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ABOUT THE EQUIVALENCE OF NULLNORMS ON BOUNDED LATTICE

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In this paper, an equivalence on the class of nullnorms on a bounded lattice based on the equality of the orders induced by nullnorms is introduced. The set of all incomparable elements w.r.t. the order induced by nullnorms is investigated. Finally, the recently posed open problems have been solved.

Keywords: nullnorm, nounded lattice, partial order, equivalence

Classification: 03E72, 03B52

1. INTRODUCTION

Nullnorms generalizing the notions of t-norms and t-conorms [5, 19] introduced in [4] and [17] are interesting not only from a theoretical point of view, but also for their applications in several fields like experts systems, neural networks, fuzzy quantifiers [18].

Recently, the order generating problem from a logical operator has been attractive for many researchers [6, 8, 11, 12, 13, 14]. In this sense, in [11], a partial order called as the T-partial order induced by a t-norm has been introduced. As further works, in the studies [1] and [6], the orders, denoted by $\preceq_V$ and $\preceq_U$ respectively, on a bounded lattice has been defined and studied their properties. Also, in [1], it has been shown that the orders $\preceq_U$ and $\preceq_V$ do not coincide in general. In [1] again, the set of incomparable elements w.r.t. the order $\preceq_V$ for any nullnorm on $[0, 1]$ has been defined and investigated.

In the present paper, we introduce an equivalence on the class of nullnorms on a bounded lattice L based on the equality of the orders induced by nullnorms. The main aim of this paper is to present the relations between the equivalence classes of nullnorms and the equivalence classes of their underlying t-norms and t-conorms. The paper is organized as follows: In Section 2, we shortly recall some basic notions and results. In Section 3, we define an equivalence on the class of nullnorms on a bounded lattice with a zero element a and we determine that two idempotent nullnorms are equivalent. Also, we show that for the equivalence of two nullnorms a necessary and sufficient condition is the equivalence of their underlying t-norms and t-conorms, respectively. We obtain a necessary and sufficient condition for nullnorms and their ϕ-conjugates to be

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in the same equivalence classes. In Section 4, we define the set of all incomparable elements w.r.t. $\preceq_V$, denoted by K_V and characterized exactly. We present some relations between the set K_V and the sets of all incomparable elements w.r.t. the order induced by the corresponding underlying t-norm and t-conorm, denoted by K_{T_V} and K_{S_V}, respectively. Finally, for the open problem posed in the study [1] stated as "Is K_F, F is a nullnom, always an interval or can it be a union of intervals?", we give an example illustrating that K_F need not be an interval and it can be a union of some intervals. Also, for the open problem given in the study [1] stated as "Given an $(a, b) \subsetneq (0, 1)$, can we find a nullnorm F such that $K_F = (a, b)$?", by Theorem 4.18, we proof that for any $(u, v) \subsetneq (0, 1)$ not consisting an element $a \in [0, 1]$ there exists a nullnorm V on $[0, 1]$ with the zero element a such that $K_V = (u, v)$.

2. NOTATIONS, DEFINITIONS AND A REVIEW OF PREVIOUS RESULTS

Definition 2.1. (Karaçal and Kesicioğlu [11], Ma and Wu [16])
An operation T (S) on a bounded lattice L is called a triangular norm (triangular conorm) if it is commutative, associative, increasing with respect to the both variables and has a neutral element 1 (0).

Definition 2.2. (Karaçal and Kesicioğlu [11], Kesicioğlu et al. [13])
A t-norm T (or a t-conorm S) on a bounded lattice L is divisible if the following condition holds:

For all $x, y \in L$ with $x \leq y$ there is $z \in L$ such that $x = T(y, z)$ (or $y = S(x, z)$).

Definition 2.3. (Karaçal et al. [9])
Let $(L, \leq, 0, 1)$ be a bounded lattice. An operation $V : L^2 \to L$ is called a nullnorm on L, if it is commutative, associative, increasing with respect to the both variables and has a zero (absorbing) element $a \in L$ such that for all $x \leq a$, $V(x, 0) = x$ and for all $x \geq a$, $V(x, 1) = x$.

In this study, the notation $\mathcal{V}(a)$ will be used for the set of all nullnorms on L with a zero element $a \in L$.

Definition 2.4. (Karaçal and Mesiar [10])
Let $(L, \leq, 0, 1)$ be a bounded lattice. An operation $U : L^2 \to L$ is called a uninorm on L, if it is commutative, associative, increasing with respect to the both variables and has a neutral element $e \in L$.

Proposition 2.5. (Karaçal et al. [9])
Let $(L, \leq, 0, 1)$ be a bounded lattice, and $V \in \mathcal{V}(a)$. Then

(i) $S_V = V \mid_{[0,a]^2} : [0, a]^2 \to [0, a]$ is a t-conorm on $[0, a]$.

(ii) $T_V = V \mid_{[a,1]^2} : [a, 1]^2 \to [a, 1]$ is a t-norm on $[a, 1]$.

S_V and T_V given in Proposition 2.5 are called the underlying t-conorm and t-norm of V, respectively.

In the whole of the paper, we will use T_V for the underlying t-norm and S_V for the underlying t-conorm of a given nullnorm V.

Definition 2.6. (Grabisch et al. [7])
Let $(L, \leq, 0, 1)$ be a bounded lattice and $V \in \mathcal{V}(a)$. An element $x \in L$ is called an idempotent element of V if $V(x,x) = x$.

Moreover a nullnorm is called idempotent nullnorm whenever $V(x,x) = x$ for all $x \in L$.

Definition 2.7. (Baczyński and Jayaram [2], Kesicioğlu and Mesiar [14], Ma and Wu [16]) Let $(L, \leq, 0, 1)$ be a bounded lattice. A decreasing function $N : L \to L$ is called a negation if $N(0) = 1$ and $N(1) = 0$. A negation N on L is called strong if it is an involution, i. e., $N(N(x)) = x$, for all $x \in L$.

Definition 2.8. (Baczyński and Jayaram [2])
Let T be a t-norm on a bounded lattice L and N be a strong negation on L. The t-conorm S defined by

$$S(x,y) = N(T(N(x), N(y))),\ x, y \in L$$

is called the N-dual t-conorm to T on L.

Definition 2.9. (Karaçal and Kesicioğlu [11])
Let L be a bounded lattice, T be a t-norm on L. The order defined as the following is called a $T-$ *partial order* (triangular order) for the t-norm T:

$$x \preceq_T y \Leftrightarrow T(\ell, y) = x \text{ for some } \ell \in L.$$

Similarly, the notion $S-$ *partial order* can be defined as follows:

Definition 2.10. (Ertuğrul et al. [6])
Let L be a bounded lattice, S be a t-conorm on L. The order defined as the following is called a $S-$ *partial order* for t-conorm S:

$$x \preceq_S y \Leftrightarrow S(\ell, x) = y \text{ for some } \ell \in L.$$

Definition 2.11. (Kesicioğlu et al. [13])
Let $(L, \leq, 0, 1)$ be a given bounded lattice. Define a relation $\sim$ on the class of all t-norms on $(L, \leq, 0, 1)$ by $T_1 \sim T_2$ if and only if the T_1- partial order coincides with the T_2- partial order.

Lemma 2.12. (Kesicioğlu et al. [13])
The relation $\sim$ given in Definition 3.1 is an equivalence relation.

Also, an equivalence relation for two t-conorms can be given as similar to the equivalence relation in Definition 2.11.

Definition 2.13. (Aşıcı [1])
Let $(L, \leq, 0, 1)$ be a bounded lattice and $V \in \mathcal{V}(a)$. The order defined as the following

is called a V-partial order for V: For every $x, y \in L$

$$x \preceq_V y \Leftrightarrow \begin{cases} \text{if} \quad x, y \in [0,a] \quad \text{and there exists} \quad k \in [0,a] \quad \text{such that} \\ \quad V(k,x) = y \quad \text{or,} \\ \text{if} \quad x, y \in [a,1] \quad \text{and there exists} \quad \ell \in [a,1] \quad \text{such that} \\ \quad V(y,\ell) = x \quad \text{or,} \\ \text{if} \quad (x,y) \in L^* \quad \text{and} \quad x \leq y, \end{cases} \tag{1}$$

where $I_a = \{x \in L \mid x \| a\}$ and $L^* = [0,a] \times [a,1] \cup [0,a] \times I_a \cup [a,1] \times [0,a] \cup [a,1] \times I_a \cup I_a \times [0,a] \cup I_a \times [a,1] \cup I_a \times I_a$.

Here, note that the notation $x||y$ denotes that x and y are incomparable.

Proposition 2.14. (Aşıcı [1])
Let $(L, \leq, 0, 1)$ be a bounded lattice and $V \in \mathcal{V}(a)$. If $x \preceq_V y$ for any $x, y \in L$, then $x \leq y$.

Definition 2.15. (Klement et al. [15])
If T is a t-norm on the unit interval $[0,1]$ and $\phi : [0,1] \to [0,1]$ an order-preserving bijection, then the operation $T_\phi : [0,1]^2 \to [0,1]$ given by

$$T_\phi(x,y) = \phi^{-1}(T(\phi(x), \phi(y)))$$

is also a t-norm. This t-norm is called ϕ-conjugate of T.

The ϕ-conjugate of a t-norm (nullnorm, t-conorm) on a bounded lattice is defined as similar to Definition 2.15.

Definition 2.16. (Birkhoff [3])
Let $(L, \leq, 0, 1)$ be a bounded lattice. If there exists an element $y \in L$ for an element $x \in L$ such that $x \wedge y = 0$ and $x \vee y = 1$, then the element y is called as a complement of x.

L is called as a complemented lattice if all elements have complements.

L is called relatively complemented if all intervals are complemented.

3. THE EQUIVALENCE CLASS

Definition 3.1. Let $(L, \leq, 0, 1)$ be a given bounded lattice. Define a relation $\sim$ on the class $\mathcal{V}(a)$ by $V_1 \sim V_2$ if and only if the V_1- partial order coincides with the V_2- partial order.

The next result is obvious.

Proposition 3.2. The relation $\sim$ given in Definition 3.1 is an equivalence relation.

Definition 3.3. Let $(L, \leq, 0, 1)$ be a bounded lattice and $V \in \mathcal{V}(a)$. We denote by $\overline{V}$ the $\sim$ equivalence class linked to V, i. e.

$$\overline{V} = \{V^{'} \mid \quad V^{'} \sim V\}.$$

Proposition 3.4. Let $(L, \leq, 0, 1)$ be a bounded lattice and $V_1, V_2 \in \mathcal{V}(a)$. If the underlying t-norms and t-conorms of V_1 and V_2 are divisible, then $V_1 \sim V_2$.

Proof. Let the underlying t-norms and t-conorms of V_1 and V_2 be divisible. Then, it follows $V_1 \sim V_2$ from $\preceq_{V_1} = \leq = \preceq_{V_2}$, by Proposition 5 in [1]. □

Corollary 3.5. Let V_1 and V_2 be two nullnorms on $[0, 1]$. If their underlying t-norms and t-conorms are continuous, then they are equal under the relation $\sim$.

The converse of Proposition 3.4 need not be true. Let us investigate the following example.

Example 3.6. Take two functions V_1 and V_2 on $[0, 1]$ defined as follows:

$$V_1(x,y) = \begin{cases} y & x = 0 \quad \text{and} \quad y \leq 1/2, \\ x & y = 0 \quad \text{and} \quad x \leq 1/2, \\ \min(x,y) & (x,y) \in [\frac{1}{2}, 1]^2, \\ \frac{1}{2} & \text{otherwise,} \end{cases}$$

and

$$V_2(x,y) = \begin{cases} y & x = 0 \quad \text{and} \quad y \leq 1/2, \\ x & y = 0 \quad \text{and} \quad x \leq 1/2, \\ xy & (x,y) \in [\frac{1}{2}, 1]^2, \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

By Corollary 7 in [9], V_1 and V_2 are nullnorms on $[0, 1]$ with zero element $1/2$, and $V_1, V_2 \mid_{[0,\frac{1}{2}]^2} = S_D$, $V_1 \mid_{[\frac{1}{2},1]^2} = T_M$, $V_2 \mid_{[\frac{1}{2},1]^2} = T_P$. Also, it can be easily seen that $\preceq_{V_1} = \preceq_{V_2}$, i.e., $V_1 \sim V_2$. Although, the underlying t-conorm of V_1 and V_2, S_D, is not continuous.

Theorem 3.7. Let $(L, \leq, 0, 1)$ be a bounded lattice and $V_1, V_2 \in \mathcal{V}(a)$. Then, $T_{V_1} \sim T_{V_2}$ and $S_{V_1} \sim S_{V_2}$ if and only if $V_1 \sim V_2$.

Proof. Let $x \preceq_{V_1} y$ for any $x, y \in L$. Suppose that $x, y \in [0, a]$. Then, there exists an element ℓ of $[0, a]$ such that

$$y = V_1(\ell, x) = V_1 \mid_{[0,a]^2} (\ell, x) = S_{V_1}(\ell, x).$$

Thus, we have that $x \preceq_{S_{V_1}} y$. Since $S_{V_1} \sim S_{V_2}$, it is obtained that $x \preceq_{S_{V_2}} y$. Then, there exists an element $\ell^* \in [0, a]$ such that

$$y = S_{V_2}(\ell^*, x) = V_2 \mid_{[0,a]^2} (\ell^*, x) = V_2(\ell^*, x),$$

which implies that $x \preceq_{V_2} y$.

Let $x, y \in [a, 1]$. Then, there exists an element $\ell \in [a, 1]$ such that

$$x = V_1(\ell, y) = V_1 \mid_{[a,1]^2} (\ell, y) = T_{V_1}(\ell, y),$$

whence $x \preceq_{T_{V_1}} y$. Since $T_{V_1} \sim T_{V_2}$, we have that $x \preceq_{T_{V_2}} y$. Then, there exists an element $\ell^* \in [a,1]$ such that

$$x = T_{V_2}(\ell^*, y) = V_2 \mid_{[a,1]^2} (\ell^*, y) = V_2(\ell^*, y),$$

which implies that $x \preceq_{V_2} y$.

Suppose that $x, y \notin [0,a]$ and $x, y \notin [a,1]$. Then, $(x,y) \in L^*$. Thus, we have that $x \leq y$ from $x \preceq_{V_1} y$. Since $x \leq y$ and $(x,y) \in L^*$, by the definition of $\preceq_{V_2}$, we obtain that $x \preceq_{V_2} y$. So, we have that $x \preceq_{V_1} y$ which implies that $x \preceq_{V_2} y$ for any $x, y \in L$.

It is clear that the similar arguments are also true when V_1 is replaced by V_2. That is, $x \preceq_{V_2} y$ implies that $x \preceq_{V_1} y$ for any $x, y \in L$. Then, we have that $\preceq_{V_1} = \preceq_{V_2}$, whence $V_1 \sim V_2$.

Conversely, let $V_1 \sim V_2$. Then, $\preceq_{V_1} = \preceq_{V_2}$. Since $\preceq_{V_1}|_{[0,a]^2} = \preceq_{V_2}|_{[0,a]^2}$ and $\preceq_{V_1}|_{[a,1]^2} = \preceq_{V_2}|_{[a,1]^2}$, we have that $\preceq_{S_{V_1}} = \preceq_{S_{V_2}}$ and $\preceq_{T_{V_1}} = \preceq_{T_{V_2}}$, which implies that $S_{V_1} \sim S_{V_2}$ and $T_{V_1} \sim T_{V_2}$. $\square$

Proposition 3.8. (Kesicioğlu et al. [12]) Let $(L, \leq, 0, 1)$ be a bounded lattice , S_1 and S_2 the N-dual t-conorms of two t-norms T_1 and T_2 on L, respectively. Then, $T_1 \sim T_2$ iff $S_1 \sim S_2$.

Corollary 3.9. Let $(L, \leq, 0, 1)$ be a bounded lattice and $V_1, V_2 \in \mathcal{V}(a)$. Let S_{V_1} and S_{V_2} be the N-dual t-conorms of T_{V_1} and T_{V_2}, respectively. Then, $T_{V_1} \sim T_{V_2}$ iff $V_1 \sim V_2$.

Proposition 3.10. Let T (S) be a t-norm (t-conorm) on a bounded lattice $(L, \leq, 0, 1)$ and let $\phi : L \to L$ be a $\leq$-preserving bijection. The following statements are equivalent:

(i) $T \sim T_\phi$ $(S \sim S_\phi)$,

(ii) ϕ is $\preceq_T$ $(\preceq_S)$-preserving. That is, for all $x, y \in L$, $x \preceq_T y$ $(x \preceq_S y)$ iff $\phi(x) \preceq_T \phi(y)$ $(\phi(x) \preceq_S \phi(y))$.

Proof. Since the proof is similar to the case $L = [0,1]$, Proposition 5 in [13], we omit its proof. $\square$

Proposition 3.11. Let $(L, \leq, 0, 1)$ be a bounded lattice, $V \in \mathcal{V}(a)$ and let ϕ be a $\leq$-preserving bijection on L with $\phi(a) = a$. Then, ϕ is $\preceq_{T_V}$ and $\preceq_{S_V}$-preserving iff ϕ is $\preceq_V$-preserving.

Proof. Let ϕ be $\preceq_{T_V}$ and $\preceq_{S_V}$-preserving. Let $x \preceq_V y$ for any $x, y \in L$.

• Suppose that $x, y \in [0,a]$. In this case, it is clear that $\phi(x), \phi(y) \in [0,a]$ from $\phi(a) = a$. Thus, since ϕ is $\preceq_{S_V}$-preserving, the following equivalents hold:

$$\begin{aligned} x \preceq_V y \Leftrightarrow x \preceq_{S_V} y \quad &\Leftrightarrow \quad \phi(x) \preceq_{S_V} \phi(y) \\ &\Leftrightarrow \quad \phi(x) \preceq_V \phi(y). \end{aligned}$$

• Let $x, y \in [a, 1]$. By $\phi(a) = a$, it is obvious that $\phi(x), \phi(y) \in [a, 1]$. Since ϕ is $\preceq_{T_V}$-preserving,

$$\begin{aligned} x \preceq_V y \Leftrightarrow x \preceq_{T_V} y \quad &\Leftrightarrow \quad \phi(x) \preceq_{T_V} \phi(y) \\ &\Leftrightarrow \quad \phi(x) \preceq_V \phi(y) \end{aligned}$$

hold.

• Let $x, y \notin [0, a]$ and $x, y \notin [a, 1]$. Clearly, $(x, y) \in L^*$, whence $(\phi(x), \phi(y)) \in L^*$. In this case,

$$\begin{aligned} x \preceq_V y \Leftrightarrow x \leq y \quad &\Leftrightarrow \quad \phi(x) \leq \phi(y) \\ &\Leftrightarrow \quad \phi(x) \preceq_V \phi(y) \end{aligned}$$

hold. Therefore, ϕ is $\preceq_V$-preserving.

Conversely, let ϕ be $\preceq_V$-preserving. We shall show that ϕ is $\preceq_{S_V}$-preserving. Since for any $x, y \in [0, a]$, $\phi(x), \phi(y) \in [0, a]$ by $\phi(a) = a$, the following equalities hold:

$$\begin{aligned} x \preceq_{S_V} y \quad &\Leftrightarrow \quad S_V(\ell, x) = y, \quad \text{for some} \quad \ell \in [0, a] \\ &\Leftrightarrow \quad V(\ell, x) = S_V(\ell, x) = y, \quad \text{for some} \quad \ell \in [0, a] \\ &\Leftrightarrow \quad x \preceq_V y \\ &\Leftrightarrow \quad \phi(x) \preceq_V \phi(y) \\ &\Leftrightarrow \quad V(\ell^*, \phi(x)) = \phi(y), \quad \text{for some} \quad \ell^* \in [0, a] \\ &\Leftrightarrow \quad \phi(y) = V(\ell^*, \phi(x)) = S_V(\ell^*, \phi(x)), \quad \text{for some} \quad \ell^* \in [0, a] \\ &\Leftrightarrow \quad \phi(x) \preceq_{S_V} \phi(y). \end{aligned}$$

Thus, ϕ is $\preceq_{S_V}$-preserving.

Similarly, it can be shown that ϕ is $\preceq_{T_V}$-preserving. □

Theorem 3.12. Let $(L, \leq, 0, 1)$ be a bounded lattice, $V \in \mathcal{V}(a)$ and let $\phi : L \to L$ be a $\leq$-preserving bijection with $\phi(a) = a$. Then, $V \sim V_\phi$ iff ϕ is $\preceq_V$-preserving.

Proof. For $S_V = V\mid_{[0,a]^2}$ and $T_V = V\mid_{[a,1]^2}$, it is clear that $(V_\phi)\mid_{[0,a]^2} = (S_V)_\phi$ and $(V_\phi)\mid_{[a,1]^2} = (T_V)_\phi$. Then, the following equalities hold:

$$\begin{aligned} \phi \quad \text{is} \preceq_V \text{-preserving} \quad &\Leftrightarrow \quad \phi \quad \text{is} \preceq_{S_V} \quad \text{and} \quad \preceq_{T_V} \text{-preserving} \\ &\qquad \text{(by Proposition 3.11)} \\ &\Leftrightarrow \quad S_V \sim (S_V)_\phi \quad \text{and} \quad T_V \sim (T_V)_\phi \\ &\qquad \text{(by Proposition 3.10)} \\ &\Leftrightarrow \quad V \sim V_\phi \text{ (by Theorem 3.7)} \end{aligned}$$

□

4. THE SET K_V OF ALL INCOMPARABLE ELEMENTS W.R.T. THE ORDER $\preceq_V$

Let $(L, \leq, 0, 1)$ be a bounded lattice and M be a nullnorm (t-norm,t-conorm) on L. In the whole study, we will use the notation $x||_{\preceq_M} y$ for any incomparable elements x and y w.r.t. the order $\preceq_M$.

Definition 4.1. Let $(L, \leq, 0, 1)$ be a bounded lattice and $V \in \mathcal{V}(a)$. Let K_V be defined as

$$K_V = \{x \in L \quad | \quad (\exists y \in L) \quad (x||_{\preceq_V} y)\}.$$

Clearly, K_{S_V} and K_{T_V} are defined as

$$K_{S_V} = \{x \in [0, a] \quad | \quad (\exists y \in [0, a]) \quad (x||_{\preceq_{S_V}} y)\}$$

and

$$K_{T_V} = \{x \in [a, 1] \quad | \quad (\exists y \in [a, 1]) \quad (x||_{\preceq_{T_V}} y)\}.$$

By Definition 4.1, if L is not a chain, it is clear that $K_V \neq \emptyset$. The converse of this claim may fail. Let us investigate the following example.

Example 4.2. Consider the lattice $L = \{0, x, y, a, 1\}$, with $0 < x < y < a < 1$.
Define the function $V : L^2 \to L$ as in Table 1:

V	0	x	y	a	1
0	0	x	y	a	a
x	x	a	a	a	a
y	y	a	a	a	a
a	a	a	a	a	a
1	a	a	a	a	1

Tab. 1. The nullnorm V on L.

Obviously, V is a nullnorm and the order $\preceq_V$ is depicted on Figure 1:

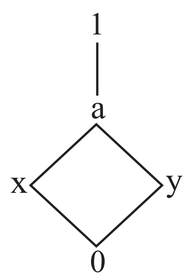

Fig. 1. $(L, \preceq_V)$.

As it can be seen from the Figure 1, $K_V = \{x, y\} \neq \emptyset$.

Proposition 4.3. Let $(L, \leq, 0, 1)$ be a bounded lattice and $V \in \mathcal{V}(a)$. Then,

$$K_{S_V} \subseteq K_V \cap [0, a] \quad \text{and} \quad K_{T_V} \subseteq K_V \cap [a, 1].$$

Proof. Let $x \in K_{S_V}$. Then, there exists an element $y \in [0, a]$ such that

$$x \| _{\preceq_{S_V}} y.$$

If $x \preceq_V y$, there would exist an element $\ell \in [0, a]$ such that

$$V(\ell, x) = y.$$

Thus, since $y = V(\ell, x) = S_V(\ell, x)$, we would have that $x \preceq_{S_V} y$, a contradiction. If $y \preceq_V x$, there would obtain a similar contradiction since $x \| _{\preceq_{S_V}} y$. Thus, it must be $x \not\preceq_V y$ and $y \not\preceq_V x$, that is, $x \| _{\preceq_V} y$. Then, $x \in K_V$. Hence, $K_{S_V} \subseteq K_V \cap [0, a]$.

Similarly, it can be shown that $K_{T_V} \subseteq K_V \cap [a, 1]$. □

Remark 4.4. In Proposition 4.3, it need not be $K_{S_V} = K_V \cap [0, a]$ and $K_{T_V} = K_V \cap [a, 1]$. Let us investigate the following example.

Example 4.5. Consider the lattice $L = \{0, x, a, y, 1\}$ whose lattice diagram is depicted on Figure 2:

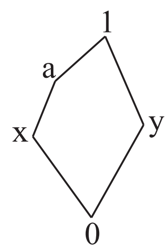

Fig. 2. $(L, \leq)$.

Define a function $V : L^2 \to L$ as in Table 2:

V	0	x	y	a	1
0	0	x	a	a	a
x	x	x	a	a	a
y	a	a	a	a	a
a	a	a	a	a	a
1	a	a	a	a	a

Tab. 2. The nullnorm V on L.

It can be easily shown that V is a nullnorm on L and $\preceq_V = \leq$. Also, since $K_{S_V} = \emptyset$ and $K_V \cap [0, a] = \{a, x\}$, we have that $\emptyset = K_{S_V} \neq K_V \cap [0, a] = \{a, x\}$.

In the following Proposition, we give a necessary and sufficient condition for the equality in Proposition 4.3.

Proposition 4.6. Let $(L, \leq, 0, 1)$ be a bounded lattice and $V \in \mathcal{V}(a)$. If $I_a = \emptyset$, then

$$K_{S_V} = K_V \cap [0, a] \quad \text{and} \quad K_{T_V} = K_V \cap [a, 1].$$

Conversely, if $K_{S_V} = K_V \cap [0, a]$ or $K_{T_V} = K_V \cap [a, 1]$, then $I_a = \emptyset$.

Proof. By Proposition 4.3, we know that $K_{S_V} \subseteq K_V \cap [0, a]$ and $K_{T_V} \subseteq K_V \cap [a, 1]$. Now, let us prove that the converses of them are also true. Let $x \in K_V \cap [0, a]$. Then, $x \in K_V$ and $x \in [0, a]$. By the definition of K_V, there exists an element $y \in L$ such that

$$x||_{\preceq_V} y.$$

Since $I_a = \emptyset$, for $y \in L$ either $y \leq a$ or $a \leq y$.

Suppose that $a \leq y$. Since $x \leq a \leq y$, it is clear that $x \preceq_V y$ by the definition of the order $\preceq_V$. This contradicts that $x||_{\preceq_V} y$. Hence, it must be $y \leq a$.

If $x \preceq_{S_V} y$, then there would exist an element $\ell \in [0, a]$ such that

$$S_V(\ell, x) = y.$$

Since $y = S_V(\ell, x) = V \mid_{[0,a]^2} (\ell, x) = V(\ell, x)$, we have that $x \preceq_V y$, which contradicts $x||_{\preceq_V} y$. Similarly, if $y \preceq_{S_V} x$, we would obtain a similar contradiction. Then, $x \npreceq_{S_V} y$ and $y \npreceq_{S_V} x$, which implies that $x \in K_{S_V}$. Thus, we have that $K_V \cap [0, a] \subseteq K_{S_V}$.

In a similar way, it can be shown that $K_{T_V} = K_V \cap [a, 1]$.

Conversely, let $K_{S_V} = K_V \cap [0, a]$ and $I_a \neq \emptyset$. Then, there exists at least an element $y \in I_a$. Since $y||a$, obviously $a \in K_V$. Since $a \in K_V \cap [0, a] = K_{S_V}$, we have that $a \in K_{S_V}$, a contradiction. Therefore, it must be $I_a = \emptyset$. □

Corollary 4.7. If V is a nullnorm on $[0, 1]$, then $K_{S_V} = K_V \cap [0, a]$ and $K_{T_V} = K_V \cap [a, 1]$.

Proposition 4.8. Let $(L, \leq, 0, 1)$ be a bounded lattice and $V \in \mathcal{V}(a)$. If $K_V = \emptyset$, then $\preceq_V = \leq$.

Proof. Let $K_V = \emptyset$. For any $x, y \in L$, by Proposition 2.14, it is clear that $x \preceq_V y$ implies that $x \leq y$. Conversely, let $x \leq y$ for any $x, y \in L$. Suppose that $x \npreceq_V y$. If $y \preceq_V x$, it would be obtained $y \leq x$, whence $x = y$. This contradicts that $x \npreceq_V y$. Then, it must be $y \npreceq_V x$. Since $x \npreceq_V y$ and $y \npreceq_V x$, we have that $x \in K_V$ which contradicts that $K_V = \emptyset$. Then, it must be $x \preceq_V y$. Thus, $\preceq_V$ and $\leq$ coincide. □

Remark 4.9. The converse of Proposition 4.8 need not be true. Let us investigate the following example.

Example 4.10. Consider the lattice $(L, \leq, 0, 1)$ and the nullnorm V on L as in Example 4.5. Then, it can be easily seen that $\preceq_V = \leq$ but $K_V = \{a, x, y\} \neq \emptyset$.

Proposition 4.11. Let $(L, \leq, 0, 1)$ be a complemented lattice and $V \in \mathcal{V}(a)$. Then, $K_V = L \setminus \{0, 1\}$.

Proof. It is clear that $K_V \subseteq L \setminus \{0, 1\}$. Conversely, let $x \in L \setminus \{0, 1\}$. Since L is a complemented lattice, for $x \in L \setminus \{0, 1\}$, there exists an element $x' \in L \setminus \{0, 1\}$ such that $x \wedge x' = 0$ and $x \vee x' = 1$. If $x \leq x'$, then it would be $x \wedge x' = x = 0$ and $x \vee x' = x' = 1$, a contradiction. If $x' \leq x$, then a similar contradiction would be obtained. Hence, it must be $x||x'$. Then, it is clear that $x||_{\preceq_V} x'$, whence $x \in K_V$. Therefore, $L \setminus \{0, 1\} \subseteq K_V$, which completes the proof. □

Corollary 4.12. Let $(L, \leq, 0, 1)$ be a bounded lattice and $V \in \mathcal{V}(a)$. If L is a relatively complemented lattice, then $K_V = L \setminus \{0, 1\}$.

Theorem 4.13. Let $(L, \leq, 0, 1)$ be a bounded lattice and $V \in \mathcal{V}(a)$. Then,

$$K_V = K_{S_V} \cup K_{T_V} \cup I_a \cup M,$$

where $M = \{x \in L \quad | \quad x||y \quad \text{for some} \quad y \in I_a\}$.

Proof. Since $K_{S_V}, K_{T_V}, I_a, M \subseteq K_V$, it is clear that $K_{S_V} \cup K_{T_V} \cup I_a \cup M \subseteq K_V$. Conversely, let $x \in K_V$ be arbitrary. Then, there exists an element $y \in L$ such that

$$x||_{\preceq_V} y.$$

- Let $x \in [0, a]$. If $y \in [a, 1]$, then it would be $x \leq a \leq y$. By the definition of $\preceq_V$, we would have that $x \preceq_V y$, which is a contradiction. Then, either $y \in [0, a]$ or $y||a$.

Let $y \in [0, a]$. In this case, three possible cases for x: $x < y$ or $y < x$ or $x||y$.

Let $x < y$. Suppose that $x \preceq_{S_V} y$. Then, there exists an element $\ell \in [0, a]$ such that $S_V(\ell, x) = y$. Since $y = S_V(\ell, x) = V(\ell, x)$, it is clear that $x \preceq_V y$, a contradiction. If $y \preceq_{S_V} x$, it would be $y \leq x$, which contradicts that $x < y$. So, we have that $x \in K_{S_V}$ from $x \not\preceq_{S_V} y$ and $y \not\preceq_{S_V} x$.

Let $y < x$ and $y \preceq_{S_V} x$. Then, there exists an element $\ell' \in [0, a]$ such that

$$S_V(\ell', y) = x.$$

Thus, since $x = S_V(\ell', y) = V(\ell', y)$, it is clear that $y \preceq_V x$, a contradiction. Therefore, it must be $y \not\preceq_{S_V} x$. If $x \preceq_{S_V} y$, it would be $x \leq y$, which contradicts that $y < x$. Then, we have that $x \in K_{S_V}$ from $x \not\preceq_{S_V} y$ and $y \not\preceq_{S_V} x$.

Let $x||y$. Since $x, y \in [0, a]$ and $x||y$, it is obvious that $x \in K_{S_V}$.

Suppose that $y||a$.

Let $x \leq y$. Since $(x, y) \in L^*$ and $x \leq y$, by the definition of $\preceq_V$, we have that $x \preceq_V y$, contradiction.

Since $y||a$, it is clear that $y \leq x$ does not hold. Otherwise, we would obtain that $y \leq a$. Hence, it must be $x||y$. Since $x||y$ and $y||a$, we have that $x \in M$.

- Let $x \in [a, 1]$.

If $y \in [0,a]$, it would be $y \preceq_V x$ from $y \leq x$ and $(y,x) \in L^*$, a contradiction. Then, either $y \in [a,1]$ or $y||a$.

Suppose that $y \in [a,1]$. If $y||x$, then it is obvious that $x \in K_{T_V}$.

Let $x < y$. If $x \preceq_{T_V} y$, there would exist an element $\ell \in [a,1]$ such that $T_V(\ell, y) = x$. Since $x = T_V(\ell, y) = V \mid_{[a,1]^2} (\ell, y) = V(\ell, y)$, we have that $x \preceq_V y$, which is a contradiction. If $y \preceq_{T_V} x$, it would be $y \leq x$, which is a contradiction to $x < y$. Since $x \npreceq_{T_V} y$ and $y \npreceq_{T_V} x$, we have that $x \in K_{T_V}$.

Let $y < x$. Suppose that $y \preceq_{T_V} x$. Then, there exists an element ℓ of $[a,1]$ such that

$$y = T_V(\ell, x).$$

Then, we obtain that $y \preceq_V x$ from $y = T_V(\ell, x) = V \mid_{[a,1]^2} (\ell, x) = V(\ell, x)$. This is clearly a contradiction. If $x \preceq_{T_V} y$, it would be $x \leq y$, a contradiction to $y < x$. Then, we have that $x \in K_{T_V}$ from $x \npreceq_{T_V} y$ and $y \npreceq_{T_V} x$.

Suppose that $y||a$. If $x < y$, it would be $a \leq x \leq y$, which contradicts that $y||a$.

If $y < x$, it would be $y \preceq_V x$ by the definition of $\preceq_V$ since $(y,x) \in L^*$ and $y < x$. This contradicts that $x||_{\preceq_V} y$. Then, it must be $x||y$. In this case, since $y||a$ and $x||y$, it is obtained that $x \in M$.

- Let $x||a$. Then, it is clear that $x \in I_a$.

Therefore, we have that $x \in K_{S_V} \cup K_{T_V} \cup I_a \cup M$ for any $x \in K_V$. □

Corollary 4.14. Let $(L, \leq, 0, 1)$ be a bounded lattice and $V \in \mathcal{V}(a)$. If $I_a = \emptyset$, then $K_V = K_{S_V} \cup K_{T_V}$.

Proof. Let $I_a = \emptyset$. Clearly, $M = \emptyset$. By the equality in Theorem 4.13, we immediately obtain that $K_V = K_{S_V} \cup K_{T_V}$. □

Corollary 4.15. Let V be a nullnorm on $[0,1]$ with a zero element a. Then, $K_V = K_{S_V} \cup K_{T_V}$.

The one of the open problems posed in the study [1] has been stated as "Is K_V always an interval or can it be a union of intervals?". The following example is an answer of this question which shows that the set K_V need not be an interval but K_V can be a union of some intervals.

Example 4.16. Consider the function $V : [0,1]^2 \to [0,1]$ given as follow:

$$V(x,y) = \begin{cases} S_D(x,y) & (x,y) \in [0,a]^2, \\ T_D(x,y) & (x,y) \in [a,1]^2, \\ a & \text{otherwise,} \end{cases}$$

where $a \in (0,1)$, $S_D : [0,a]^2 \to [0,a]$ is the drastic sum and $T_D : [a,1]^2 \to [a,1]$ is the drastic product. By [9], V is a nullnorm with zero element a on $[0,1]$. Also, $K_{T_D} = (a,1)$ and $K_{S_D} = (0,a)$.

Now, let us show that $K_{T_D} = (a,1)$. It is clear that $K_{T_D} \subseteq (a,1)$. Let $x \in (a,1)$. For any $y \in (a,1)$ with $x < y$, it is obvious that $x \npreceq_{T_D} y$. Otherwise, it would be $T_D(\ell, y) = x$, for some $\ell \in [a, 1]$. Since $x \neq y$, it must be $\ell \neq 1$, whence we have that $x = a$, which is a contradiction. Then, $x \in K_{T_D}$. Similarly, it can be shown that $K_{S_D} = (0, a)$. By Corollary 4.15, since $K_V = K_{S_V} \cup K_{T_V}$, we have that $K_V = (0, a) \cup (a, 1)$. This shows that K_V can be a union of some intervals.

Remark 4.17. Let $(x, y) \subsetneq (0, 1)$. If $a \in (x, y)$ for any $a \in [0, 1]$, there doesn't exist a nullnorm V on $[0, 1]$ with the zero element a such that $K_V = (x, y)$. Indeed,
for any $x \leq a$, since $V(a, x) = a$, we have that $x \preceq_V a$. On the other hand, for any $x \geq a$, since $V(a, x) = a$, it is obtained that $a \preceq_V x$. Then, the zero element a of a nullnorm V is comparable to any element x of $[0, 1]$ w.r.t. $\preceq_V$. Thus, $a \notin K_V$. If there exists a nullnorm V on $[0, 1]$ with the zero element $a \in (x, y)$ such that $K_V = (x, y)$, it would be $a \in K_V$, which is a contradiction.

The following theorem is an answer for the open problem in [1] stated as "Given an $(a, b) \subsetneq (0, 1)$, can we find a nullnorm F such that $K_F = (a, b)$?".

Theorem 4.18. Let $(u, v) \subsetneq (0, 1)$ be any subinterval which doesn't consist an element $a \in [0, 1]$. Then, there exists a nullnorm $V : [0, 1]^2 \to [0, 1]$ with a zero element a such that $K_V = (u, v)$.

Proof. Let $(u, v) \subsetneq (0, 1)$ be any subinterval. Since $a \notin (u, v)$, it is clear that $(u, v) \subseteq (0, a)$ or $(u, v) \subseteq (a, 1)$.
Suppose that $(u, v) \subseteq (a, 1)$. Take the drastic sum S_M on $[0, a]$ defined by: for any $x, y \in [0, a]$,

$$S_M(x, y) = \max(x, y).$$

Consider the following function T on $[a, 1]$: for any $x, y \in [a, 1]$,

$$T(x, y) = \begin{cases} y & x = v \quad \text{and} \quad y \in [u, v], \\ x & y = v \quad \text{and} \quad x \in [u, v], \\ u & (x, y) \in [u, v)^2, \\ \min(x, y) & \text{otherwise}. \end{cases}$$

It can be easily seen that T is a t-norm on $[a, 1]$. By [9], the function V defined by

$$V(x, y) = \begin{cases} S_M(x, y) & (x, y) \in [0, a]^2, \\ T(x, y) & (x, y) \in [a, 1]^2, \\ a & \text{otherwise}, \end{cases}$$

is a nullnorm on $[0, 1]$. Since S_M is continuous, it is clear that $\preceq_{S_M} = \leq$. Then, we have that $K_{S_V} = K_{S_M} = \emptyset$. Now, let us show that $K_{T_V} = K_T = (u, v)$.

Let $x \in (u, v)$. For any $y \in (u, v)$ with $x < y$, it is obvious that $x \npreceq_T y$. Otherwise, there would exist an element ℓ of $[a, 1]$ such that $T(\ell, y) = x$. Since $x \neq y$, $\ell \neq v$. Also, since $x \neq u$, $\ell \notin [u, v)$. Thus, $x = T(\ell, y) = \min(\ell, y)$, whence it must be $\ell = x$. Then, $x = T(\ell, y) = T(x, y) = u$, a contradiction. Hence, for any $y \in (u, v)$ with $x < y$, we have that $x \npreceq_T y$, i.e., $x \in K_T$.

Conversely, let $x \in K_T$ but $x \notin (u,v)$. Then, $x \leq u$ or $x \geq v$. For any $y \in [a,1]$ with $x < y$, since $T(x,y) = \min(x,y) = x$, we have that $x \preceq_T y$, which contradicts that $x \in K_T$. Then, we obtain that $K_T = (u,v)$.

By Corollary 4.15, it is obtained that $K_V = K_{T_V} \cup K_{S_V} = K_T \cup K_{S_M} = (u,v) \cup \emptyset = (u,v)$.

Let $(u,v) \subseteq (0,a)$. Consider the following function S on $[0,a]$ such that for any $x,y \in [0,a]$,

$$S(x,y) = \begin{cases} y & x = u \quad \text{and} \quad y \in [u,v], \\ x & y = u \quad \text{and} \quad x \in [u,v], \\ v & (x,y) \in (u,v]^2, \\ \max(x,y) & \text{otherwise.} \end{cases}$$

It can be easily shown that S is a t-conorm on $[0,a]$. Also, take the minimum t-norm T_M on $[a,1]$ defined by $T_M(x,y) = \min(x,y)$ for any $x,y \in [a,1]$. By [9], the function V defined by

$$V(x,y) = \begin{cases} S(x,y) & (x,y) \in [0,a]^2, \\ T_M(x,y) & (x,y) \in [a,1]^2, \\ a & \text{otherwise,} \end{cases}$$

is a nullnorm on $[0,1]$. It can be easily seen that $K_{T_V} = K_{T_M} = \emptyset$ and $K_{S_V} = K_S = (u,v)$. By Corollary 4.15, we have that $K_V = K_S \cup K_{T_M} = (u,v)$. □

5. CONCLUDING REMARKS

In this paper, it has been shown that two nullnorms are equivalent if and only if their corresponding underlying t-norms and t-conorms are also equivalent. Defining the set of all incomparable elements w.r.t. the order induced by nullnorms, denoted by K_V, the set has been characterized. The open problems posed in the study [1] have been exactly solved.

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