Petr Vodstrčil; Jiří Bouchala; Marta Jarošová; Zdeněk Dostál

On conditioning of Schur complements of H-TFETI clusters for 2D problems governed by Laplacian

Applications of Mathematics, Vol. 62 (2017), No. 6, 699–718

Persistent URL: `http://dml.cz/dmlcz/147004`

62 (2017) APPLICATIONS OF MATHEMATICS No. 6, 699–718

ON CONDITIONING OF SCHUR COMPLEMENTS OF H-TFETI CLUSTERS FOR 2D PROBLEMS GOVERNED BY LAPLACIAN

Petr Vodstrčil, Jiří Bouchala,
Marta Jarošová, Zdeněk Dostál, Ostrava

Received July 15, 2017. First published December 13, 2017.

Abstract. Bounds on the spectrum of the Schur complements of subdomain stiffness matrices with respect to the interior variables are key ingredients in the analysis of many domain decomposition methods. Here we are interested in the analysis of floating clusters, i.e. subdomains without prescribed Dirichlet conditions that are decomposed into still smaller subdomains glued on primal level in some nodes and/or by some averages. We give the estimates of the regular condition number of the Schur complements of the clusters arising in the discretization of problems governed by 2D Laplacian. The estimates depend on the decomposition and discretization parameters and gluing conditions. We also show how to plug the results into the analysis of H-TFETI methods and compare the estimates with numerical experiments. The results are useful for the analysis and implementation of powerful massively parallel scalable algorithms for the solution of variational inequalities.

Keywords: two-level domain decomposition; hybrid FETI; Schur complement; bounds on the spectrum

MSC 2010: 34B16, 34C25

1. Introduction

Variants of FETI (finite element tearing and interconnecting) methods introduced by Farhat and Roux [7], [8] belong to the most powerful methods for massively parallel solution of large discretized elliptic partial differential equations. The basic

The research has been supported by Grant of SGS No. SP2017/122, VSB—Technical University of Ostrava, Czech Republic and by The Ministry of Education, Youth and Sports from the National Programme of Sustainability (NPU II) project IT4Innovations excellence in science—LQ1602 and from the Large Infrastructures for Research, Experimental Development and Innovations project IT4Innovations National Supercomputing Center LM2015070.

DOI: 10.21136/AM.2017.0193-17

idea is to decompose the domain into subdomains which are joined by Lagrange multipliers. After eliminating the primal variables, the original problem reduces to a smaller global problem and a number of local problems that can be solved in parallel. Moreover, if applied to any variational inequality, the duality transforms the inequality constraints into bound constraints. The conditioning of the global problem depends on the conditioning of Schur complements of local stiffness matrices with respect to interior variables. The bounds on the spectra of local Schur complements are well known (see e.g. Brenner [1] or Pechstein [13]) and are the essential ingredients of the analysis of any FETI-type method (see e.g. Farhat, Mandel, and Roux [6] or Tosseli and Widlund [14]).

Here we are interested in the estimates that are necessary for the analysis of H-TFETI (hybrid total FETI, see e.g. [4]) without preconditioner. This method exploits the decomposition of domains into floating clusters, i.e. the subdomains without Dirichlet conditions that are decomposed into still smaller subdomains glued on the primal level in some nodes and/or averages. The two-level structure of the stiffness matrices arising from applications of H-TFETI complies well with the hierarchical organization of modern supercomputers—the clusters and their subdomains can be naturally associated with the nodes and their cores, respectively. We give the estimates in terms of the discretization and decomposition parameters of the regular condition number of the Schur complements of clusters arising from the discretization of problems governed by 2D Laplacian. We consider various gluing conditions, plug the results into the analysis of H-TFETI methods, and compare the results with numerical experiments.

Let us mention that the interest in the conditioning of FETI without preconditioners is motivated by the possibility to work with bound or separable inequality constraints. The preconditioning transforms these constraints into more general constraints (see e.g. Dostál et al. [4]). Let us mention that H-TFETI has been successfully applied to the solution of nonlinear elliptic problems discretized by some hundred billions of nodal variables [4].

2. Domain decomposition, subdomains and clusters

Let us consider a problem governed by the Laplacian on the 2D unit square Ω with the boundary Γ, such as the Poisson equation

$$\Delta u = f \tag{2.1}$$

with homogeneous Dirichlet, Neumann, or Signorini conditions. For the application of TFETI, let us decompose Ω into square subdomains Ω_i of equal side-length H_s with the boundaries Γ_i as in Fig. 1, $i = 1, \ldots, n_s$, $n_s = 1/H_s^2$.

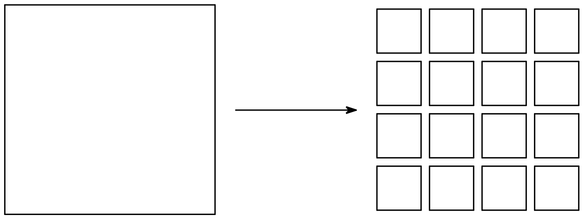

Figure 1. Decomposition of Ω into subdomains Ω_i.

In each Ω_i, we introduce a regular triangularization with the discretization parameter h and the pattern depicted in Fig. 2. We assume matching discretization of subdomains, so the nodes should coincide on the interface. The total number of nodes is denoted by n.

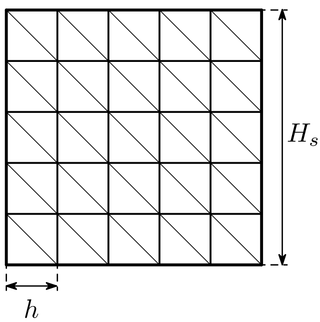

Figure 2. Subdomain Ω_i and its triangularization.

For each subdomain Ω_i, we introduce the standard finite element linear basis functions $\varphi_j^i(x)$ and set up local stiffness matrices K^i, local nodal displacement vectors u^i, and local load vectors f^i,

$$K = \operatorname{diag}(K^1, \ldots, K^{n_s}), \quad u = \begin{bmatrix} u^1 \\ \vdots \\ u^{n_s} \end{bmatrix}, \quad f = \begin{bmatrix} f^1 \\ \vdots \\ f^{n_s} \end{bmatrix}, \quad i = 1, \ldots, n_s.$$

The basis functions span the space $V_h(\Omega_i)$ with the elements

$$u_h^i(x) = \sum_j u_j^i \varphi_j^i(x), \quad x \in \Omega_i.$$

Notice that each local stiffness matrix K^i is symmetric positive semidefinite (SPS) with the kernel spanned by the vector

$$e_i = [1, \ldots, 1]^{\mathrm{T}}.$$

The solution of the discretized problem (2.1) can be obtained by the solution of a constrained quadratic programming problem

$$\min\left(\tfrac{1}{2}x^{\mathrm{T}}Kx - f^{\mathrm{T}}x\right) \quad \text{subject to} \quad B_E x = o \quad \text{and} \quad B_I x \leqslant o, \tag{2.2}$$

where B_E represents gluing of the subdomains and the Dirichlet boundary conditions, B_I enforces the Signorini conditions, and o denotes zero vector. For example, the continuity of the solution in the corners of interior subdomains is enforced by three rows of B with nonzero entries placed in four columns corresponding to the global indices of the corner variables,

$$\begin{bmatrix} \dots & 1 & \dots & -1 & \dots & 0 & \dots & 0 & \dots \\ \dots & 0 & \dots & 0 & \dots & 1 & \dots & -1 & \dots \\ \dots & 1 & \dots & 1 & \dots & -1 & \dots & -1 & \dots \end{bmatrix}.$$

Matrices B_E and B_I can be considered as submatrices of matrix B with the column blocks complying with the block structure of K, i.e.

$$B = \begin{bmatrix} B_E \\ B_I \end{bmatrix} = [B_1, \dots, B_{n_s}].$$

Both, original FETI algorithm (also referred to as FETI1) [7] and TFETI [3], were proposed for linear problems. The idea was to switch to the constrained dual problem in Lagrange multipliers and solve it iteratively with preconditioning by the projector

$$P = I - G(G^{\mathrm{T}}G)^{+}G^{\mathrm{T}}, \quad G = BR,$$

where $(G^{\mathrm{T}}G)^{+}$ denotes a left generalized inverse. Recal that if A is any square matrix, then

$$AA^{+}A = A.$$

The method was later adapted to the solution of variational inequalities [2], [4]. The dual problem reads

$$\min\left(\tfrac{1}{2}\lambda^{\mathrm{T}}F\lambda - d^{\mathrm{T}}\lambda\right) \quad \text{subject to } G\lambda = o \text{ and } \lambda_I \geqslant o, \tag{2.3}$$

where

$$F = BK^{+}B^{\mathrm{T}}, \quad G = R^{\mathrm{T}}B^{\mathrm{T}}, \quad \lambda = [\lambda_E^{\mathrm{T}}, \lambda_I^{\mathrm{T}}]^{\mathrm{T}},$$

and denotes the Lagrange multipliers with the components enforcing the equality and inequality constraints, respectively, and d is a vector the specification of which is not relevant in this paper. The regular condition number $\overline{\kappa}(F) = \kappa(F|\operatorname{Im} F)$ was shown in [6] to satisfy

$$\overline{\kappa}(F) \leqslant CH/h, \tag{2.4}$$

with a constant C independent of the decomposition and discretization parameters H and h, respectively. This estimate guarantees an optimal complexity of the method provided the cost of the action of P does not dominate the cost of the iteration.

To overcome the latter limitation and to reduce the dimension of $G^{\mathrm{T}}G$ without compromising the number of subdomains, Farhat, Lesoinne, and Pierson [5] and Klawonn and Rheinbach [9], [10] proposed to enforce some gluing constraints explicitly. For example, to glue four adjacent subdomains in the only common node

$$x \in \overline{\Omega}_i \cap \overline{\Omega}_j \cap \overline{\Omega}_k \cap \overline{\Omega}_l,$$

it is enough to replace the four columns of identity matrix which correspond to x by a vector with four appropriately placed ones to get the matrix L which transforms global variables $\widetilde{u}$ to u,

$$u = L\widetilde{u}.$$

It can be checked that the stiffness matrix $\widetilde{K}$ of Ω and the constraint matrices $\widetilde{B}_E$ and $\widetilde{B}_I$ in remaining variables are given by

$$\widetilde{K} = L^{\mathrm{T}}KL, \quad \widetilde{B}_E = B_E L, \quad \widetilde{B}_I = B_I L.$$

Moreover, the kernel of the stiffness matrix $\widetilde{K}^m$ of the cluster obtained by gluing the four subdomains in one node is only one dimensional, while the dimension of the kernel of $\operatorname{diag}(K^i, K^j, K^k, K^l)$ is four. The cluster resulting from the gluing of four adjacent subdomains in corners is depicted in Fig. 3.

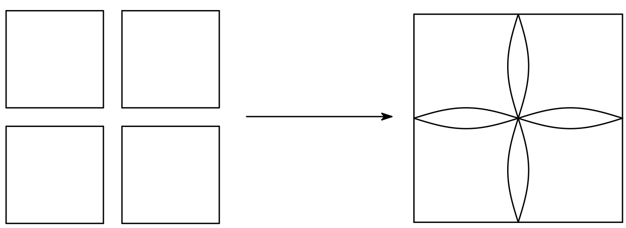

Figure 3. The gluing of four subdomains in adjacent corners.

More generally, we can split the equality constraints into two blocks B_{EP} and B_{ED},

$$B_E = \begin{bmatrix} B_{EP} \\ B_{ED} \end{bmatrix},$$

and use B_{EP} to eliminate some primal variables. The constraint matrix B_{EP} can define not only gluing of some nodes but also more general relations between the variables on neighbouring edges such as zero average of the sum of variables in the interior of adjacent edges. The details of implementation are a bit tricky, but well known, see e.g. Klawonn and Rheinbach [10], Lee [11], or Dostál et al. [4], Chap. 19. In this way, we can implement a decomposition of Ω into clusters $\widetilde{\Omega}^i$, $i = 1, \ldots, n_c$. As a result, we shall obtain the H-TFETI problem to find

$$\min(\tfrac{1}{2}\lambda^{\mathrm{T}}\widetilde{F}\lambda - \tilde{d}^{\mathrm{T}}\lambda) \quad \text{subject to } \widetilde{G}\lambda = o \text{ and } \lambda_I \geqslant o, \tag{2.5}$$

where

$$\widetilde{F} = \widetilde{B}\widetilde{K}^{+}\widetilde{B}^{\mathrm{T}}, \quad \widetilde{K} = \operatorname{diag}(\widetilde{K}^1, \ldots, \widetilde{K}^{n_c}), \quad \widetilde{G} = \widetilde{R}^{\mathrm{T}}\widetilde{B}^{\mathrm{T}}, \quad \lambda = [\lambda_{ED}^{\mathrm{T}}, \lambda_I^{\mathrm{T}}]^{\mathrm{T}}$$

and now denotes the Lagrange multipliers associated with the components enforcing the inequality and remaining equality constraints of the modified problem described by a matrix $\widetilde{B}$, respectively. The preconditioning is carried out by a modified projector

$$\widetilde{P} = I - \widetilde{G}(\widetilde{G}^{\mathrm{T}}\widetilde{G})^{+}\widetilde{G}.$$

In the next sections, we shall show that it is possible, at least for small clusters, to get similar optimality results for the solution of (2.5) as those reported above for TFETI based algorithms for (2.3). Let us point out that there are better results for TFETI and H-TFETI with standard preconditioners (see e.g. [13] or [14]), but these results do not support optimality of algorithms for variational inequalities. The reason is that there is no algorithm for the solution of quadratic programming problems with general inequality constraints that enjoy the rate of convergence in terms of conditioning of the Hessian.

3. Schur complements of clusters and general estimates

The condition number of $\widetilde{F}$ can be estimated in two steps. The first one is a subject of the following simple lemma.

Lemma 3.1. *Let there be constants* $0 < C_1 < C_2$ *such that for each* $\lambda \in \mathbb{R}^m$,

$$C_1\|\lambda\|^2 \leqslant \|\widetilde{B}^{\mathrm{T}}\lambda\|^2 \leqslant C_2\|\lambda\|^2, \tag{3.1}$$

where $\|\cdot\|$ *denotes the Euclidean norm. Then*

$$\overline{\kappa}(\widetilde{P}\widetilde{F}\widetilde{P}) \leqslant \frac{C_2 \max\{\|\widetilde{K}^i\| \colon\ i = 1, \ldots, n_c\}}{C_1 \min\{\overline{\lambda}_{\min}(\widetilde{K}^i) \colon\ i = 1, \ldots, n_c\}}. \tag{3.2}$$

P r o o f. The proof of this lemma is rather trivial; it uses only the observations that if $\lambda \in \operatorname{Im} \widetilde{P}$, then $\widetilde{G}^{\mathrm{T}}\lambda = o$, i.e. $\widetilde{B}^{\mathrm{T}}\lambda \in \operatorname{Im} K$, that the nonzero eigenvalues of $\widetilde{K}$ are reciprocal to the corresponding eigenvalues of $\widetilde{K}^{+}$, and that the spectrum $\sigma(\widetilde{K})$ of $\widetilde{K}$ satisfies

$$\sigma(\widetilde{K}) = \bigcup_{i=1}^{n_c} \sigma(\widetilde{K}^i).$$

□

However, the estimate given by Lemma 3.1 is a bit pessimistic. The reason is that $\operatorname{Im} \widetilde{B}$ is spanned by the vectors that have zero components corresponding to the variables in the interior of Ω_i. To enhance this observation, let us define the (extended) skeleton Σ of the decomposition by

$$\Sigma := \bigcup_{i=1}^{n_s} \Gamma_i$$

and decompose the set of indices $\mathcal{N} = \{1, \ldots, n\}$ into the indices of skeleton nodes $\mathcal{S}$ and subdomain interior nodes $\mathcal{I}$. For any matrix $A \in \mathbb{R}^{m\times n}$ and the subsets $\mathcal{I} \subseteq \{i = 1, \ldots, m\}$ and $\mathcal{J} \subseteq \{j = 1, \ldots, n\}$, let $A_{\mathcal{I}\mathcal{J}}$ denote a submatrix of A with the rows $i \in \mathcal{I}$ and $j \in \mathcal{J}$. Then it is easy to check that

$$(\widetilde{K}^{+})_{\mathcal{S}\mathcal{S}} = \widetilde{S}^{+}, \quad \widetilde{S} = \widetilde{K}_{\mathcal{S}\mathcal{S}} - \widetilde{K}_{\mathcal{S}\mathcal{I}}\widetilde{K}_{\mathcal{I}\mathcal{I}}^{-1}\widetilde{K}_{\mathcal{I}\mathcal{S}}.$$

Matrix $\widetilde{S}$ is called the Schur complement of the block of subdomain interior variables. The same formula holds for the clusters, i.e.

$$((\widetilde{K}^i)^{+})_{\mathcal{S}_i\mathcal{S}_i} = (\widetilde{S}^i)^{+}, \quad \widetilde{S}^i = \widetilde{K}^i_{\mathcal{S}_i\mathcal{S}_i} - \widetilde{K}^i_{\mathcal{S}_i\mathcal{I}_i}(\widetilde{K}^i_{\mathcal{I}_i\mathcal{I}_i})^{-1}\widetilde{K}^i_{\mathcal{I}_i\mathcal{S}_i}, \quad i = 1, \ldots, n_c,$$

where $\mathcal{S}_i$ and $\mathcal{J}_i$ denote the boundary and interior indices of the subdomains of clusters, respectively. We can enhance these observations into the following corollary.

Corollary 3.1. *Let the assumptions of Lemma* 3.1 *be satisfied. Then*

$$\overline{\kappa}(\widetilde{P}\widetilde{F}\widetilde{P}) \leqslant \frac{C_2 \max\{\|\widetilde{S}^i\|\colon\ i = 1, \ldots, n_c\}}{C_1 \min\{\overline{\lambda}_{\min}(\widetilde{S}^i)\colon\ i = 1, \ldots, n_c\}}. \tag{3.3}$$

It is useful to observe that the local Schur complements $\widetilde{S}^i$ are closely related to the harmonic extensions of functions from the boundary of subdomains Ω_i. In particular, if

$$u_h^i(x) = \sum_{j\in\mathcal{S}_i} u_j^i \varphi_j(x)$$

is a discrete harmonic function with the indices of the interior and boundary displacements in $\mathcal{I}_i$ and $\mathcal{B}_i$, respectively, so that

$$K^i_{\mathcal{I}_i\mathcal{I}_i}(u^i_h)_{\mathcal{I}_i} + K^i_{\mathcal{I}_i\mathcal{B}_i}(u^i_h)_{\mathcal{B}_i} = o,$$

then

$$(u^i_{\mathcal{B}})^{\mathrm{T}} S^i u^i_{\mathcal{B}} = (u^i)^{\mathrm{T}} K^i u^i = \int_{\Omega^i} \|\nabla u^i_h\|^2 \,\mathrm{d}\Omega. \tag{3.4}$$

A similar relation holds for the clusters and their skeletons. It follows that we can bound the spectrum of $\widetilde{F}$ by the analysis of $V_h(\Omega)$.

4. Gluing by corners

Let us consider a square subdomain Ω_i of side length H_s with the discretization introduced in Section 2 (see Fig. 2).

Theorem 4.1. *Let $u_h \in V_h(\Omega_i)$ denote a discrete harmonic function, i.e.*

$$\forall\, v_h \in V_h(\Omega_i) \cap H^1_0(\Omega_i)\colon \int_{\Omega_i} \nabla u_h \cdot \nabla v_h \,\mathrm{d}\Omega = 0,$$

where the dot denotes the Euclidean scalar product in $\mathbb{R}^2$. Then

$$\int_{\Omega_i} \|\nabla u_h\|^2 \,\mathrm{d}\Omega \leqslant 3 \sum_{x_k \in \Gamma_i} (u_h(x_k))^2 \tag{4.1}$$

(we sum the squares of values of u_h in all vertices of the triangles on the boundary of Ω_i).

Proof. Let $\tilde{u}_h \in V_h(\Omega_i)$ be defined in the vertices of the triangles by

$$\tilde{u}_h(x_k) := \begin{cases} u_h(x_k), & x_k \in \Gamma_i, \\ 0, & x_k \in \overline{\Omega}_i \setminus \Gamma_i. \end{cases}$$

Let T be a triangle of a triangulation of Ω_i with the vertices denoted by x_i, x_j, x_k, so that the right angle is at the vertex x_k. Then

$$\int_T \|\nabla \tilde{u}_h\|^2 \,\mathrm{d}\Omega = \tfrac{1}{2}[(\tilde{u}_h(x_i) - \tilde{u}_h(x_k))^2 + (\tilde{u}_h(x_j) - \tilde{u}_h(x_k))^2]. \tag{4.2}$$

If

$$\tilde{u}_h(x_i) \neq 0 \neq \tilde{u}_h(x_k) \quad \text{or} \quad \tilde{u}_h(x_j) \neq 0 \neq \tilde{u}_h(x_k),$$

then substituting of

$$\begin{aligned} &\big(\tilde{u}_h(x_i)-\tilde{u}_h(x_k)\big)^2 \leqslant 2\big((\tilde{u}_h(x_i))^2+(\tilde{u}_h(x_k))^2\big)\\ \text{or}\quad &\big(\tilde{u}_h(x_j)-\tilde{u}_h(x_k)\big)^2 \leqslant 2\big((\tilde{u}_h(x_j))^2+(\tilde{u}_h(x_k))^2\big)\end{aligned}$$

into (4.2) yields

$$\int_{\Omega_i}\|\nabla u_h\|^2\,\mathrm{d}\Omega \leqslant \int_{\Omega_i}\|\nabla \tilde{u}_h\|^2\,\mathrm{d}\Omega \leqslant 3\sum_{x_k\in\Gamma_i}(\tilde{u}_h(x_k))^2 = 3\sum_{x_k\in\Gamma_i}(u_h(x_k))^2.$$

□

Lemma 4.1. *There exists a constant $C>0$ independent of H_s and h such that*

$$\|u_h-\overline{u}_h\|^2_{L^\infty(\Omega_i)} \leqslant C\Big(1+\ln\frac{H_s}{h}\Big)\int_{\Omega_i}\|\nabla u_h\|^2\,\mathrm{d}\Omega$$

for every $u_h\in V_h(\Omega_i)$, where

$$\overline{u}_h = \frac{1}{H_s^2}\int_{\Omega_i} u_h\,\mathrm{d}\Omega.$$

Proof. See [12], Corollary 3.2. □

Lemma 4.2. *There exists a constant $C>0$ independent of H_s and h such that for every $u_h\in V_h(\Omega_i)$ and $\alpha\in\Big[\min\limits_{x\in\overline{\Omega}_i}u_h(x),\ \max\limits_{x\in\overline{\Omega}_i}u_h(x)\Big]$*

$$\|u_h-\alpha\|^2_{L^\infty(\Omega_i)} \leqslant C\Big(1+\ln\frac{H_s}{h}\Big)\int_{\Omega_i}\|\nabla u_h\|^2\,\mathrm{d}\Omega.$$

Proof. The proof is an easy consequence of Lemma 4.1, because

$$\begin{aligned}\|u_h-\alpha\|_{L^\infty(\Omega_i)} &= \max\Big\{\max_{x\in\overline{\Omega}_i}u_h(x)-\alpha,\ \alpha-\min_{x\in\overline{\Omega}_i}u_h(x)\Big\}\\ &\leqslant \max_{x\in\overline{\Omega}_i}u_h(x)-\min_{x\in\overline{\Omega}_i}u_h(x)\\ &\leqslant 2\max\Big\{\max_{x\in\overline{\Omega}_i}u_h(x)-\overline{u}_h,\ \overline{u}_h-\min_{x\in\overline{\Omega}_i}u_h(x)\Big\}\\ &= 2\|u_h-\overline{u}_h\|_{L^\infty(\Omega_i)}.\end{aligned}$$

□

To simplify the notation, let us now denote by Ω a cluster of four equal non-overlapping square subdomains $\Omega_1, \ldots, \Omega_4$ of side length H_s, so that $\bigcap_{i=1}^{4} \overline{\Omega}_i = \{x_0\}$ (see Fig. 4), and let Σ denote the skeleton of Ω. Define

$$\begin{aligned}
V_h(\Omega) &:= \{u_h = (u_h^1, u_h^2, u_h^3, u_h^4)\colon\ u_h^i \in V_h(\Omega_i)\},\\
\int_\Omega \|\nabla u_h\|^2 \,\mathrm{d}\Omega &:= \sum_{i=1}^{4} \int_{\Omega_i} \|\nabla u_h^i\|^2 \,\mathrm{d}\Omega,\\
\sum_{x_j\in\Sigma} (u_h(x_j))^2 &:= \sum_{i=1}^{4} \sum_{x_k\in\Gamma_i} (u_h^i(x_k))^2,\\
\|u_h\|_{L^\infty(\Omega)} &:= \max\{\|u_h^1\|_{L^\infty(\Omega_1)}, \ldots, \|u_h^4\|_{L^\infty(\Omega_4)}\},\\
\max_{x\in\overline{\Omega}} u_h(x) &:= \max\Bigl\{\max_{x\in\overline{\Omega}_1} u_h^1(x), \ldots, \max_{x\in\overline{\Omega}_4} u_h^4(x)\Bigr\},\\
\min_{x\in\overline{\Omega}} u_h(x) &:= \min\Bigl\{\min_{x\in\overline{\Omega}_1} u_h^1(x), \ldots, \min_{x\in\overline{\Omega}_4} u_h^4(x)\Bigr\}.
\end{aligned}$$

In this and the following section, we use simplified notations to improve readability. The following theorem is the main result of this section.

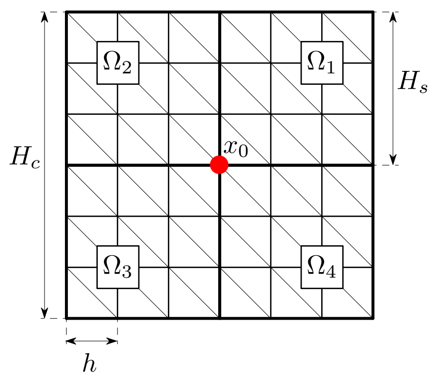

Figure 4. Cluster Ω joined in one point.

Theorem 4.2. *The following statements are true for the cluster obtained by gluing the subdomains in a corner.*

(i) *There exists a constant $C > 0$ independent of H_s and h such that*

$$\Bigl(\max_{x\in\overline{\Omega}} u_h(x) - \min_{x\in\overline{\Omega}} u_h(x)\Bigr)^2 \leqslant C\Bigl(1 + \ln\frac{H_s}{h}\Bigr) \int_\Omega \|\nabla u_h\|^2 \,\mathrm{d}\Omega$$

for every $u_h \in V_h(\Omega)$ which satisfies

$$u_h^1(x_0) = u_h^2(x_0) = u_h^3(x_0) = u_h^4(x_0).$$

(ii) *If u_h satisfies also*

$$\int_\Sigma u_h \,\mathrm{d}\Gamma := \sum_{i=1}^{4} \int_{\Gamma_i} u_h^i \,\mathrm{d}\Gamma = 0,$$

then there exists a constant $C > 0$ independent of H_s and h such that

$$C \frac{h}{H_s\big(1 + \ln(H_s/h)\big)} \sum_{x_j \in \Sigma} (u_h(x_j))^2 \leqslant \int_\Omega \|\nabla u_h\|^2 \,\mathrm{d}\Omega. \tag{4.3}$$

P r o o f of Theorem 4.2. (i) Let i and j be such that

$$\max_{x \in \overline{\Omega}} u_h(x) = \max_{x \in \overline{\Omega}_i} u_h^i(x) \quad \text{and} \quad \min_{x \in \overline{\Omega}} u_h(x) = \min_{x \in \overline{\Omega}_j} u_h^j(x).$$

Using Lemma 4.2, we get

$$\begin{aligned}
\Big(\max_{x \in \overline{\Omega}} u_h(x) - \min_{x \in \overline{\Omega}} u_h(x)\Big)^2 &= \Big(\Big(\max_{x \in \overline{\Omega}_i} u_h^i(x) - \alpha\Big) + (\alpha - \min_{x \in \overline{\Omega}_j} u_h^j(x))\Big)^2 \\
&\leqslant 2(\|u_h^i - \alpha\|_{L^\infty(\Omega_i)})^2 + 2(\|u_h^j - \alpha\|_{L^\infty(\Omega_j)})^2 \\
&\leqslant C\Big(1 + \ln \frac{H_s}{h}\Big) \int_\Omega \|\nabla u_h\|^2 \,\mathrm{d}\Omega,
\end{aligned}$$

where

$$\alpha := u_h^1(x_0) = u_h^2(x_0) = u_h^3(x_0) = u_h^4(x_0).$$

(ii) If $\int_\Sigma u_h \,\mathrm{d}\Gamma = 0$, it follows that

$$\min_{x \in \overline{\Omega}} u_h(x) \leqslant 0 \leqslant \max_{x \in \overline{\Omega}} u_h(x)$$

and therefore,

$$\|u_h\|^2_{L^\infty(\Omega)} \leqslant \Big(\max_{x \in \overline{\Omega}} u_h(x) - \min_{x \in \overline{\Omega}} u_h(x)\Big)^2.$$

Now it suffices to use (i) and

$$\sum_{x_j \in \Sigma} (u_h(x_j))^2 \leqslant 16 \frac{H_s}{h} \|u_h\|^2_{L^\infty(\Omega)}.$$

□

Applying (4.1) and (4.3) to the harmonic function u_h satisfying $\int_\Sigma u_h \, \mathrm{d}\Gamma = 0$, we get

$$\frac{C_1 h}{H_s}\Big(1+\ln\frac{H_s}{h}\Big)^{-1}\sum_{x_j\in\Sigma}(u_h(x_j))^2 \leqslant \int_\Omega \|\nabla u_h\|^2 \leqslant 3\sum_{x_j\in\Sigma}(u_h(x_j))^2. \tag{4.4}$$

Using the above inequalities, (3.4) and simple manipulations, we get the following corollary.

Corollary 4.1. *Let the assumption of Lemma* 3.1 *be satisfied. Then there are constants C_1 and C_2 independent of H_s and h such that*

$$C_1 \leqslant \overline{\lambda}_{\min}(\widetilde{P}\widetilde{F}\widetilde{P}) \leqslant \|\widetilde{P}\widetilde{F}\widetilde{P}\| \leqslant C_2\frac{H_s}{h}\Big(1+\ln\frac{H_s}{h}\Big). \tag{4.5}$$

Remark. If $BB^{\mathrm{T}} = I$, then (4.5) holds with $C_1 = 1/3$.

The following example shows that we can neither exclude the term $(1+\ln(/H_s h))$ nor replace it with a term of lower order in inequality (4.3).

Example 4.1. Let us consider the domains

$$\begin{aligned} \Omega_1 &= (0,H_s)\times(0,H_s), & \Omega_2 &= (-H_s,0)\times(0,H_s),\\ \Omega_3 &= (-H_s,0)\times(-H_s,0), & \Omega_4 &= (0,H_s)\times(-H_s,0) \end{aligned}$$

and $u_h = (u_h^1,\dots,u_h^4) \in V_h(\Omega)$ such that

$$\begin{aligned} u_h^1(ih,jh) &:= \ln(i+j+1) \quad \text{for } i,j\in\Big\{0,\dots,\frac{H_s}{h}\Big\},\\ u_h^2(x,y) &:= 0 \quad \text{for } (x,y)\in\overline{\Omega}_2,\\ u_h^3(x,y) &:= -u_h^1(-x,-y) \quad \text{for } (x,y)\in\overline{\Omega}_3,\\ u_h^4(x,y) &:= 0 \quad \text{for } (x,y)\in\overline{\Omega}_4. \end{aligned}$$

Then u_h satisfies the assumptions of Theorem 4.2 and for every (admissible) $h > 0$ we have (for simplicity we denote $m = H_s/h$)

$$\sum_{x_j\in\Gamma_1}(u_h^1(x_j))^2 \geqslant \ln^2(m+1)+\ldots+\ln^2(2m+1) \geqslant m\ln^2 m. \tag{4.6}$$

Moreover, using Lagrange's Mean Value Theorem, we obtain for every $h > 0$

$$\begin{aligned}
(4.7)\quad \int_{\Omega_1} \|\nabla u_h^1\|^2 \,\mathrm{d}\Omega &= \sum_{i,j=1}^{m} [(\ln(i+j+1) - \ln(i+j))^2 + (\ln(i+j) - \ln(i+j-1))^2] \\
&\leqslant \sum_{i,j=1}^{m} \Big[\frac{1}{(i+j)^2} + \frac{1}{(i+j-1)^2}\Big] \leqslant 2\sum_{i,j=1}^{m} \frac{1}{(i+j-1)^2} \\
&\leqslant 2 \sum_{\substack{i,j\in\mathbb{N}\\ i+j\leqslant 2m}} \frac{1}{(i+j-1)^2} = 2\sum_{k=2}^{2m} (k-1)\cdot\frac{1}{(k-1)^2} = 2\sum_{k=2}^{2m}\frac{1}{k-1} \\
&\leqslant 2(1+\ln(2m-1)).
\end{aligned}$$

Combining (4.6) and (4.7), we obtain

$$\begin{aligned}
\frac{\sum_{x_j\in\Sigma}(u_h(x_j))^2}{H_s h^{-1}\int_\Omega \|\nabla u_h\|^2\,\mathrm{d}\Omega} = \frac{2\sum_{x_j\in\Gamma_1}(u_h^1(x_j))^2}{2H_s h^{-1}\int_{\Omega_1}\|\nabla u_h^1\|^2\,\mathrm{d}\Omega} &\geqslant \frac{\ln^2 H_s/h}{2(1+\ln(2H_s/h-1))} \\
&\approx \Big(1+\ln\frac{H_s}{h}\Big) \quad \text{for } h\to 0+\,.
\end{aligned}$$

5. Gluing by averages

Let us consider a square subdomain Ω_i of side length H_s and let one of its sides be denoted by Φ. We denote by $x_0, x_1, \ldots, x_n$ the nodes of the discretization on Φ with the discretization parameter h and set $n = H_s/h$ as in Fig. 5. For $u_h \in V_h(\Omega_i)$ we denote the average value in the interior of Φ by

$$p_\Phi(u_h) := \frac{u_h(x_1) + u_h(x_2) + \ldots + u_h(x_{n-1})}{n-1}.$$

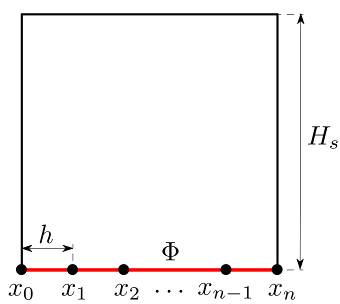

Figure 5. A subdomain with nodes on one side.

Lemma 5.1. *Let $H_s = 1$. Then there exists a constant $C > 0$ independent of h such that for every $u_h \in V_h(\Omega_i)$*

$$\int_{\Gamma_i} (u_h)^2 \,\mathrm{d}\Gamma \leqslant C\Big(\int_{\Omega_i} \|\nabla u_h\|^2 \,\mathrm{d}\Omega + p_\Phi^2(u_h)\Big).$$

P r o o f. First notice that due to the definition of $p_\Phi(u_h)$ and the triangle inequality,

$$\begin{aligned}(u_h(x_k) - p_\Phi(u_h))^2 &\leqslant \Big(\max_{j\in\{1,\dots,n-1\}} |u_h(x_k) - u_h(x_j)|\Big)^2 \leqslant \Big(\sum_{i=1}^{n} |u_h(x_i) - u_h(x_{i-1})|\Big)^2\\ &\leqslant n\sum_{i=1}^{n} (u_h(x_i) - u_h(x_{i-1}))^2 \leqslant 2n \int_{\Omega_i} \|\nabla u_h\|^2 \,\mathrm{d}\Omega.\end{aligned}$$

It follows that

$$\begin{aligned}\Big(\int_\Phi u_h \,\mathrm{d}\Gamma\Big)^2 &= h^2\Big(\tfrac12 u_h(x_0) + \underbrace{u_h(x_1) + u_h(x_2) + \ldots + u_h(x_{n-1})}_{=(n-1)p_\Phi(u_h)} + \tfrac12 u_h(x_n)\Big)^2\\ &= h^2(\tfrac12(u_h(x_0) - p_\Phi(u_h)) + \tfrac12(u_h(x_n) - p_\Phi(u_h)) + np_\Phi(u_h))^2\\ &\leqslant 3h^2(\tfrac14(u_h(x_0) - p_\Phi(u_h))^2 + \tfrac14(u_h(x_n) - p_\Phi(u_h))^2 + n^2p_\Phi^2(u_h))\\ &\leqslant 3h^2\Big(n\int_{\Omega_i} \|\nabla u_h\|^2 \,\mathrm{d}\Omega + n^2p_\Phi^2(u_h)\Big) \leqslant 3\Big(\int_{\Omega_i} \|\nabla u_h\|^2 \,\mathrm{d}\Omega + p_\Phi^2(u_h)\Big),\end{aligned}$$

i.e.

$$\Big(\int_\Phi u_h \,\mathrm{d}\Gamma\Big)^2 \leqslant 3\Big(\int_{\Omega_i} \|\nabla u_h\|^2 \,\mathrm{d}\Omega + p_\Phi^2(u_h)\Big). \tag{5.1}$$

The remaining part of the proof is a direct consequence of the Poincaré inequality, the trace theorem and formula (5.1):

$$\begin{aligned}\int_{\Gamma_i} (u_h)^2 \,\mathrm{d}\Gamma \leqslant C_1\|u_h\|^2_{H^1(\Omega_i)} &\leqslant C_2\Big(\int_{\Omega_i} \|\nabla u_h\|^2 \,\mathrm{d}\Omega + \Big(\int_\Phi u_h \,\mathrm{d}\Gamma\Big)^2\Big)\\ &\leqslant C\Big(\int_{\Omega_i} \|\nabla u_h\|^2 \,\mathrm{d}\Omega + p_\Phi^2(u_h)\Big).\end{aligned}$$

□

Now we again consider a cluster Ω comprising four equal square subdomains $\Omega_1, \ldots, \Omega_4$ of side H_s such that $\bigcap_{i=1}^{4} \overline{\Omega}_i = \{x_0\}$ (as in Fig. 4),

$$V_h(\Omega) := \{u_h = (u_h^1, u_h^2, u_h^3, u_h^4)\colon\ u_h^i \in V_h(\Omega_i)\},$$

and denote $\Phi_{i,j} = \overline{\Omega}_i \cap \overline{\Omega}_j$.

Theorem 5.1. *There exists a constant $C > 0$ independent of H_s and h such that for every $u_h \in V_h(\Omega)$ which satisfies*

$$\int_\Sigma u_h \,\mathrm{d}\Gamma := \sum_{i=1}^{4} \int_{\Gamma_i} u_h^i \,\mathrm{d}\Gamma = 0$$

and for every $\Phi_{i,j} \in \{\Phi_{2,3}, \Phi_{3,4}, \Phi_{1,4}\}$ which satisfies

$$p_{\Phi_{i,j}}(u_h^i) = p_{\Phi_{i,j}}(u_h^j)$$

we have

$$\int_\Sigma (u_h)^2 \,\mathrm{d}\Gamma \leqslant C H_s \int_\Omega \|\nabla u_h\|^2 \,\mathrm{d}\Omega.$$

Corollary 5.1. *Under the above assumptions there is a constant $C > 0$ independent of H_s and h such that*

$$C \frac{h}{H_s} \sum_{x_j \in \Sigma} (u_h(x_j))^2 \leqslant \int_\Omega \|\nabla u_h\|^2 \,\mathrm{d}\Omega. \tag{5.2}$$

P r o o f of Theorem 5.1. First assume that $H_s = 1$ (the general case can be directly obtained using the substitution $x = H_s y$). If $v_h = (v_h^1, v_h^2, v_h^3, v_h^4) \in V_h(\Omega)$ and $p_{\Phi_{i,j}}(v_h^i) = p_{\Phi_{i,j}}(v_h^j)$ for $\Phi_{i,j} \in \{\Phi_{2,3}, \Phi_{3,4}, \Phi_{1,4}\}$, then by Lemma 5.1

$$\begin{aligned}
p^2_{\Phi_{i,j}}(v_h^i) &= \Big(\frac{v_h^j(x_1) + v_h^j(x_2) + \ldots + v_h^j(x_{n-1})}{n-1}\Big)^2 \\
&\leqslant \frac{1}{n-1}\big((v_h^j(x_1))^2 + (v_h^j(x_2))^2 + \ldots + (v_h^j(x_{n-1}))^2\big) \\
&\leqslant \frac{3}{h(n-1)} \int_{\Gamma_j} (v_h^j)^2 \,\mathrm{d}\Gamma \leqslant \frac{6}{hn} \int_{\Gamma_j} (v_h^j)^2 \,\mathrm{d}\Gamma = 6 \int_{\Gamma_j} (v_h^j)^2 \,\mathrm{d}\Gamma \\
&\leqslant C_1 \bigg(\int_{\Omega_j} \|\nabla v_h^j\|^2 \,\mathrm{d}\Omega + p^2_{\Phi_{3,4}}(v_h^3) \bigg),
\end{aligned}$$

and therefore also

$$\int_\Sigma (v_h)^2 \,\mathrm{d}\Gamma \leqslant C_2\biggl(\int_{\Omega_1} \|\nabla v_h^1\|^2 \,\mathrm{d}\Omega + p^2_{\Phi_{1,4}}(v_h^1) + \int_{\Omega_2} \|\nabla v_h^2\|^2 \,\mathrm{d}\Omega + p^2_{\Phi_{2,3}}(v_h^2)$$
$$+ \int_{\Omega_3} \|\nabla v_h^3\|^2 \,\mathrm{d}\Omega + p^2_{\Phi_{3,4}}(v_h^3) + \int_{\Omega_4} \|\nabla v_h^4\|^2 \,\mathrm{d}\Omega + p^2_{\Phi_{3,4}}(v_h^4)\biggr)$$
$$\leqslant C\biggl(\int_\Omega \|\nabla v_h\|^2 \,\mathrm{d}\Omega + p^2_{\Phi_{3,4}}(v_h^3)\biggr).$$

To finish the proof it is enough to choose $v_h := u_h - p_{\Phi_{3,4}}(u_h^3)$ and notice that

$$\int_\Sigma (v_h)^2 \,\mathrm{d}\Gamma \geqslant \int_\Sigma (u_h)^2 \,\mathrm{d}\Gamma - 2p_{\Phi_{3,4}}(u_h^3) \underbrace{\int_\Sigma u_h \,\mathrm{d}\Gamma}_{=0} = \int_\Sigma (u_h)^2 \,\mathrm{d}\Gamma,$$
$$\int_\Omega \|\nabla v_h\|^2 \,\mathrm{d}\Omega = \int_\Omega \|\nabla u_h\|^2 \,\mathrm{d}\Omega, \quad p^2_{\Phi_{3,4}}(v_h^3) = 0.$$

□

Applying (5.2) and (4.1) to the harmonic function u_h, we get

(5.3)
$$C_1 \frac{h}{H_s} \sum_{x_j \in \Sigma} (u_h(x_j))^2 \leqslant \int_\Omega \|\nabla u_h\|^2 \leqslant 3 \sum_{x_j \in \Sigma} (u_h(x_j))^2.$$

Using the above inequalities, (3.4) and simple manipulations, we get the following corollary.

Corollary 5.2. *Let the assumption of Lemma* 3.1 *be satisfied. Then there are constants C_1 and C_2 independent of H_s and h such that*

(5.4)
$$C_1 \leqslant \overline{\lambda}_{\min}(\widetilde{P}\widetilde{F}\widetilde{P}) \leqslant \|\widetilde{P}\widetilde{F}\widetilde{P}\| \leqslant C_2 \frac{H_s}{h}.$$

6. Numerical experiments

To compare our estimate with the specific values, we computed the bounds on the nonzero eigenvalues of the Schur complement of 2×2 clusters with $H_s = 1/4$, $H_c = 1/2$, and varying discretization parameter h. We considered clusters obtained by gluing corners and edge averages. To compute the averages, the variables of each edge were transformed using the transformation matrix, see [10]. The results are in Table 1 and in Fig. 6.

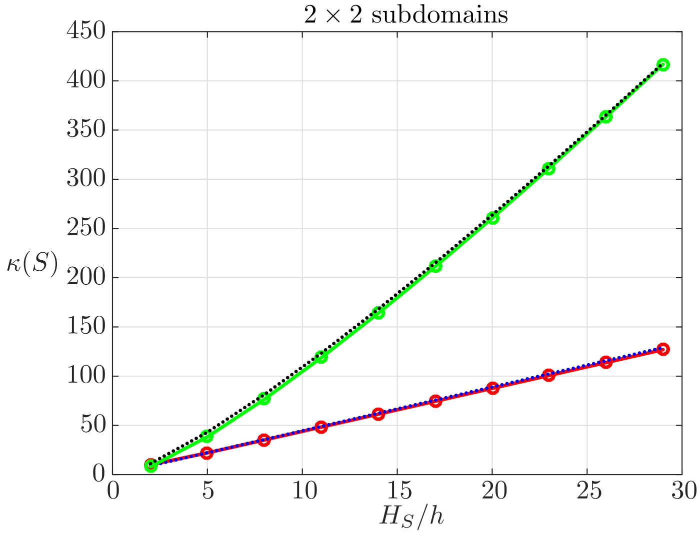

Figure 6. Regular condition number $\overline{\kappa}(\widetilde{S})$ for 2×2 subdomains and changing H_s/h: dark lower graph-averages, light upper graph-corners, dotted lines-fitted bounds $O(H_s/h)$ and $O((1 + \log(H_s/h))H_s/h)$.

	Averages			Corners		
H_s/h	$\kappa(S)$	$\lambda_{\max}$	$\lambda_{\min}$	$\kappa(S)$	$\lambda_{\max}$	$\lambda_{\min}$
5	22.0798	2.6720	0.1210	38.3458	2.6720	0.0697
11	48.2900	2.7904	0.0578	118.8414	2.7904	0.0235
17	74.5157	2.8118	0.0377	211.3836	2.8118	0.0133
23	100.7273	2.8191	0.0280	311.3911	2.8191	0.0091
29	126.9323	2.8225	0.0222	416.8811	2.8225	0.0068

Table 1. Effective condition numbers and extremal eigenvalues for changing H_s/h and 2×2 subdomains.

The results are in agreement with the theoretical results, in particular, it is possible to observe the nonlinear effect in the estimates for the clusters glued in corners. To illustrate the conditioning of larger clusters, we carried out the computations also for 4×4 clusters (see Fig. 7 and Table 2). The results indicate fast deterioration of the regular condition number for the clusters glued by corners, but promising results for the clusters glued by averages. In the latter case, the condition number increased (as compared with 2×2 cluster) by some sixty percent, which indicates that improved parallelization may be very effective.

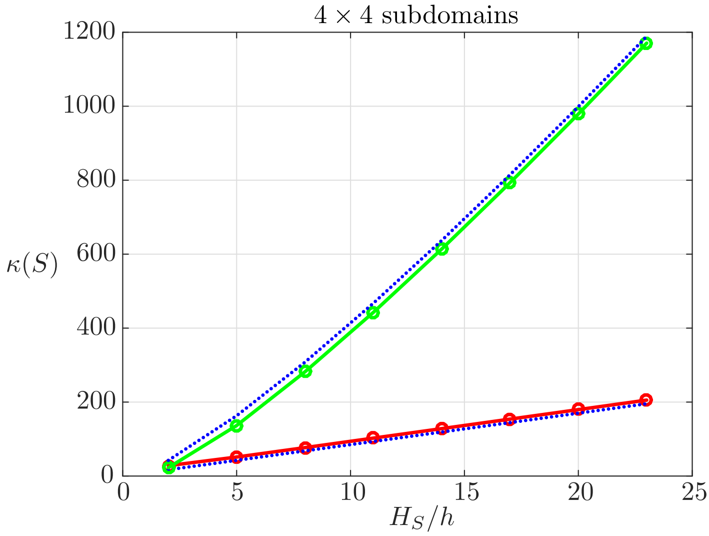

Figure 7. Effective condition number $\kappa(S)$ for 4×4 subdomains and changing H_s/h: dark lower graph-averages, light upper graph-corners, dotted lines-fitted bounds $O(H_s/h)$ and $O((1 + \log(H_s/h))H_s/h)$.

	Averages			Corners		
H_s/h	$\kappa(S)$	$\lambda_{\max}$	$\lambda_{\min}$	$\kappa(S)$	$\lambda_{\max}$	$\lambda_{\min}$
5	51.418	2.672	0.052	136.633	2.672	0.020
11	102.526	2.790	0.027	442.471	2.790	0.006
17	153.916	2.812	0.018	793.049	2.812	0.004
23	205.364	2.819	0.014	1170.667	2.819	0.002

Table 2. Effective condition numbers and extremal eigenvalues for changing H_s/h and 4×4 subdomains.

It seems that H-TFETI without preconditioners can be very effective method for the nonlinear problems but more research is necessary.

7. Conclusions

We have established bounds on the regular condition number of the Schur complements of floating 2×2 clusters arising from gluing the subdomains in nodes and averages. In the first case, we showed that our estimates cannot be qualitatively improved. We considered the Schur complements of the stiffness matrices arising from the discretization of problems governed by the Laplace operator. The research was motivated by the effort to understand massively parallel H-TFETI algorithms for the solution of elliptic problems described by variational inequalities which have already proved to be effective for the problems discretized by billions of nodal variables [4]. It seems that the application of H-TFETI can overcome the bottleneck of TFETI associated with the dimension of the coarse grid and can increase the scalability from the current tens of thousands of cores to hundreds of thousands of cores.

References

[1] *S. C. Brenner*: The condition number of the Schur complement in domain decomposition. Numer. Math. *83* (1999), 187–203. zbl MR doi

[2] *Z. Dostál, F. A. M. Gomes Neto, S. A. Santos*: Duality-based domain decomposition with natural coarse-space for variational inequalities. J. Comput. Appl. Math. *126* (2000), 397–415. zbl MR doi

[3] *Z. Dostál, D. Horák, R. Kučera*: Total FETI—an easier implementable variant of the FETI method for numerical solution of elliptic PDE. Commun. Numer. Methods Eng. *22* (2006), 1155–1162. zbl MR doi

[4] *Z. Dostál, T. Kozubek, M. Sadowská, V. Vondrák*: Scalable Algorithms for Contact Problems. Advances in Mechanics and Mathematics 36, Springer, New York, 2016. zbl MR doi

[5] *C. Farhat, M. Lesoinne, K. Pierson*: A scalable dual-primal domain decomposition method. Numer. Linear Algebra Appl. *7* (2000), 687–714. zbl MR doi

[6] *C. Farhat, J. Mandel, F.-X. Roux*: Optimal convergence properties of the FETI domain decomposition method. Comput. Methods Appl. Mech. Eng. *115* (1994), 365–385. MR doi

[7] *C. Farhat, F.-X. Roux*: A method of finite element tearing and interconnecting and its parallel solution algorithm. Int. J. Numer. Methods Eng. *32* (1991), 1205–1227. zbl MR doi

[8] *C. Farhat, F.-X. Roux*: An unconventional domain decomposition method for an efficient parallel solution of large-scale finite element systems. SIAM J. Sci. Stat. Comput. *13* (1992), 379–396. zbl MR doi

[9] *A. Klawonn, O. Rheinbach*: A hybrid approach to 3-level FETI. Proc. Appl. Math. Mech. *8* (2008), 10841–10843. doi

[10] *A. Klawonn, O. Rheinbach*: Highly scalable parallel domain decomposition methods with an application to biomechanics. ZAMM, Z. Angew. Math. Mech. *90* (2010), 5–32. zbl MR doi

[11] *J. Lee*: Two domain decomposition methods for auxiliary linear problems for a multibody variational inequality. SIAM J. Sci. Comput. *35* (2013), A1350–A1375. zbl MR doi

[12] *D. Lukáš, J. Bouchala, P. Vodstrčil, L. Malý*: 2-dimensional primal domain decomposition theory in detail. Appl. Math., Praha *60* (2015), 265–283. zbl MR doi

[13] *C. Pechstein*: Finite and Boundary Element Tearing and Interconnecting Solvers for Multiscale Problems. Lecture Notes in Computational Science and Engineering 90, Springer, Berlin, 2013. zbl MR doi

[14] *A. Toselli, O. B. Widlund*: Domain Decomposition Methods—Algorithms and Theory. Springer Series on Computational Mathematics 34, Springer, Berlin, 2005. zbl MR

Authors' address: *Petr Vodstrčil*, *Jiří Bouchala*, *Marta Jarošová*, *Zdeněk Dostál*, VŠB—Technical University of Ostrava, 17. listopadu 2172/15, 708 00 Ostrava, Czech Republic, e-mails: petr.vodstrcil@vsb.cz, jiri.bouchala@vsb.cz, marta.jarosova@vsb.cz, zdenek.dostal@vsb.cz.