1}\varphi(t_k)I_{(t_k,t_{k+1}]}(r)\langle Y_r,B^*\xi\rangle_V,\quad r\in[0,t],\ n\in\mathbb{N}.$$

Then $\Phi_n\in\operatorname{Dom}\delta_H$,

$$\Phi_n(r)\underset{n\to\infty}{\longrightarrow}\varphi(r)\langle Y_r,B^*\xi\rangle_V$$

for any fixed r, ω, and

$$
\begin{aligned}
|\Phi_n(r)| &= \left|\sum_{k=0}^{n-1} \varphi(t_k) I_{(t_k,t_{k+1}]}(r) \langle Y_r, B^*\xi\rangle_V\right| \\
&\leqslant \|\varphi\|_{\mathcal{C}^1([0,T])} \sup_{r\in[0,T]} \|Y_r\|_V \|B^*\xi\|_V \sum_{k=0}^{n-1} I_{(t_k,t_{k+1}]}(r) \\
&= \|\varphi\|_{\mathcal{C}^1([0,T])} \sup_{r\in[0,T]} \|Y_r\|_V \|B^*\xi\|_V.
\end{aligned}
$$

By the Lebesgue dominated convergence theorem $\Phi_n \in L^2(\Omega; L^2([0,t]; V))$ and

$$
\Phi_n \underset{n\to\infty}{\longrightarrow} \varphi\langle Y, B^*\xi\rangle_V \quad \text{in } L^2(\Omega; L^2([0,t]; V)).
$$

By the closedness of Skorokhod integral $Y_t^1 = \int_0^t \varphi(r)\langle Y_r, B^*\xi\rangle_V \, \mathrm{d}B_r^H$ and equality (6.2) holds for ζ of the form (6.3).

2nd step: Let $\zeta \in \mathcal{C}^1([0,T]; D^*)$. Then there exists a sequence $\{\zeta_n, n \in \mathbb{N}\} \subset \mathcal{C}^1([0,T]; D^*)$ of elementary functions of the form (6.3) such that $\zeta_n \underset{n\to\infty}{\longrightarrow} \zeta$ in $\mathcal{C}^1([0,T]; D^*)$. The aim is to pass to the limit in the equation

$$
\langle Y_t, \zeta_n(t)\rangle_V = \int_0^t \langle Y_r, A^*\zeta_n(r) + \zeta'(r)\rangle_V \, \mathrm{d}r + \int_0^t \langle Y_r, B^*\zeta_n(r)\rangle_V \, \mathrm{d}B_r^H
$$

in $L^2(\Omega)$. Clearly,

$$
|\langle Y_t, \zeta_n(t) - \zeta(t)\rangle_V| \leqslant \sup_{r\in[0,T]} \|Y_r\|_V \|\zeta_n - \zeta\|_{\mathcal{C}^1([0,T];D^*)} \underset{n\to\infty}{\longrightarrow} 0
$$

and

$$
\begin{aligned}
&\mathbb{E}\left[\left(\int_0^t \langle Y_r, A^*(\zeta_n(r) - \zeta(r)) + (\zeta_n'(r) - \zeta'(r))\rangle_V \, \mathrm{d}r\right)^2\right] \\
&\qquad \leqslant \mathbb{E}\Big[\sup_{r\in[0,T]} \|Y_r\|_V^2\Big] \|\zeta_n - \zeta\|_{\mathcal{C}^1([0,T];D^*)}^2 T^2 \underset{n\to\infty}{\longrightarrow} 0,
\end{aligned}
$$

thus $\int_0^t \langle Y_r, B^*\zeta_n(r)\rangle_V \, \mathrm{d}B_r^H \underset{n\to\infty}{\longrightarrow} Y_t^2$ in $L^2(\Omega)$. By the closedness of Skorokhod integral, $Y_t^2 = \int_0^t \langle Y_r, B^*\zeta(r)\rangle_V \, \mathrm{d}B_r^H$, because

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^t \langle Y_r, B^*(\zeta_n(r) - \zeta(r))\rangle_V^2 \, \mathrm{d}r\right] \\
&\qquad \leqslant \mathbb{E}\Big[\sup_{r\in[0,T]} \|Y_r\|_V^2\Big] \|B^*\|_{\mathcal{L}(V)}^2 \|\zeta_n - \zeta\|_{\mathcal{C}^1([0,T];D^*)}^2 T \underset{n\to\infty}{\longrightarrow} 0
\end{aligned}
$$

and $\langle Y, B^*\zeta_n\rangle_V \in \operatorname{Dom}\delta_H$ for any $n \in \mathbb{N}$. $\square$

Now, we are able to prove the uniqueness result.

Theorem 6.2. *Under the assumptions of Theorem* 4.7 *the solution to the equation* (6.1) *is unique in the space* $\mathcal{M}$.

P r o o f. Let X^M be the solution to the equation

$$X_t^M = U_Y(t,0)x + \int_0^t U_Y(t,r)F(r)\,\mathrm{d}r, \quad t \in [0,T],$$

which is also a weak one to (6.1) (Theorem 4.7), where

$$U_Y(t,s) = S_B(B_t^H - B_s^H)U(t-s,0), \quad s \leqslant t \leqslant T,$$

(for more details see (3.5)). Using the notation from Section 3,

$$\|U_Y(t,s)\|_{\mathcal{L}(V)} \leqslant M_B \exp\{2\omega_B \|B^H\|_{\mathcal{C}([0,T])}\} C_U, \quad 0 \leqslant s \leqslant t \leqslant T,$$

yields

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \|X_t^M\|_V^2\Big] \leqslant 2C_U^2 M_B^2 (\|x\|_V^2 \mathbb{E}\exp\{2\omega_B\|B^H\|_{\mathcal{C}([0,T])}\} + \|F\|_{L^2([0,T])}^2 T\,\mathbb{E}\exp\{4\omega_B \|B^H\|_{\mathcal{C}([0,T])}\}) < \infty,$$

by the Fernique Theorem. Therefore, $X^M \in \mathcal{M}$ (the continuity of trajectories is guaranteed by Theorem 4.1).

Take another weak solution $X^1 \in \mathcal{M}$ to (6.1) and define

$$\overline{X} := X^1 - X^M.$$

Then $\overline{X}$ is a weak solution to the equation

$$\mathrm{d}\overline{X}_t = A\overline{X}_t\,\mathrm{d}t + B\overline{X}_t\,\mathrm{d}B_t^H, \quad \overline{X}_0 = 0. \tag{6.5}$$

Hence, applying Lemma 6.1 to $\langle \overline{X}_t, \xi\rangle_V$ for any fixed $\xi \in D^*$ and $\zeta(s) = S_A^*(t-s)\xi$, $s \in [0,t]$, we obtain

$$\begin{aligned}\langle \overline{X}_t, \xi\rangle_V &= \int_0^t \langle \overline{X}_r, A^* S_A^*(t-r)\xi - S_A^*(t-r)A^*\xi\rangle_V\,\mathrm{d}r \\ &\quad + \int_0^t \langle \overline{X}_r, B^* S_A^*(t-r)\xi\rangle_V\,\mathrm{d}B_r^H \\ &= \int_0^t \langle S_A(t-r)B\overline{X}_r, \xi\rangle_V\,\mathrm{d}B_r^H = \Big\langle \int_0^t S_A(t-r)B\overline{X}_r\,\mathrm{d}B_r^H, \xi\Big\rangle_V.\end{aligned}$$

Therefore,

$$\overline{X}_t = \int_0^t S_A(t-r)B\overline{X}_r \,\mathrm{d}B_r^H, \quad t \in [0,T],$$

so $\overline{X}$ is also a mild solution to the equation (6.5) and by the uniqueness result in Theorem 5.3 we have

$$\overline{X}_t = \int_0^t S_A(t-r)B\overline{X}_r \,\mathrm{d}B_r^H = S_B(B_t^H)U(t,0)0 = 0,$$

hence $X^1 = X^M$. $\square$

Corollary 6.3. *The weak solution* $\{S_B(B_t^H)U(t,0)x,\ t \in [0,T]\}$ *to the equation* (5.1) *is unique in* $\mathcal{M}$.

In particular, the solution

$$X_t = \exp\{bB_t^H - \tfrac{1}{2}b^2t^{2H} + at\}x, \quad t \in [0,T],$$

to the one-dimensional equation

$$\mathrm{d}X_t = aX_t \,\mathrm{d}t + bX_t \,\mathrm{d}B_t^H, \quad X_0 = x,$$

is unique in $\mathcal{M}$.

Acknowledgements. The authors would like to thank M. Jolis for her help with Section 5, J. Seidler and anonymous referees for valuable suggestions and comments.

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