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# ON THE GENERALIZATION OF TWO RESULTS OF CAO AND ZHANG

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*Abstract.* This paper studies the uniqueness of meromorphic functions

$$f^n \prod_{i=1}^{k} (f^{(i)})^{n_i} \quad \text{and} \quad g^n \prod_{i=1}^{k} (g^{(i)})^{n_i}$$

that share two values, where $n, n_k, k \in \mathbb{N}$, $n_i \in \mathbb{N} \cup \{0\}$, $i = 1, 2, \ldots, k-1$. The results significantly rectify, improve and generalize the results due to Cao and Zhang (2012).

*Keywords*: uniqueness; meromorphic function; weighted sharing; nonlinear differential polynomials

*MSC 2010*: 30D35

## 1. Introduction, definitions and results

In this paper by meromorphic functions we shall always mean meromorphic functions in the complex plane.

Let $f$ and $g$ be two non-constant meromorphic functions and let $a \in \mathbb{C}$. We say that $f$ and $g$ share $a$ CM (counting multiplicities) provided that $f - a$ and $g - a$ have the same zeros with the same multiplicities. Similarly, we say that $f$ and $g$ share $a$ IM (ignoring multiplicities) provided that $f - a$ and $g - a$ have the same zeros ignoring multiplicities. In addition we say that $f$ and $g$ share $\infty$ CM, if $1/f$ and $1/g$ share 0 CM, and we say that $f$ and $g$ share $\infty$ IM, if $1/f$ and $1/g$ share 0 IM.

We adopt the standard notation of value distribution theory (see [8]). We denote by $T(r)$ the maximum of $T(r, f)$ and $T(r, g)$. The symbol $S(r)$ denotes any quantity satisfying $S(r) = o(T(r))$ as $r \to \infty$, outside of a possible exceptional set of finite

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linear measure. A meromorphic function $a(z)$ is called a small function with respect to $f$ provided that $T(r,a) = S(r,f)$.

Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. Let $a(z)$ be a small function with respect to $f(z)$ and $g(z)$. We say that $f(z)$ and $g(z)$ share $a(z)$ CM if $f(z)-a(z)$ and $g(z)-a(z)$ have the same zeros with the same multiplicities, and we say that $f(z)$, $g(z)$ share $a(z)$ IM if we do not consider the multiplicities. For the sake of simplicity we also use the notation

$$n_i^* := \begin{cases} 0 & \text{if } n_i = 0, \\ 1 & \text{if } n_i \neq 0, \end{cases}$$

and

$$n_i^{**} = \begin{cases} 0 & \text{if } n_i = 0, \\ n_i & \text{if } n_i \neq 0. \end{cases}$$

A finite value $z_0$ is called a fixed point of $f$ if $f(z_0) = z_0$ or $z_0$ is a zero of $f(z) - z$. The following well known theorem in value distribution theory was posed by Hayman and settled by several authors almost at the same time ([2], [4]).

**Theorem A.** *Let $f(z)$ be a transcendental meromorphic function, $n \in \mathbb{N}$. Then $f^n f' = 1$ has infinitely many solutions.*

To investigate the uniqueness result corresponding to Theorem A, both Fang and Hua [5], Yang and Hua [16] obtained the following result.

**Theorem B.** *Let $f$ and $g$ be two non-constant entire (meromorphic) functions, $n \in \mathbb{N}$ such that $n \geqslant 6$ ($n \geqslant 11$). If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where $c_1, c_2, c \in \mathbb{C}$ satisfy $4(c_1c_2)^{n+1}c^2 = -1$, or $f \equiv tg$ for a constant $t$ such that $t^{n+1} = 1$.*

Considering the uniqueness question of entire or meromorphic functions having fixed points, Fang and Qiu [6] obtained the following theorem.

**Theorem C.** *Let $f$ and $g$ be two non-constant meromorphic (entire) functions, $n \in \mathbb{N}$ such that $n \geqslant 11$ ($n \geqslant 6$). If $f^n f' - z$ and $g^n g' - z$ share 0 CM, then either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where $c_1, c_2, c \in \mathbb{C}$ satisfy $4(c_1c_2)^{n+1}c^2 = -1$, or $f \equiv tg$ for a constant $t$ such that $t^{n+1} = 1$.*

We recall the following result by Xu, Yi and Zhang [13]

**Theorem D.** *Let $f$ be a transcendental meromorphic function, $n \in \mathbb{N} \setminus \{1\}$, $k \in \mathbb{N}$. Then $f^n f^{(k)}$ takes every finite nonzero value infinitely many times or has infinitely many fixed points.*

Recently, Cao and Zhang [3] proved the following theorems.

**Theorem E.** *Let $f$ and $g$ be two transcendental meromorphic functions, whose zeros are of multiplicities at least $k$, where $k \in \mathbb{N}$. Let $n \in \mathbb{N}$ be such that $n > \max\{2k-1, k+4/k+4\}$. If $f^n f^{(k)}$ and $g^n g^{(k)}$ share $z$ CM, $f$ and $g$ share $\infty$ IM, then one of the following two conclusions holds:*
(i) $f^n f^{(k)} \equiv g^n g^{(k)}$;
(ii) $f(z) = c_1 \mathrm{e}^{cz^2}$, $g(z) = c_2 \mathrm{e}^{-cz^2}$, *where $c_1, c_2, c \in \mathbb{C}$ such that $4(c_1c_2)^{n+1}c^2 = -1$.*

**Theorem F.** *Let $f$ and $g$ be two non-constant meromorphic functions, whose zeros are of multiplicities at least $k+1$, where $k \in \mathbb{N}$ is such that $k \leqslant 5$. Let $n \in \mathbb{N}$ be such that $n \geqslant 10$. If $f^n f^{(k)}$ and $g^n g^{(k)}$ share 1 CM, $f^{(k)}$ and $g^{(k)}$ share 0 CM, $f$ and $g$ share $\infty$ IM, then one of the following two conclusions holds:*
(i) $f \equiv tg$, *where $t$ is a constant such that $t^{n+1} = 1$;*
(ii) $f(z) = c_3 \mathrm{e}^{dz}$, $g(z) = c_4 \mathrm{e}^{-dz}$, *where $c_3, c_4, d \in \mathbb{C}$ are such that $(-1)^k (c_3c_4)^{n+1} \times d^{2k} = 1$.*

Remark 1.1. Theorems E (Theorem 1.2 in [3]) and F (Theorem 1.3 in [3]) are new and seem fine. However, in the statements of both the Theorems E and F there are some contradiction. It is assumed that $f$ and $g$ have zeros of multiplicities at least $k$ in Theorem E and $k+1$ in Theorem F. But further authors concluded that "$f(z) = c_1 \mathrm{e}^{cz^2}$, $g(z) = c_2 \mathrm{e}^{-cz^2}$, where $c_1, c_2, c \in \mathbb{C}$ are such that $4(c_1c_2)^{n+1}c^2 = -1$" in Theorem E and "$f(z) = c_3 \mathrm{e}^{dz}$, $g(z) = c_4 \mathrm{e}^{-dz}$, where $c_3, c_4, d \in \mathbb{C}$ are such that $(-1)^k (c_3c_4)^{n+1} d^{2k} = 1$" in Theorem F. Here we see that $f$ and $g$ have no zeros, so their multiplicities are equal to $k = 0$. Furthermore, it is assumed that $k \in \mathbb{N}$, but in both Theorems E and F the case $k = 0$ is also considered, which is very strange.

The above discussion is sufficient enough to make oneself inquisitive to investigate the accurate forms of Theorems E and F. Also it is quite natural to ask the following questions.

Question 1.2. Can one remove the condition "zeros of $f$ and $g$ are of multiplicities at least $k(k+1)$, where $k \in \mathbb{N}$" in Theorem E (Theorem F) keeping all the conclusions intact?

Question 1.3. Does Theorem F hold for $k \geqslant 6$?

We now explain the notation of weighted sharing as introduced in [10].

**Definition 1.1** ([10]). Let $k \in \mathbb{N} \cup \{0\} \cup \{\infty\}$. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all $a$-points of $f$, where an $a$-point of multiplicity $m$ is counted $m$ times if $m \leqslant k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that $f$, $g$ share the value $a$ with weight $k$.

We write $f$, $g$ share $(a, k)$ to mean that $f$, $g$ share the value $a$ with weight $k$. Clearly if $f$, $g$ share $(a, k)$, then $f$, $g$ share $(a, p)$ for any integer $p$, $0 \leqslant p < k$. Also we note that $f$, $g$ share a value $a$ IM or CM if and only if $f$, $g$ share $(a, 0)$ or $(a, \infty)$, respectively.

## 2. Main results

In this paper, taking the possible answers of the above questions into background we obtain the following results which significantly rectify, improve and generalize Theorems E and F. Throughout this paper we use the following notation:

$$s = \sum_{i=1}^{k} n_i^{**}, \quad t = \sum_{i=1}^{k} n_i^{*}, \quad m = \sum_{i=1}^{k} i n_i^{*} \quad \text{and} \quad m_1 = \sum_{i=1}^{k} i n_i^{**},$$

where $n_i \in \mathbb{N} \cup \{0\}$, $i = 1, 2, \ldots, k-1$ and $n_k, k \in \mathbb{N}$. Also it is clear that $m_1 \leqslant sm$.

In this paper we always use $p(z)$ to denote a nonzero polynomial such that either $\deg(p) \leqslant n + s - 1$ or the zeros of $p(z)$ are of multiplicities at most $n-1$, i.e.,

$$p(z) = a_n (z - z_1)^{l_1} (z - z_2)^{l_2} \ldots (z - z_t)^{l_t}, \tag{2.1}$$

where $a_n \in \mathbb{C} \setminus \{0\}$, $z_i \in \mathbb{C}$, $i = 1, 2, \ldots, t$ are distinct and $l_1, l_2, \ldots, l_t \in \mathbb{N}$. Here we see that either $\sum\limits_{i=1}^{t} l_i \leqslant n + s - 1$ or $l_i \leqslant n - 1$ for all $i = 1, 2, \ldots, t$.

**Theorem 2.1.** *Let $f$, $g$ be two transcendental meromorphic functions, let $n, n_k, k \in \mathbb{N}$, $n_i \in \mathbb{N} \cup \{0\}$, $i = 1, 2, \ldots, k-1$ be such that $n > 2s + m + 2t + 2$ and let $p(z)$ be defined as in (2.1). Let $f^n \prod\limits_{i=1}^{k} (f^{(i)})^{n_i} - p(z)$ and $g^n \prod\limits_{i=1}^{k} (g^{(i)})^{n_i} - p(z)$ share $(0, k_1)$, where $k_1 = (3 + m_1 - s)/(n + s + m_1 - 2m - 1) + 3$, and $f$ and $g$ share $\infty$ IM.*

*Suppose $p(z)$ is not a constant. Then*

(1) *when each $l_i$ is a multiple of $n_1$, $i = 1, 2, \ldots, t$, where $l_i$ is defined as in (2.1), then one of the following two conclusions holds:*

  (1.1) *$f^n \prod\limits_{i=1}^{k} (f^{(i)})^{n_i} \equiv g^n \prod\limits_{i=1}^{k} (g^{(i)})^{n_i}$. In particular, when $f$, $g$ share $0$ CM and $f(z)/g(z) \neq e^{az+b}$, where $a, b \in \mathbb{C}$ $(a \neq 0)$, then $f \equiv tg$, where $t$ is a constant such that $t^{n+s} = 1$;*

(1.2) $f(z)=c_1\mathrm{e}^{cQ(z)}$, $g(z)=c_2\mathrm{e}^{-cQ(z)}$, *where* $Q(z)=\int_0^z p^{1/n_1}(t)\,\mathrm{d}t$, $c_1,c_2,c\in\mathbb{C}$ *are such that* $c^{2n_1}(c_1c_2)^{n+n_1}=(-1)^{n_1}$,

(2) *when at least one of* $l_i$ *is not a multiple of* $n_1$, $i=1,2,\ldots,t$, *then*

$$f^n\prod_{i=1}^k(f^{(i)})^{n_i}\equiv g^n\prod_{i=1}^k(g^{(i)})^{n_i}.$$

*In particular, when* $f$, $g$ *share* 0 CM *and* $f(z)/g(z)\neq \mathrm{e}^{az+b}$, *where* $a,b\in\mathbb{C}$ $(a\neq 0)$, *then* $f\equiv tg$, *where* $t$ *is a constant such that* $t^{n+s}=1$.

*Suppose* $p(z)=b\in\mathbb{C}\setminus\{0\}$. *Then one of the following two conclusions holds:*

(i) $f^n\prod\limits_{i=1}^k(f^{(i)})^{n_i}\equiv g^n\prod\limits_{i=1}^k(g^{(i)})^{n_i}$. *In particular, when* $f$, $g$ *share* 0 CM *and* $f(z)/g(z)\neq \mathrm{e}^{az+b}$, *where* $a,b\in\mathbb{C}$ $(a\neq 0)$, *then* $f\equiv tg$, *where* $t$ *is a constant such that* $t^{n+s}=1$;

(ii) $f(z)=c_3\mathrm{e}^{cz}$, $g(z)=c_4\mathrm{e}^{-cz}$, *where* $c_3,c_4,c\in\mathbb{C}$ *are such that* $(-1)^{m_1}(c_3c_4)^{n+s}\times c^{2m_1}=b^2$.

Remark 2.1. Instead of $f$ and $g$ share 0 CM, one can assume that $f^{(k)}$ and $g^{(k)}$ share 0 CM in Theorem 2.1 when $n_i=0$, $i=1,2,\ldots,k-1$.

We now explain some definitions and notation which are used in the paper.

**Definition 2.1** ([12]). Let $p\in\mathbb{N}$ and $a\in\mathbb{C}\cup\{\infty\}$.

(i) $N(r,a;f\mid\geqslant p)$ ($\overline{N}(r,a;f\mid\geqslant p)$) denotes the counting function (reduced counting function) of those $a$-points of $f$ whose multiplicities are not less than $p$.

(ii) $N(r,a;f\mid\leqslant p)$ ($\overline{N}(r,a;f\mid\leqslant p)$) denotes the counting function (reduced counting function) of those $a$-points of $f$ whose multiplicities are not greater than $p$.

**Definition 2.2.** We denote by $\overline{N}(r,a;f\mid=k)$ the reduced counting function of those $a$-points of $f$ whose multiplicities are exactly $k$, where $k\in\mathbb{N}$.

**Definition 2.3** ([19]). For $a\in\mathbb{C}\cup\{\infty\}$ and $p\in\mathbb{N}$ we denote by $N_p(r,a;f)$ the sum $\overline{N}(r,a;f)+\overline{N}(r,a;f\mid\geqslant 2)+\ldots+\overline{N}(r,a;f\mid\geqslant p)$. Clearly $N_1(r,a;f)=\overline{N}(r,a;f)$.

**Definition 2.4** ([1]). Let $f$ and $g$ be two non-constant meromorphic functions such that $f$ and $g$ share the value 1 IM. Let $z_0$ be a 1-point of $f$ with multiplicity $p$, a 1-point of $g$ with multiplicity $q$. We denote by $\overline{N}_L(r,1;f)$ the counting function of those 1-points of $f$ and $g$ where $p>q$, by $N_E^{1)}(r,1;f)$ the counting function of those 1-points of $f$ and $g$ where $p=q=1$ and by $\overline{N}_E^{(2}(r,1;f)$ the counting function of those 1-points of $f$ and $g$ where $p=q\geqslant 2$, each point in these counting functions being counted only once. In the same way we can define $\overline{N}_L(r,1;g)$, $N_E^{1)}(r,1;g)$, $\overline{N}_E^{(2}(r,1;g)$.

**Definition 2.5** ([10]). Let $f$, $g$ share a value $a$ IM. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those $a$-points of $f$ whose multiplicities differ from the multiplicities of the corresponding $a$-points of $g$. Clearly

$$\overline{N}_*(r,a;f,g) \equiv \overline{N}_*(r,a;g,f) = \overline{N}_L(r,a;f) + \overline{N}_L(r,a;g).$$

## 3. Lemmas

Let $F$, $G$ be two non-constant meromorphic functions. Henceforth we shall denote by $H$ and $V$ the functions

$$H = \Big(\frac{F''}{F'} - \frac{2F'}{F-1}\Big) - \Big(\frac{G''}{G'} - \frac{2G'}{G-1}\Big) \tag{3.1}$$

and

$$V = \Big(\frac{F'}{F-1} - \frac{F'}{F}\Big) - \Big(\frac{G'}{G-1} - \frac{G'}{G}\Big) = \frac{F'}{F(F-1)} - \frac{G'}{G(G-1)}. \tag{3.2}$$

**Lemma 3.1** ([20]). *Let $f$ be a non-constant meromorphic function and $p,k \in \mathbb{N}$, then*

$$N_p(r,0;f^{(k)}) \leqslant N_{p+k}(r,0;f) + k\overline{N}(r,\infty;f) + S(r,f).$$

**Lemma 3.2** ([11]). *If $N(r,0;f^{(k)}\,|f\neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of $f$, where a zero of $f^{(k)}$ is counted according to its multiplicity, then*

$$N(r,0;f^{(k)}\,|f\neq 0) \leqslant k\overline{N}(r,\infty;f) + N(r,0;f\,|<k) + k\overline{N}(r,0;f\,|\geqslant k) + S(r,f).$$

**Lemma 3.3** ([8]). *Suppose that $f$ is a non-constant meromorphic function, $k \in \mathbb{N}\setminus\{1\}$. If*

$$N(r,\infty;f) + N(r,0;f) + N(r,0;f^{(k)}) = S\Big(r,\frac{f'}{f}\Big),$$

*then $f(z) = \mathrm{e}^{az+b}$, where $a \in \mathbb{C}\setminus\{0\}$, $b \in \mathbb{C}$.*

**Lemma 3.4** ([15]). *Let $f$ be a non-constant meromorphic function and $P(f) = a_0 + a_1 f + a_2 f^2 + \ldots + a_n f^n$, where $a_0, a_1, \ldots, a_{n-1} \in \mathbb{C}$ and $a_n \in \mathbb{C}\setminus\{0\}$. Then $T(r,P(f)) = nT(r,f) + O(1)$.*

**Lemma 3.5.** *Let $f$ be a transcendental meromorphic function and $n, n_k, k \in \mathbb{N}$ and $n_i \in \mathbb{N} \cup \{0\}$, $i = 1, 2, \ldots, k-1$. Then $\varphi = f^n (f')^{n_1} \ldots (f^{(k)})^{n_k}$ is non-constant.*

P r o o f. Suppose $\varphi$ is constant. Then $\overline{N}(r, 0; f) = 0$ and $\overline{N}(r, \infty; f) = 0$. Also we see that

$$\left(\frac{1}{f}\right)^{n+s} \equiv \frac{(f')^{n_1} \ldots (f^{(k)})^{n_k}}{f^s} \frac{1}{\varphi}.$$

Then by Lemma 3.4 we have

$$\begin{aligned}(n+s)T(r, f) &\leqslant \sum_{i=1}^{k} n_i^* T\Big(r, \frac{f^{(i)}}{f}\Big) + T\Big(r, \frac{1}{\varphi}\Big) + O(1) \\ &\leqslant \sum_{i=1}^{k} n_i^* N\Big(r, \infty; \frac{f^{(i)}}{f}\Big) + S(r, f) \\ &\leqslant \sum_{i=1}^{k} n_i^* (N_i(r, 0; f) + i\overline{N}(r, \infty; f)) + S(r, f) = S(r, f),\end{aligned}$$

which is impossible. Hence $\varphi$ is non-constant. This completes the proof. □

**Lemma 3.6** ([17]). *Let $f_j$, $j = 1, 2, 3$ be meromorphic and $f_1$ non-constant. Suppose that*

$$\sum_{j=1}^{3} f_j \equiv 1$$

*and*

$$\sum_{j=1}^{3} N(r, 0; f_j) + 2\sum_{j=1}^{3} \overline{N}(r, \infty; f_j) < (\lambda + o(1))T(r),$$

*as $r \to \infty$, $r \in I$, $\lambda < 1$ and $T(r) = \max\limits_{1 \leqslant j \leqslant 3} T(r, f_j)$. Then $f_2 \equiv 1$ or $f_3 \equiv 1$.*

**Lemma 3.7** ([17], Theorem 1.24). *Let $f$ be a non-constant meromorphic function and let $k \in \mathbb{N}$. Suppose that $f^{(k)} \not\equiv 0$, then*

$$N(r, 0; f^{(k)}) \leqslant N(r, 0; f) + k\overline{N}(r, \infty; f) + S(r, f).$$

**Lemma 3.8** ([7]). *Let $f(z)$ be a non-constant entire function and let $k \in \mathbb{N} \setminus \{1\}$. If $f(z) f^{(k)}(z) \neq 0$, then $f(z) = \mathrm{e}^{az+b}$, where $a \in \mathbb{C} \setminus \{0\}$, $b \in \mathbb{C}$.*

**Lemma 3.9** ([8], [18]). *Let $f$ be a non-constant meromorphic function and let $a_1(z)$, $a_2(z)$ be two meromorphic functions such that $T(r, a_i) = S(r, f)$, $i = 1, 2$. Then*

$$T(r, f) \leqslant \overline{N}(r, \infty; f) + \overline{N}(r, a_1; f) + \overline{N}(r, a_2; f) + S(r, f).$$

**Lemma 3.10.** *Let $f$, $g$ be two non-constant meromorphic functions and* $F = f^n \prod_{i=1}^{k} (f^{(i)})^{n_i}$, $G = g^n \prod_{i=1}^{k} (g^{(i)})^{n_i}$, *where* $n, n_k, k \in \mathbb{N}$, $n_i \in \mathbb{N} \cup \{0\}$, $i = 1, 2, \ldots, k-1$. *Suppose* $H \not\equiv 0$. *If* $F$, $G$ *share* $(1, k_1)$, $f$, $g$ *share* $(\infty, p)$, *where* $k_1 \in \mathbb{N} \cup \{0\} \cup \{\infty\}$, $p \in \mathbb{N} \cup \{0\} \cup \{\infty\}$, *then*

$$((n+s)(p+1)+m_1-1)N(r,\infty;f \mid\geqslant p+1) \leqslant \overline{N}(r,0;F) + \overline{N}(r,0;G) + \overline{N}_*(r,1;F,G) + S(r).$$

P r o o f. Suppose $\infty$ is an e.v.P of both $f$ and $g$, then the lemma follows immediately.

Next suppose $\infty$ is not an e.v.P of $f$ and $g$. We assert that $V \not\equiv 0$. If not, suppose $V \equiv 0$. Then by integration we obtain

$$1 - \frac{1}{F} \equiv A\Big(1 - \frac{1}{G}\Big).$$

It means that if $z_0$ is a pole of $f$ then it is a pole of $g$. Hence from the definition of $F$ and $G$ we have $1/F(z_0) = 0$ and $1/G(z_0) = 0$. So $A = 1$ and hence $F \equiv G$. Consequently $H \equiv 0$, which contradicts our assumption. Hence $V \not\equiv 0$. Let $z_0$ be a pole of $f$ with multiplicity $q$ and a pole of $g$ with multiplicity $r$. If both $q$ and $r$ are $\leqslant p$, then $q = r$ but when both $q$ and $r$ are $\geqslant p+1$, they may or may not be equal. Clearly $z_0$ is a pole of $F$ with multiplicity $(n+s)q+m_1$ and a pole of $G$ with multiplicity $(n+s)r+m_1$. We note that there is no pole of $F$ and $G$ of order $t_1$ satisfying $(n+s)p+m_1+1 \leqslant t_1 \leqslant (n+s)(p+1)+m_1-1$. Since $f$ and $g$ share $(\infty, p)$, from the definition of $V$ it is clear that $z_0$ is a zero of $V$ with multiplicity at least $(n+s)(p+1)+m_1-1$.

So from the definition of $V$ we have

$$\begin{aligned}((n+s)(p+1)+m_1-1)\overline{N}(r,\infty;f \mid\geqslant p+1) &\leqslant N(r,0;V) \leqslant N(r,\infty;V) + S(r,f) + S(r,g)\\ &\leqslant \overline{N}(r,0;F) + \overline{N}(r,0;G) + \overline{N}_*(r,1;F,G) + S(r).\end{aligned}$$

This completes the proof. □

**Lemma 3.11.** *Let $f$, $g$ be two non-constant meromorphic functions,* $n, n_k, k \in \mathbb{N}$, $n_i \in \mathbb{N} \cup \{0\}$, $i = 1, 2, \ldots, k-1$. *Suppose* $H \not\equiv 0$. *If* $F$, $G$ *share* $(1, k_1)$ *and* $f$, $g$ *share* $(\infty, 0)$, *where* $F$ *and* $G$ *are given as in Lemma* 3.10, $k_1 \in \mathbb{N} \cup \{0\} \cup \{\infty\}$, *then*

$$\overline{N}(r,\infty;f) \leqslant \frac{2(t+1)}{n+s+m_1-2m-1}T(r) + \frac{1}{n+s+m_1-2m-1}\overline{N}_*(r,1;F,G) + S(r).$$

P r o o f. Using Lemmas 3.2 and 3.10 for $p=0$ we get

$$\begin{aligned}(n+s+m_1-1)\overline{N}(r,\infty;f)\\ &\leqslant \overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\ &\leqslant \overline{N}(r,0;f)+\sum_{i=1}^{k} n_i^*\overline{N}(r,0;f^{(i)}\,|f\neq 0)+\overline{N}(r,0;g)\\ &\quad+\sum_{i=1}^{k} n_i^*\overline{N}(r,0;g^{(i)}\,|g\neq 0)+\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\ &\leqslant \overline{N}(r,0;f)+m\overline{N}(r,\infty;f)+tN(r,0;f)+\overline{N}(r,0;g)+m\overline{N}(r,\infty;g)\\ &\quad+tN(r,0;g)+\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\ &\leqslant 2(t+1)T(r)+2m\overline{N}(r,\infty;f)+\overline{N}_*(r,1;F,G)+S(r).\end{aligned}$$

Hence the lemma follows. □

**Lemma 3.12.** *Let $f$ be a non-constant meromorphic function, $F=f^n\prod\limits_{i=1}^{k}(f^{(i)})^{n_i}$, where $n,n_k,k\in\mathbb{N}$ and $n_i\in\mathbb{N}\cup\{0\}$, $i=1,2,\ldots,k-1$ are such that $n>s$. Then*

$$(n-s)T(r,f)\leqslant T(r,F)-sN(r,\infty;f)-N\biggl(r,0;\prod_{i=1}^{k}(f^{(i)})^{n_i}\biggr)+S(r,f).$$

P r o o f. Note that

$$\begin{aligned}N(r,\infty;F)&=N(r,\infty;f^n)+N\biggl(r,\infty;\prod_{i=1}^{k}(f^{(i)})^{n_i}\biggr)\\ &=N(r,\infty;f^n)+sN(r,\infty;f)+m_1\overline{N}(r,\infty;f)+S(r,f).\end{aligned}$$

That is,

$$N(r,\infty;f^n)=N(r,\infty;F)-sN(r,\infty;f)-m_1\overline{N}(r,\infty;f)+S(r,f).$$

Also

$$\begin{aligned}m(r,f^n)&=m\biggl(r,\frac{F}{\prod_{i=1}^{k}(f^{(i)})^{n_i}}\biggr)\leqslant m(r,F)+m\biggl(r,\frac{1}{\prod_{i=1}^{k}(f^{(i)})^{n_i}}\biggr)+S(r,f)\\ &=m(r,F)+T\biggl(r,\prod_{i=1}^{k}(f^{(k)})^{n_k}\biggr)-N\biggl(r,0;\prod_{i=1}^{k}(f^{(k)})^{n_i}\biggr)+S(r,f)\end{aligned}$$

$$= m(r,F) + N\left(r,\infty;\prod_{i=1}^{k}(f^{(i)})^{n_i}\right) + m\left(r,\prod_{i=1}^{k}(f^{(i)})^{n_i}\right)$$
$$- N\left(r,0;\prod_{i=1}^{k}(f^{(i)})^{n_i}\right) + S(r,f)$$
$$\leqslant m(r,F) + sN(r,\infty;f) + m_1\overline{N}(r,\infty;f) + m\left(r,\frac{1}{f^s}\prod_{i=1}^{k}(f^{(i)})^{n_i}\right)$$
$$+ m(r,f^s) - N\left(r,0;\prod_{i=1}^{k}(f^{(i)})^{n_i}\right) + S(r,f)$$
$$= m(r,F) + sT(r,f) + m_1\overline{N}(r,\infty;f) - N\left(r,0;\prod_{i=1}^{k}(f^{(i)})^{n_i}\right) + S(r,f).$$

Now

$$nT(r,f) = N(r,\infty;f^n) + m(r,f^n)$$
$$\leqslant T(r,F) + sT(r,f) - sN(r,\infty;f) - N\left(r,0;\prod_{i=1}^{k}(f^{(i)})^{n_i}\right) + S(r,f),$$

i.e.,

$$(n-s)T(r,f) \leqslant T(r,F) - sN(r,\infty;f) - N\left(r,0;\prod_{i=1}^{k}(f^{(i)})^{n_i}\right) + S(r,f).$$

This completes the proof. □

**Lemma 3.13.** *Let $f$ be a transcendental meromorphic function, $n, n_k, k \in \mathbb{N}$, $n_i \in \mathbb{N} \cup \{0\}$, $i = 1, 2, \ldots, k-1$ and let $a(z)$ ($\not\equiv 0, \infty$) be a small function of $f$. If $n > s+1$, then $f^n(f')^{n_1}\ldots(f^{(k)})^{n_k} - a(z)$ has infinitely many zeros.*

P r o o f. Let $F = f^n(f')^{n_1}\ldots(f^{(k)})^{n_k}$. Now in view of Lemma 3.12 and the second theorem for small functions (see [14]) we get

$$(n-s)T(r,f) \leqslant T(r,F) - sN(r,\infty;f) - N(r,0;(f')^{n_1}\ldots(f^{(k)})^{n_k}) + S(r,f)$$
$$\leqslant \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,a(z);F) - sN(r,\infty;f)$$
$$- N(r,0;(f')^{n_1}\ldots(f^{(k)})^{n_k}) + (\varepsilon + o(1))T(r,f)$$
$$\leqslant \overline{N}(r,0;f) + \overline{N}(r,a(z);F) + (\varepsilon + o(1))T(r,f)$$
$$\leqslant T(r,f) + \overline{N}(r,a(z);F) + (\varepsilon + o(1))T(r,f)$$

for all $\varepsilon > 0$. Take $\varepsilon < 1$. Since $n > s+1$, from the above one can easily see that $F - a(z)$ has infinitely many zeros. This completes the proof. □

**Lemma 3.14** ([9]). *Let $f$ and $g$ be two non-constant meromorphic functions. Suppose that $f$ and $g$ share 0 and $\infty$ CM, $f^{(k)}$ and $g^{(k)}$ share 0 CM for $k=1,2,\dots,6$. Then $f$ and $g$ satisfy one of the following conditions:*

(i) $f\equiv tg$, *where $t$ ($\neq 0$) is a constant,*

(ii) $f(z)=\mathrm{e}^{az+b}$, $g(z)=\mathrm{e}^{cz+d}$, *where $a$, $b$, $c$ and $d$ are constants such that $ac\neq 0$,*

(iii) $f(z)=a/(1-b\mathrm{e}^{\alpha(z)})$, $g(z)=a/(\mathrm{e}^{-\alpha(z)}-b)$, *where $a$, $b$ are nonzero constants and $\alpha(z)$ is a non-constant entire function,*

(iv) $f(z)=a(1-b\mathrm{e}^{cz})$, $g(z)=d(\mathrm{e}^{-cz}-b)$, *where $a$, $b$, $c$ and $d$ are nonzero constants.*

**Lemma 3.15.** *Let $f$ and $g$ be two transcendental meromorphic functions such that $f(z)/g(z)\neq \mathrm{e}^{az+b}$, where $a,b\in\mathbb{C}$ ($a\neq 0$) and let $n,n_k,k\in\mathbb{N}$, $n_i\in\mathbb{N}\cup\{0\}$, $i=1,2,\dots,k-1$ be such that $n\geqslant 2$. Suppose $f$ and $g$ share 0 CM and $\infty$ IM. If $f^n\prod\limits_{i=1}^{k}(f^{(i)})^{n_i}\equiv g^n\prod\limits_{i=1}^{k}(g^{(i)})^{n_i}$, then $f\equiv tg$, where $t$ is a constant such that $t^{n+s}=1$.*

P r o o f. Suppose

$$f^n\prod_{i=1}^{k}(f^{(i)})^{n_i}\equiv g^n\prod_{i=1}^{k}(g^{(i)})^{n_i}, \tag{3.3}$$

i.e.,

$$\frac{f^n}{g^n}\equiv\prod_{i=1}^{k}(g^{(i)})^{n_i}\Big/\prod_{i=1}^{k}(f^{(i)})^{n_i}. \tag{3.4}$$

Since $f$ and $g$ share $\infty$ IM, it follows from (3.3) that $f$ and $g$ share $\infty$ CM and so $f^{(i)}$ and $g^{(i)}$ share $\infty$ CM, where $i=1,2,\dots,k$. Let $h_1=f/g$ and $h_2=\prod\limits_{i=1}^{k}(f^{(i)})^{n_i}\Big/\prod\limits_{i=1}^{k}(g^{(i)})^{n_i}$. Since $f$ and $g$ share 0 CM, it follows that $h_1\neq 0,\infty$ and $h_2\neq 0,\infty$. From (3.4) we see that

$$h_1^n h_2\equiv 1. \tag{3.5}$$

First we suppose $h_1$ is a non-constant entire function. Clearly $h_2$ is also a non-constant entire function. Let $F_1=h_1^n$ and $G_1=h$. From (3.5) we get

$$F_1G_1\equiv 1. \tag{3.6}$$

Clearly $F_1\not\equiv dG_1$, where $d$ is a nonzero constant, otherwise $F_1$ would be a constant and so $h_1$ would be a constant. Since $F_1\neq 0,\infty$ and $G_1\neq 0,\infty$ there exist two non-constant entire functions $\alpha$ and $\beta$ such that $F_1=\mathrm{e}^{\alpha}$ and $G_1=\mathrm{e}^{\beta}$. Now from (3.6) we see that $\alpha + \beta = C$, where $C \in \mathbb{C}$. Therefore $\alpha' = -\beta'$. Note that $F_1' = \alpha' e^{\alpha}$ and $G_1' = \beta' e^{\beta}$. This shows that $F_1'$ and $G_1'$ share 0 CM. Note that $F_1 \neq 0, \infty$, $G_1 \neq 0, \infty$ and $F_1 \not\equiv dG_1$, where $d$ is a nonzero constant. Now in view of Lemma 3.14 we have to consider the case

$$F_1(z) = c_1 e^{az} \quad \text{and} \quad G_1(z) = c_2 e^{-az},$$

where $a, c_1, c_2$ are nonzero constants such that $c_1 c_2 = 1$. Since $(f(z)/g(z))^n = c_1 e^{az}$, it follows that

$$\frac{f(z)}{g(z)} = t_1 e^{a/nz} = t_1 e^{cz}, \tag{3.7}$$

where $c, t_1$ are nonzero constants such that $t_1^n = c_1$ and $c = a/n$.

Now from (3.7) we arrive at a contradiction. Hence $h_1$ is constant. Then from (3.3) we get $h_1^{n+s} = 1$. Therefore we have $f \equiv tg$, where $t$ is a constant such that $t^{n+s} = 1$. This completes the proof. □

Remark 3.1. Instead of $f$ and $g$ share 0 CM, one can assume that $f^{(k)}$ and $g^{(k)}$ share 0 CM in Lemma 3.15 when $n_i = 0$, $i = 1, 2, \ldots, k-1$.

**Lemma 3.16.** *Let $f$, $g$ be two transcendental meromorphic functions and let $f^n \prod_{i=1}^{k} (f^{(i)})^{n_i} - p(z)$ and $g^n \prod_{i=1}^{k} (g^{(i)})^{n_i} - p(z)$ share 0 CM and $f$, $g$ share $\infty$ IM, where $p(z)$ is defined as in (2.1) and $n, n_k \in \mathbb{N}$, $n_i \in \mathbb{N} \cup \{0\}$. Suppose $f^n (f')^{n_1} \ldots (f^{(k)})^{n_k} g^n (g')^{n_1} \ldots (g^{(k)})^{n_k} \equiv p^2$.*

(i) *If $p(z)$ is not a constant and $l_i$ is a multiple of $n_1$ for all $i = 1, 2, \ldots, t$, where $l_i$ is defined as in (2.1), then $f(z) = c_1 e^{cQ(z)}$, $g(z) = c_2 e^{-cQ(z)}$, where $Q(z) = \int_0^z p^{1/n_1}(t)\,dt$, $c_1, c_2, c \in \mathbb{C}$ are such that $c^{2n_1}(c_1 c_2)^{n+n_1} = (-1)^{n_1}$,*

(ii) *if $p(z) = b \in \mathbb{C} \setminus \{0\}$, then $f(z) = c_3 e^{dz}$, $g(z) = c_4 e^{-dz}$, where $c_3, c_4, d \in \mathbb{C}$ are such that $(-1)^{m_1}(c_3 c_4)^{n+s} d^{2m_1} = b^2$ .*

Proof. Suppose

$$f^n (f')^{n_1} \ldots (f^{(k)})^{n_k} g^n (g')^{n_1} \ldots (g^{(k)})^{n_k} \equiv p^2. \tag{3.8}$$

Since $f$ and $g$ share $\infty$ IM, from (3.8) one can easily see that $f$ and $g$ are transcendental entire functions. We now consider the following cases.

*Case 1.* Let $\deg(p(z)) = l \in \mathbb{N}$. From (3.8) it follows that $N(r, 0; f) = O(\log r)$ and $N(r, 0; g) = O(\log r)$. Let

$$F_1 = \frac{f^n (f')^{n_1} \ldots (f^{(k)})^{n_k}}{p} \quad \text{and} \quad G_1 = \frac{g^n (g')^{n_1} \ldots (g^{(k)})^{n_k}}{p}. \tag{3.9}$$

From (3.8) we get

$$F_1G_1 \equiv 1. \tag{3.10}$$

By Lemma 3.5, we have $F_1 \not\equiv cG_1$, where $c \in \mathbb{C}\setminus\{0\}$. Let

$$\Phi = \frac{f^n(f')^{n_1}\dots(f^{(k)})^{n_k}-p}{g^n(g')^{n_1}\dots(g^{(k)})^{n_k}-p}. \tag{3.11}$$

We deduce from (3.11) that

$$\Phi \equiv \mathrm{e}^{\beta}, \tag{3.12}$$

where $\beta$ is an entire function. Let $f_1 = F_1$, $f_2 = -\mathrm{e}^{\beta}G_1$ and $f_3 = \mathrm{e}^{\beta}$. Here $f_1$ is transcendental. Now from (3.12) we have $f_1+f_2+f_3 \equiv 1$. Hence by Lemma 3.7 we get

$$\begin{aligned}\sum_{j=1}^{3} N(r,0;f_j) + 2\sum_{j=1}^{3}\overline{N}(r,\infty;f_j) &\leqslant N(r,0;F_1) + N(r,0;\mathrm{e}^{\beta}G_1) + O(\log r)\\ &\leqslant (\lambda+o(1))T(r),\end{aligned}$$

as $r\to\infty$, $r\in I$, $\lambda<1$ and $T(r) = \max\limits_{1\leqslant j\leqslant 3} T(r,f_j)$. So by Lemma 3.6 we get either $\mathrm{e}^{\beta}G_1 \equiv -1$ or $\mathrm{e}^{\beta}\equiv 1$. But here the only possibility is that $\mathrm{e}^{\beta}G_1\equiv -1$, i.e., $g^n(g')^{n_1}\dots(g^{(k)})^{n_k} \equiv -\mathrm{e}^{-\beta}p(z)$ and so from (3.8) we obtain $F_1 \equiv \mathrm{e}^{\gamma_1}G_1$, i.e.,

$$f^n(f')^{n_1}\dots(f^{(k)})^{n_k} \equiv \mathrm{e}^{\gamma_1} g^n(g')^{n_1}\dots(g^{(k)})^{n_k},$$

where $\gamma_1$ is a non-constant entire function. Then from (3.8) we get

$$f^n(f')^{n_1}\dots(f^{(k)})^{n_k} \equiv c\mathrm{e}^{\gamma_1/2}p(z), g^n(g')^{n_1}\dots(g^{(k)})^{n_k} \equiv c\mathrm{e}^{-\gamma_1/2}p(z), \tag{3.13}$$

where $c\pm 1$. This shows that $f^n(f')^{n_1}\dots(f^{(k)})^{n_k}$ and $g^n(g')^{n_1}\dots(g^{(k)})^{n_k}$ share 0 CM. Since $N(r,0;f) = O(\log r)$ and $N(r,0;g) = O(\log r)$, so we can take

$$f(z) = h_1(z)\mathrm{e}^{\alpha(z)}, \quad g(z) = h_2(z)\mathrm{e}^{\beta(z)}, \tag{3.14}$$

where $h_1$ and $h_2$ are nonzero polynomials and $\alpha$, $\beta$ are two non-constant entire functions. We deduce from (3.8) and (3.14) that either both $\alpha$ and $\beta$ are transcendental entire functions or both $\alpha$ and $\beta$ are polynomials. We now consider the following cases.

*Subcase 1.1.* Let $k \in \mathbb{N} \setminus \{1\}$. First we suppose both $\alpha$ and $\beta$ are transcendental entire functions. Let $\alpha_1 = \alpha' + h_1'/h_1$ and $\beta_1 = \beta' + h_2'/h_2$. Clearly both $\alpha_1$ and $\beta_1$ are transcendental. Note that

$$S(r, \alpha_1) = S\Big(r, \frac{f'}{f}\Big), \quad S(r, \beta_1) = S\Big(r, \frac{g'}{g}\Big).$$

Moreover, we see that

$$N(r, 0; f^n (f')^{n_1} \dots (f^{(k)})^{n_k}) \leqslant N(r, 0; p^2) = O(\log r),$$
$$N(r, 0; g^n (g')^{n_1} \dots (g^{(k)})^{n_k}) \leqslant N(r, 0; p^2) = O(\log r).$$

From these inequalities and using (3.14) we have

$$N(r, \infty; f) + N(r, 0; f) + N(r, 0; f^{(k)}) = S(r, \alpha_1) = S\Big(r, \frac{f'}{f}\Big) \tag{3.15}$$

and

$$N(r, \infty; g) + N(r, 0; g) + N(r, 0; g^{(k)}) = S(r, \beta_1) = S\Big(r, \frac{g'}{g}\Big). \tag{3.16}$$

Then from (3.15), (3.16) and Lemma 3.3 we have

$$f(z) = \mathrm{e}^{az+b}, \quad g(z) = \mathrm{e}^{cz+d}, \tag{3.17}$$

where $a, c \in \mathbb{C} \setminus \{0\}$, $b, d \in \mathbb{C}$. But these types of $f$ and $g$ do not agree with the relation (3.8). Next we suppose both $\alpha$ and $\beta$ are polynomials. Also from (3.8) we get $\alpha + \beta \equiv C$ i.e., $\alpha' \equiv -\beta'$. Therefore $\deg(\alpha) = \deg(\beta)$. We deduce from (3.14) that

$$\begin{aligned} f^n (f')^{n_1} \dots (f^{(k)})^{n_k} &\equiv A h_1^n \prod_{i=1}^{k} (h_1 (\alpha')^i + P_{i-1}(\alpha', h_1'))^{n_i} \mathrm{e}^{(n+s)\alpha} \\ &\equiv p(z) \mathrm{e}^{(n+s)\alpha}, \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} g^n (g')^{n_1} \dots (g^{(k)})^{n_k} &\equiv B h_2^n \prod_{i=1}^{k} (h_2 (\beta')^i + Q_{i-1}(\beta', h_2'))^{n_i} \mathrm{e}^{(n+s)\beta} \\ &\equiv p(z) \mathrm{e}^{(n+s)\beta}, \end{aligned} \tag{3.19}$$

where $A, B \in \mathbb{C} \setminus \{0\}$, and $P_{i-1}(\alpha', h_1')$ and $Q_{i-1}(\beta', h_2')$, $i = 1, 2, \dots, k$ are differential polynomials in $\alpha', h_1'$ and $\beta', h_2'$, respectively.

Since $p(z)$ is a polynomial, from (3.18) and (3.19) we conclude that both $h_1, h_2 \in \mathbb{C}\setminus\{0\}$. So we can rewrite $f$ and $g$ as

$$f = \mathrm{e}^{\gamma_2}, \quad g = \mathrm{e}^{\delta_2}, \tag{3.20}$$

where $\gamma_2 + \delta_2 \equiv C \in \mathbb{C}\setminus\{0\}$ and $\deg(\gamma_2) = \deg(\delta_2)$. Clearly $\gamma_2' \equiv -\delta_2'$. If $\deg(\gamma_2) = \deg(\delta_2) = 1$, we then again get a contradiction from (3.8). Next we suppose $\deg(\gamma_2) = \deg(\delta_2) \geqslant 2$. We deduce from (3.20) that

$$\begin{aligned}
f' &= \gamma_2'\mathrm{e}^{\gamma_2},\\
f'' &= ((\gamma_2')^2 + \gamma_2'')\mathrm{e}^{\gamma_2},\\
f''' &= ((\gamma_2')^3 + 3\gamma_2'\gamma_2'' + \gamma_2''')\mathrm{e}^{\gamma_2},\\
f^{(iv)} &= ((\gamma_2')^4 + 6(\gamma_2')^2\gamma_2'' + 3(\gamma_2'')^2 + 4\gamma_2'\gamma_2''' + \gamma_2^{(iv)})\mathrm{e}^{\gamma_2},\\
f^{(v)} &= ((\gamma_2')^5 + 10(\gamma_2')^3\gamma_2'' + 15\gamma_2'(\gamma_2'')^2 + 10(\gamma_2')^2\gamma_2''' + 10\gamma_2''\gamma_2''' + 5\gamma_2'\gamma_2^{(iv)} + \gamma_2^{(v)})\mathrm{e}^{\gamma_2},\\
&\ \ \vdots\\
f^{(k)} &= \Big((\gamma_2')^k + \frac{k(k-1)}{2}(\gamma_2')^{k-2}\gamma_2'' + P_{k-2}(\gamma_2')\Big)\mathrm{e}^{\gamma_2}.
\end{aligned}$$

Similarly we get

$$\begin{aligned}
g^{(k)} &= \Big((\delta_2')^k + \frac{k(k-1)}{2}(\delta_2')^{k-2}\delta_2'' + P_{k-2}(\delta_2')\Big)\mathrm{e}^{\delta_2}\\
&= \Big((-1)^k(\gamma_2')^k + \frac{k(k-1)}{2}(-1)^{k-1}(\gamma_2')^{k-2}\gamma_2'' + P_{k-2}(-\gamma_2')\Big)\mathrm{e}^{\delta_2},
\end{aligned}$$

where $P_{k-2}(\gamma_2')$ is a differential polynomial in $\gamma_2'$. Since $\deg(\gamma_2) \geqslant 2$, we observe that $\deg((\gamma_2')^k) \geqslant k\deg(\gamma_2')$ and so $(\gamma_2')^{k-2}\gamma_2''$ is either a nonzero constant or $\deg((\gamma_2')^{k-2}\gamma_2'') \geqslant (k-1)\deg(\gamma_2') - 1$. Also we see that

$$\deg((\gamma_2')^k) > \deg((\gamma_2')^{k-2}\gamma_2'') > \deg(P_{k-2}(\gamma_2')) \quad (\text{or } \deg(P_{k-2}(-\gamma_2'))).$$

Since $f$ and $g$ have no zeros, from (3.13) it follows that $(f')^{n_1}\dots(f^{(k)})^{n_k}$ and $(g')^{n_1}\dots(g^{(k)})^{n_k}$ share 0 CM and so

$$\begin{aligned}
((\gamma_2)')^{n_1}\prod_{i=2}^{k}\Big((\gamma_2')^i + \frac{i(i-1)}{2}(\gamma_2')^{i-2}\gamma_2'' + P_{i-2}(\gamma_2')\Big)^{n_i} &\equiv d(-1)^{n_1}((\gamma_2)')^{n_1}\\
\times\prod_{i=2}^{k}\Big((-1)^i(\gamma_2')^i + \frac{i(i-1)}{2}(-1)^{i-1}(\gamma_2')^{i-2}\gamma_2'' + P_{i-2}(-\gamma_2')\Big)^{n_i}&,
\end{aligned}\tag{3.21}$$

where $d \in \mathbb{C}\setminus\{0\}$.

Now from (3.21) we arrive at a contradiction since $k \geqslant 2$.

*Subcase 1.2.* Let $k = 1$. Suppose that $\alpha$ and $\beta$ are transcendental. Then from (3.8) and (3.14) we get

$$(h_1h_2)^n(h_1\alpha' + h_1')^{n_1}(h_2\beta' + h_2')^{n_1}\mathrm{e}^{(n+n_1)(\alpha+\beta)} \equiv p^2(z). \tag{3.22}$$

Let $\alpha + \beta = \gamma$ and $s_1 = n + n_1$. From (3.22) we know that $\gamma$ is not a constant since in that case we get a contradiction. Now from (3.22) we get

$$(h_1h_2)^n(h_1\alpha' + h_1')^{n_1}(h_2(\gamma' - \alpha') + h_2')^{n_1}\mathrm{e}^{s_1\gamma} \equiv p^2(z). \tag{3.23}$$

We have $T(r, \gamma') = m(r, s_1\gamma') + O(1) = m(r, (\mathrm{e}^{s_1\gamma})'/\mathrm{e}^{s_1\gamma}) = S(r, \mathrm{e}^{s_1\gamma})$. Thus from (3.23) we get

$$\begin{aligned}
T(r, \mathrm{e}^{s_1\gamma}) &\leqslant T\Big(r, \frac{p^2}{(h_1h_2)^n(h_1\alpha' + h_1')^{n_1}(h_2(\gamma' - \alpha') + h_2')^{n_1}}\Big) + O(1)\\
&\leqslant n_1T(r, \alpha') + n_1T(r, \gamma' - \alpha') + O(\log r) + O(1)\\
&\leqslant 2n_1T(r, \alpha') + S(r, \alpha') + S(r, \mathrm{e}^{s_1\gamma}),
\end{aligned}$$

which implies that $T(r, \mathrm{e}^{s_1\gamma}) = O(T(r, \alpha'))$ and so $S(r, \mathrm{e}^{s_1\gamma})$ can be replaced by $S(r, \alpha')$. Thus we get $T(r, \gamma') = S(r, \alpha')$ and so $\gamma'$ is a small function with respect to $\alpha'$. In view of (3.23) and by Lemma 3.9 we get

$$\begin{aligned}
T(r, \alpha') &\leqslant \overline{N}(r, \infty; \alpha') + \overline{N}(r, 0; h_1\alpha' + h_1') + \overline{N}(r, 0; h_2(\gamma' - \alpha') + h_2') + S(r, \alpha')\\
&\leqslant O(\log r) + S(r, \alpha'),
\end{aligned}$$

which shows that $\alpha'$ is a polynomial and so $\alpha$ is a polynomial. Similarly we can prove that $\beta$ is also a polynomial. This contradicts the fact that $\alpha$ and $\beta$ are transcendental. Next suppose without loss of generality that $\alpha$ is a polynomial and $\beta$ is a transcendental entire function. Then $\gamma$ is transcendental. So in view of (3.23) we obtain

$$\begin{aligned}
s_1T(r, \mathrm{e}^{\gamma}) &\leqslant T\Big(r, \frac{p^2}{(h_1h_2)^n(h_1\alpha' + h_1')^{n_1}(h_2(\gamma' - \alpha') + h_2')^{n_1}}\Big) + O(1)\\
&\leqslant n_1T(r, \alpha') + n_1T(r, \gamma' - \alpha') + S(r, \mathrm{e}^{\gamma})\\
&\leqslant n_1T(r, \gamma') + S(r, \mathrm{e}^{\gamma}) = S(r, \mathrm{e}^{\gamma}),
\end{aligned}$$

which leads to a contradiction. Thus both $\alpha$ and $\beta$ are polynomials. From (3.8) we conclude that $\alpha(z) + \beta(z) \equiv C \in \mathbb{C}$ and so $\alpha'(z) + \beta'(z) \equiv 0$. We deduce from (3.8) that

$$f^n(f')^{n_1} \equiv h_1^n(h_1\alpha' + h_1')^{n_1}\mathrm{e}^{(n+n_1)\alpha} \equiv p(z)\mathrm{e}^{(n+n_1)\alpha}, \tag{3.24}$$

and

$$g^n(g')^{n_1} \equiv h_2^n(h_2\beta' + h_2')^{n_1}\mathrm{e}^{(n+n_1)\beta} \equiv p(z)\mathrm{e}^{(n+n_1)\beta}. \tag{3.25}$$

Since $p(z)$ is a polynomial, from (3.24) and (3.25) we conclude that both $h_1$ and $h_2$ are nonzero constant. So we can rewrite $f$ and $g$ as

$$f = \mathrm{e}^{\gamma_3}, \quad g = \mathrm{e}^{\delta_3}. \tag{3.26}$$

Now from (3.8) we get

$$(\gamma_3')^{n_1}(\delta_3')^{n_1}\mathrm{e}^{(n+n_1)(\gamma_3+\delta_3)} \equiv p^2. \tag{3.27}$$

From (3.27) we can conclude that $\gamma_3(z) + \delta_3(z) \equiv C \in \mathbb{C}$ and so $\gamma_3'(z) + \delta_3'(z) \equiv 0$. Thus from (3.27) we get $\mathrm{e}^{(n+n_1)C}(\gamma_3')^{n_1}(\delta_3')^{n_1} \equiv p^2(z)$, i.e.,

$$(-1)^{n_1}\mathrm{e}^{(n+n_1)C}(\gamma_3')^{2n_1} \equiv p^2(z). \tag{3.28}$$

We now consider the following two subcases.

*Subcase 1.2.1.* Suppose at least one of $l_i$, $i = 1, 2, \ldots, t$ is not a multiple of $n_1$. As $\gamma_3'$ is a polynomial, from (3.28) we arrive at a contradiction.

*Subcase 1.2.2.* Suppose $l_i$ is a multiple of $n_1$ for all $i = 1, 2, \ldots, t$. By computation, from (3.28) we get

$$\gamma_3' = cp^{1/n_1}(z), \quad \delta_3' = -cp^{1/n_1}(z). \tag{3.29}$$

Hence

$$\gamma_3(z) = cQ(z) + b_1, \quad \delta_3(z) = -cQ(z) + b_2, \tag{3.30}$$

where $Q(z) = \int_0^z p^{1/n_1}(t)\,\mathrm{d}t$ and $b_1, b_2 \in \mathbb{C}$. Finally, we take $f$ and $g$ as

$$f(z) = c_1\mathrm{e}^{cQ(z)}, \quad g(z) = c_2\mathrm{e}^{-cQ(z)},$$

where $c_1, c_2 \in \mathbb{C}$ and $c \in \mathbb{C} \setminus \{0\}$ such that $c^{2n_1}(c_1c_2)^{n+n_1} = (-1)^{n_1}$.

*Case 2.* Let $p(z) = b \in \mathbb{C} \setminus \{0\}$. Then from (3.8) we get

$$f^n(f')^{n_1}\ldots(f^{(k)})^{n_k}g^n(g')^{n_1}\ldots(g^{(k)})^{n_k} \equiv b^2, \tag{3.31}$$

where $f$ and $g$ are transcendental entire functions. Clearly $f$ and $g$ have no zeros and so we can take $f$ and $g$ as

$$f = \mathrm{e}^{\alpha}, \quad g = \mathrm{e}^{\beta}, \tag{3.32}$$

where $\alpha(z)$, $\beta(z)$ are two non-constant entire functions. We now consider the following two subcases.

*Subcase 2.1.* Let $k \geqslant 2$. From (3.31) it is clear that $ff^{(k)} \neq 0$ and $gg^{(k)} \neq 0$. Then by Lemma 3.8 we have

$$f(z) = \mathrm{e}^{az+b}, \quad g(z) = \mathrm{e}^{cz+d}, \tag{3.33}$$

where $a, c \in \mathbb{C} \setminus \{0\}$, $b, d \in \mathbb{C}$. But from (3.31) we see that $a + c = 0$.

*Subcase 2.2.* Let $k = 1$. Considering Subcase 1.2 one can easily get

$$f(z) = \mathrm{e}^{az+b}, \quad g(z) = \mathrm{e}^{cz+d}, \tag{3.34}$$

where $a, c \in \mathbb{C} \setminus \{0\}$, $b, d \in \mathbb{C}$. Finally, we can take $f$ and $g$ as

$$f(z) = c_3\mathrm{e}^{dz}, \quad g(z) = c_4\mathrm{e}^{-dz},$$

where $c_3, c_4, d \in \mathbb{C} \setminus \{0\}$ are such that $(-1)^{m_1}(c_3c_4)^{n+s}d^{2m_1} = b^2$. This completes the proof. □

**Lemma 3.17.** *Let $f$ and $g$ be two transcendental meromorphic functions and let $F = f^n(f')^{n_1}\dots(f^{(k)})^{n_k}/p$ and $G = g^n(g')^{n_1}\dots(g^{(k)})^{n_k}/p$, where $p(z)$ is defined as in (2.1) and $n, n_k, k \in \mathbb{N}$, $n_i \in \mathbb{N}\cup\{0\}$, $i = 1, 2, \dots, k-1$ are such that $n > s+t+m+2$. If $f$, $g$ share $(\infty, 0)$ and $H \equiv 0$ then either*

$$f^n(f')^{n_1}\dots(f^{(k)})^{n_k}g^n(g')^{n_1}\dots(g^{(k)})^{n_k} \equiv p^2(z),$$

*where $f^n(f')^{n_1}\dots(f^{(k)})^{n_k} - p(z)$ and $g^n(g')^{n_1}\dots(g^{(k)})^{n_k} - p(z)$ share 0 CM or*

$$f^n(f')^{n_1}\dots(f^{(k)})^{n_k} \equiv g^n(g')^{n_1}\dots(g^{(k)})^{n_k}.$$

Proof. Since $H \equiv 0$, by integration we get

$$\frac{1}{F-1} \equiv \frac{bG + a - b}{G-1}, \tag{3.35}$$

where $a, b \in \mathbb{C}$ and $a \in \mathbb{C} \setminus \{0\}$. From (3.35) it is clear that $F$ and $G$ share $(1, \infty)$. We now consider the following cases.

*Case 1.* Let $b \in \mathbb{C} \setminus \{0\}$ and $a \neq b$. If $b = -1$, then from (3.35) we have

$$F \equiv \frac{-a}{G - a - 1}.$$

Therefore $\overline{N}(r, a+1; G) = \overline{N}(r, \infty; F) = \overline{N}(r, \infty; f) + \overline{N}(r, 0; p)$. So in view of Lemma 3.12 and the second fundamental theorem we get

$$\begin{aligned}(n-s)T(r,g) &\leqslant T(r,G) - sN(r,\infty;g) - N(r,0;(g')^{n_1}\dots(g^{(k)})^{n_k}) + S(r,g)\\ &\leqslant \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,a+1;G) - dN(r,\infty;g)\\ &\quad - N(r,0;(g')^{n_1}\dots(g^{(k)})^{n_k}) + S(r,g)\\ &\leqslant \overline{N}(r,0;g) + \overline{N}(r,0;(g')^{n_1}\dots(g^{(k)})^{n_k}) + \overline{N}(r,\infty;f)\\ &\quad - N(r,0;(g')^{n_1}\dots(g^{(k)})^{n_k}) + S(r,g)\\ &\leqslant \overline{N}(r,0;g) + \overline{N}(r,\infty;g) + S(r,g)\\ &\leqslant 2T(r,g) + S(r,g),\end{aligned}$$

which is a contradiction since $n > s+2$. If $b \neq -1$, from (3.35) we obtain that

$$F - \Bigl(1 + \frac{1}{b}\Bigr) \equiv \frac{-a}{b^2(G + (a-b)/b)}.$$

So $\overline{N}(r, (b-a)/b; G) = \overline{N}(r, \infty; F) = \overline{N}(r, \infty; f) + \overline{N}(r, 0; p)$. Using Lemma 3.12 and the same argument as the one used in the case when $b = -1$ we get a contradiction.

*Case 2.* Let $b \in \mathbb{C} \setminus \{0\}$ and $a = b$. If $b = -1$, then from (3.35) we have $FG \equiv 1$, i.e., $f^n(f')^{n_1}\dots(f^{(k)})^{n_k} g^n(g')^{n_1}\dots(g^{(k)})^{n_k} \equiv p^2$, where $f^n(f')^{n_1}\dots(f^{(k)})^{n_k} - p(z)$ and $g^n(g')^{n_1}\dots(g^{(k)})^{n_k} - p(z)$ share 0 CM. If $b \neq -1$, from (3.35) we have

$$\frac{1}{F} \equiv \frac{bG}{(1+b)G - 1}.$$

Therefore $\overline{N}(r, 1/(1+b); G) = \overline{N}(r, 0; F)$. So in view of Lemmas 3.2, 3.12 and the second fundamental theorem we get

$$\begin{aligned}(n-s)T(r,g) &\leqslant T(r,G) - sN(r,\infty;g) - N(r,0;(g')^{n_1}\dots(g^{(k)})^{n_k}) + S(r,g)\\ &\leqslant \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}\Bigl(r,\frac{1}{1+b};G\Bigr) - dN(r,\infty;g)\\ &\quad - N(r,0;(g')^{n_1}\dots(g^{(k)})^{n_k}) + S(r,g)\\ &\leqslant \overline{N}(r,0;g) + \overline{N}(r,0;F) + S(r,g)\\ &\leqslant \overline{N}(r,0;g) + \overline{N}(r,0;f) + \sum_{i=1}^{k} n_i^*\overline{N}(r,0;f^{(i)}\,|f\neq 0) + S(r,g)\\ &\leqslant \overline{N}(r,0;g) + \overline{N}(r,0;f) + t\overline{N}(r,0;f) + m\overline{N}(r,\infty;f) + S(r,g)\\ &\leqslant T(r,g) + T(r,f) + tT(r,f) + mT(r,f) + S(r,f) + S(r,g).\end{aligned}$$

Without loss of generality, we may suppose that there exists a set $I$ with infinite measure such that $T(r,f) \leqslant T(r,g)$ for $r \in I$. So for $r \in I$ we have $(n-s)T(r,g) \leqslant (t+m+2)T(r,g) + S(r,g)$, which is a contradiction since $n > s+t+m+2$.

*Case 3.* Let $b=0$. From (3.35) we obtain

$$F \equiv \frac{G+a-1}{a}. \tag{3.36}$$

If $a \neq 1$ then from (3.36) we obtain $\overline{N}(r,1-a;G) = \overline{N}(r,0;F)$. We can deduce a contradiction similarly to Case 2. Therefore $a=1$ and from (3.36) we obtain $F \equiv G$, i.e., $f^n(f')^{n_1}\dots(f^{(k)})^{n_k} \equiv g^n(g')^{n_1}\dots(g^{(k)})^{n_k}$. This completes the proof. □

**Lemma 3.18.** *Let $f$ and $g$ be non-constant meromorphic functions sharing $(1,k_1)$, where $k_1 \in \mathbb{N}\cup\{\infty\}\setminus\{1\}$. Then*

$$\begin{aligned} N(r,1;g) - \overline{N}(r,1;g) \geqslant \overline{N}(r,1;f\,|{=}\,2) + 2\overline{N}(r,1;f\,|{=}\,3) + \dots \\ + (k_1-1)\overline{N}(r,1;f\,|{=}\,k_1) + k_1\overline{N}_L(r,1;f) \\ + (k_1+1)\overline{N}_L(r,1;g) + k_1\overline{N}_E^{(k_1+1}(r,1;g). \end{aligned}$$

## 4. Proofs of the theorems

Proof of Theorem 2.1. Let

$$F = \frac{f^n(f')^{n_1}\dots(f^{(k)})^{n_k}}{p} \quad \text{and} \quad G = \frac{g^n(g')^{n_1}\dots(g^{(k)})^{n_k}}{p}.$$

Note that $f$ and $g$ are transcendental meromorphic functions, so $p(z)$ is a small function with respect to both $f^n(f')^{n_1}\dots(f^{(k)})^{n_k}$ and $g^n(g')^{n_1}\dots(g^{(k)})^{n_k}$. Also $F$, $G$ share $(1,k_1)$ and $f$, $g$ share $(\infty,0)$.

*Case 1.* Let $H \not\equiv 0$. From (3.1) it can be easily calculated that the possible poles of $H$ occur at

(i) multiple zeros of $F$ and $G$,

(ii) those 1 points of $F$ and $G$ whose multiplicities are different,

(iii) those poles of $F$ and $G$ whose multiplicities are different,

(iv) the zeros of $F'(G')$ which are not zeros of $F(F-1)(G(G-1))$.

Since $H$ has only simple poles we get

$$\begin{aligned} N(r,\infty;H) \leqslant \overline{N}_*(r,\infty;f,g) + \overline{N}_*(r,1;F,G) + \overline{N}(r,0;F\,|{\geqslant}\,2) \\ + \overline{N}(r,0;G\,|{\geqslant}\,2) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G'), \end{aligned} \tag{4.1}$$

where $\overline{N}_0(r,0;F')$ is the reduced counting function of those zeros of $F'$ which are not zeros of $F(F-1)$, and $\overline{N}_0(r,0;G')$ is similarly defined.

Let $z_0$ be a simple zero of $F-1$ but $p(z_0)\neq 0$. Then $z_0$ is a simple zero of $G-1$ and a zero of $H$. So

$$N(r,1;F\,|{=}\,1)\leqslant N(r,0;H)\leqslant N(r,\infty;H)+S(r,f)+S(r,g). \tag{4.2}$$

Using (4.1) and (4.2) we get

$$\begin{aligned}\overline{N}(r,1;F)&\leqslant N(r,1;F\,|{=}\,1)+\overline{N}(r,1;F\,|{\geqslant}\,2)\\&\leqslant \overline{N}_*(r,\infty;f,g)+\overline{N}(r,0;F\,|{\geqslant}\,2)+\overline{N}(r,0;G\,|{\geqslant}\,2)\\&\quad+\overline{N}_*(r,1;F,G)+\overline{N}(r,1;F\,|{\geqslant}\,2)+\overline{N}_0(r,0;F')\\&\quad+\overline{N}_0(r,0;G')+S(r,f)+S(r,g)\\&\leqslant \overline{N}(r,\infty;f)+\overline{N}(r,0;F\,|{\geqslant}\,2)+\overline{N}(r,0;G\,|{\geqslant}\,2)\\&\quad+\overline{N}_*(r,1;F,G)+\overline{N}(r,1;F\,|{\geqslant}\,2)+\overline{N}_0(r,0;F')\\&\quad+\overline{N}_0(r,0;G')+S(r,f)+S(r,g).\end{aligned}\tag{4.3}$$

Now in view of Lemmas 3.2 and 3.18 we get

$$\begin{aligned}&\overline{N}_0(r,0;G')+\overline{N}(r,1;F\,|{\geqslant}\,2)+\overline{N}_*(r,1;F,G)\\&\leqslant \overline{N}_0(r,0;G')+\overline{N}(r,1;F\,|{=}\,2)\\&\quad+\overline{N}(r,1;F\,|{=}\,3)+\ldots+\overline{N}(r,1;F\,|{=}\,k_1)+\overline{N}_E^{(k_1+1}(r,1;F)\\&\quad+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_*(r,1;F,G)\\&\leqslant \overline{N}_0(r,0;G')-\overline{N}(r,1;F\,|{=}\,3)-\ldots-(k_1-2)\overline{N}(r,1;F\,|{=}\,k_1)\\&\quad-(k_1-1)\overline{N}_L(r,1;F)-k_1\overline{N}_L(r,1;G)-(k_1-1)\overline{N}_E^{(k_1+1}(r,1;F)\\&\quad+N(r,1;G)-\overline{N}(r,1;G)+\overline{N}_*(r,1;F,G)\\&\leqslant \overline{N}_0(r,0;G')+N(r,1;G)-\overline{N}(r,1;G)\\&\quad-(k_1-2)\overline{N}_L(r,1;F)-(k_1-1)\overline{N}_L(r,1;G)\\&\leqslant N(r,0;G'\,|G\neq 0)-(k_1-2)\overline{N}_L(r,1;F)-(k_1-1)\overline{N}_L(r,1;G)\\&\leqslant \overline{N}(r,0;G)+\overline{N}(r,\infty;g)-(k_1-2)\overline{N}_L(r,1;F)-(k_1-1)\overline{N}_L(r,1;G)\\&=\overline{N}(r,0;G)+\overline{N}(r,\infty;g)-(k_1-2)\overline{N}_*(r,1;F,G)-\overline{N}_L(r,1;G),\end{aligned}\tag{4.4}$$

Hence using (4.3), (4.4) and Lemma 3.1 we get from second fundamental theorem that

$$
\begin{aligned}
(4.5)\quad T(r,F) &\leqslant \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,1;F)-N_0(r,0;F')\\
&\leqslant 2\overline{N}(r,\infty;f)+N_2(r,0;F)+\overline{N}(r,0;G\,|\geqslant 2)+\overline{N}(r,1;F\,|\geqslant 2)\\
&\quad +\overline{N}_*(r,1;F,G)+\overline{N}_0(r,0;G')+S(r,f)+S(r,g)\\
&\leqslant 3\overline{N}(r,\infty;f)+N_2(r,0;F)+N_2(r,0;G)\\
&\quad -(k_1-2)\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\
&\leqslant 3\overline{N}(r,\infty;f)+2\overline{N}(r,0;f)+N_2(r,0;(f')^{n_1}\dots(f^{(k)})^{n_k})\\
&\quad +2\overline{N}(r,0;g)+\sum_{i=1}^{k} n_i^{**}N_2(r,0;g^{(i)})\\
&\quad -(k_1-2)\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\
&\leqslant 3\overline{N}(r,\infty;f)+2\overline{N}(r,0;f)+N(r,0;(f')^{n_1}\dots(f^{(k)})^{n_k})\\
&\quad +2\overline{N}(r,0;g)+\sum_{i=1}^{k} n_i^{**}N_{i+2}(r,0;g)+\sum_{i=1}^{k} in_i^{**}\overline{N}(r,\infty;g)\\
&\quad -(k_1-2)\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\
&\leqslant (3+m_1)\overline{N}(r,\infty;f)+2\overline{N}(r,0;f)+2\overline{N}(r,0;g)\\
&\quad +sN(r,0;g)+N(r,0;(f')^{n_1}\dots(f^{(k)})^{n_k})\\
&\quad -(k_1-2)\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g).
\end{aligned}
$$

Now using Lemmas 3.11 and 3.12 we get from (4.5)

$$
\begin{aligned}
(4.6)\quad (n-s)T(r,f) &\leqslant T(r,F)-sN(r,\infty;f)-N(r,0;(f')^{n_1}\dots(f^{(k)})^{n_k})+S(r,f)\\
&\leqslant (3+m_1-s)\overline{N}(r,\infty;f)+2\overline{N}(r,0;f)+2\overline{N}(r,0;g)\\
&\quad +sN(r,0;g)-(k_1-2)\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g)\\
&\leqslant \frac{2(t+1)(3+m_1-s)}{n+s+m_1-2m-1}T(r)+(4+s)T(r)+S(r)\\
&\leqslant \Big(\frac{4n+(6+2t)m_1-8m+8}{n+s+m_1-2m-1}+s\Big)T(r)+S(r).
\end{aligned}
$$

In a similar way we can obtain

$$
(4.7)\qquad (n-s)T(r,g)\leqslant \Big(\frac{4n+(6+2t)m_1-8m+8}{n+s+m_1-2m-1}+s\Big)T(r)+S(r).
$$

Combining (4.6) and (4.7) we see that

$$
(n-s)T(r)\leqslant \Big(\frac{(s+4)n+(6+2t)m_1-8m+sm_1-2sm+s^2-s+8}{n+s+m_1-2m-1}\Big)T(r)+S(r),
$$

i.e.,

$$((n-K_1)(n-K_2))T(r) \leqslant S(r), \tag{4.8}$$

where

$$K_1 = \frac{2m+s+5-m_1+\sqrt{L}}{2} \quad \text{and} \quad K_2 = \frac{2m+s+5-m_1-\sqrt{L}}{2},$$

where

$$L = (2m+s+5-m_1)^2 + 8s^2 - 8s + 4(6+2t)m_1 + 8sm_1 - 16sm - 32m + 32.$$

Note that

$$\begin{aligned} L &= (m_1+3s)^2 + 4(6+2t)m_1 + 2s - 12sm - 4mm_1 - 10m_1 + 4m^2 - 12m + 57 \\ &\leqslant (m_1+3s)^2 + 2m_1 + 8tm_1 + 2s - 4m(m_1-m) - 12(sm-m_1) - 12m + 57 \\ &< (m_1+3s)^2 + 2(m_1+3s)(1+4t) + (1+4t)^2 = (m_1+3s+4t+1)^2. \end{aligned}$$

Therefore

$$\begin{aligned} K_1 = \frac{2m+s+5-m_1+\sqrt{L}}{2} &< \frac{2m+s+5-m_1+m_1+3s+4t+1}{2} \\ &= 2s+m+2t+3. \end{aligned}$$

Since $n > 2s+m+2t+2$, (4.8) leads to a contradiction.

*Case 2.* Let $H \equiv 0$. Then the theorem follows from Lemmas 3.17, 3.15 and 3.16. □

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