Czechoslovak Mathematical Journal

Amiran Gogatishvili; Rza Mustafayev; Tuğçe Ünver

Embeddings between weighted Copson and Cesàro function spaces

Czechoslovak Mathematical Journal, Vol. 67 (2017), No. 4, 1105–1132

Persistent URL: `http://dml.cz/dmlcz/146970`

EMBEDDINGS BETWEEN WEIGHTED COPSON AND CESÀRO FUNCTION SPACES

Amiran Gogatishvili, Praha, Rza Mustafayev, Baku, Kırıkkale,
Tuğçe Ünver, Kırıkkale

Received August 9, 2016. First published October 11, 2017.

Abstract. In this paper, characterizations of the embeddings between weighted Copson function spaces $\mathrm{Cop}_{p_1,q_1}(u_1,v_1)$ and weighted Cesàro function spaces $\mathrm{Ces}_{p_2,q_2}(u_2,v_2)$ are given. In particular, two-sided estimates of the optimal constant c in the inequality

$$\biggl(\int_0^\infty\biggl(\int_0^t f(\tau)^{p_2}v_2(\tau)\,\mathrm{d}\tau\biggr)^{q_2/p_2}u_2(t)\,\mathrm{d}t\biggr)^{1/q_2} \leqslant c\biggl(\int_0^\infty\biggl(\int_t^\infty f(\tau)^{p_1}v_1(\tau)\,\mathrm{d}\tau\biggr)^{q_1/p_1}u_1(t)\,\mathrm{d}t\biggr)^{1/q_1},$$

where $p_1,p_2,q_1,q_2\in(0,\infty)$, $p_2\leqslant q_2$ and u_1,u_2,v_1,v_2 are weights on $(0,\infty)$, are obtained. The most innovative part consists of the fact that possibly different parameters p_1 and p_2 and possibly different inner weights v_1 and v_2 are allowed. The proof is based on the combination of duality techniques with estimates of optimal constants of the embeddings between weighted Cesàro and Copson spaces and weighted Lebesgue spaces, which reduce the problem to the solutions of iterated Hardy-type inequalities.

Keywords: Cesàro and Copson function spaces; embedding; iterated Hardy inequalities

MSC 2010: 46E30, 26D10

The research of A. Gogatishvili was partly supported by the grants GA13-14743S of the Grant Agency of the Czech Republic and RVO: 67985840, by Shota Rustaveli National Science Foundation grants no. 31/48 (Operators in some function spaces and their applications in Fourier Analysis) and no. DI/9/5-100/13 (Function spaces, weighted inequalities for integral operators and problems of summability of Fourier series). The research of all authors was partly supported by the joint project between the Czech Academy of Sciences and the Scientific and Technological Research Council of Turkey.

DOI: 10.21136/CMJ.2017.0424-16

1. Introduction

Many Banach spaces, which play an important role in functional analysis and its applications, are obtained in a special way: the norms of these spaces are generated by positive sublinear operators and by L_p-norms.

In connection with Hardy and Copson operators

$$(Pf)(x) := \frac{1}{x}\int_0^x f(t)\,\mathrm{d}t \quad \text{and} \quad (Qf)(x) := \int_x^\infty \frac{f(t)}{t}\,\mathrm{d}t, \quad x > 0,$$

the classical Cesàro function space

$$\mathrm{Ces}(p) = \left\{f\colon\ \|f\|_{\mathrm{Ces}(p)} := \left(\int_0^\infty \left(\frac{1}{x}\int_0^x |f(t)|\,\mathrm{d}t\right)^p \mathrm{d}x\right)^{1/p} < \infty\right\},$$

and the classical Copson function space

$$\mathrm{Cop}(p) = \left\{f\colon\ \|f\|_{\mathrm{Cop}(p)} := \left(\int_0^\infty \left(\int_x^\infty \frac{|f(t)|}{t}\,\mathrm{d}t\right)^p \mathrm{d}x\right)^{1/p} < \infty\right\},$$

where $1 < p \leqslant \infty$, with the usual modifications if $p = \infty$, are of interest.

The classical Cesàro function spaces $\mathrm{Ces}(p)$ have been introduced in 1970 by Shiue in [39] and subsequently studied in [30]. These spaces have been defined analogously to the Cesàro sequence spaces that appeared two years earlier in [38], when the Dutch Mathematical Society posted the problem to find a representation of their dual spaces. This problem was resolved in 1974 by Jagers who in [31] gave an explicit isometric description of the dual of Cesàro sequence space. In [40], Sy, Zhang and Lee gave a description of dual spaces of $\mathrm{Ces}(p)$ spaces based on Jagers' result. In 1996, a different, isomorphic description due to Bennett appeared in [12]. For a long time, Cesàro function spaces have not attracted a lot of attention, contrary to their sequence counterparts. In fact there is quite rich literature concerning different topics studied in Cesàro sequence spaces, for instance, [21], [20], [17], [18], [16], [19]. However, recently in a series of papers [2]–[9], Astashkin and Maligranda started to study the structure of Cesàro function spaces. Among others, in [4] they investigated dual spaces for $\mathrm{Ces}(p)$ for $1 < p < \infty$. Their description can be viewed as being analogous to the one given for sequence spaces in [12] (for more detailed information about the history of classical Cesàro spaces see the recent survey paper [10]).

In [12], Theorem 21.1, Bennett observes that the classical Cesàro function space and the classical Copson function space coincide for $p > 1$. He also derives estimates for the norms of the corresponding inclusion operators. The same result, with different estimates, is due to Boas, see [14], who in fact obtained the integral analogue of the Askey-Boas theorem, see [13], Lemma 6.18, and [1]. These results are generalized in [29] using the blocking technique.

Let A be any measurable subset of $(0,\infty)$. By $\mathfrak{M}(A)$ we denote the set of all measurable functions on A. The symbol $\mathfrak{M}^+(A)$ stands for the collection of all $f \in \mathfrak{M}(A)$ which are non-negative on A. The family of all weight functions (also called just weights) on A, that is, measurable, positive and finite a.e. on A, is given by $\mathcal{W}(A)$.

For $p \in (0,\infty]$, we define the functional $\|\cdot\|_{p,A}$ on $\mathfrak{M}(A)$ by

$$\|f\|_{p,A} := \begin{cases} \left(\int_A |f(x)|^p \,\mathrm{d}x\right)^{1/p} & \text{if } p<\infty,\\ \operatorname*{ess\,sup}_A |f(x)| & \text{if } p=\infty. \end{cases}$$

If $w \in \mathcal{W}(A)$, then the weighted Lebesgue space $L_p(w,A)$ is given by

$$L_p(w,A) \equiv L_{p,w}(A) := \{f \in \mathfrak{M}(A)\colon \ \|f\|_{p,w,A} := \|fw\|_{p,A} < \infty\},$$

and it is equipped with the quasi-norm $\|\cdot\|_{p,w,A}$. When $A=(0,\infty)$, we often write simply $L_{p,w}$ and $L_p(w)$ instead of $L_{p,w}(A)$ and $L_p(w,A)$, respectively.

We adopt the following usual conventions.

Convention 1.1. (i) Throughout the paper we put $0/0=0$, $0\cdot(\pm\infty)=0$ and $1/(\pm\infty)=0$.

(ii) We put

$$p' := \begin{cases} \dfrac{p}{1-p} & \text{if } \ 0<p<1,\\ \infty & \text{if } \ p=1,\\ \dfrac{p}{p-1} & \text{if } \ 1<p<\infty,\\ 1 & \text{if } \ p=\infty. \end{cases}$$

(iii) If $I=(a,b)\subseteq\mathbb{R}$ and g is a monotone function on I, then by $g(a)$ and $g(b)$ we mean the limits $\lim\limits_{x\to a+} g(x)$ and $\lim\limits_{x\to b-} g(x)$, respectively.

To state our results we use the notation $p\to q$ for $0<p,\ q\leqslant\infty$ defined by

$$\frac{1}{p\to q} = \frac{1}{q} - \frac{1}{p} \quad \text{if } q<p,$$

and $p\to q=\infty$ if $q\geqslant p$ (see, for instance, [29], page 30).

Throughout the paper, we always denote by c and C a positive constant which is independent of main parameters but it may vary from line to line. However, a constant with subscript or superscript such as c_1 does not change in different occurrences. By $a \lesssim b$ ($b \gtrsim a$) we mean that $a \leqslant \lambda b$, where $\lambda > 0$ depends on inessential parameters. If $a \lesssim b$ and $b \lesssim a$, we write $a \approx b$ and say that a and b are equivalent. We will denote by $\mathbf{1}$ the function $\mathbf{1}(x) = 1$, $x \in \mathbb{R}$. Since the expressions in our main results are too long, to make the formulas simpler we sometimes omit the differential element $\mathrm{d}x$.

Given two quasi-normed vector spaces X and Y, we write $X = Y$ if X and Y are equal in the algebraic and the topological sense (their quasi-norms are equivalent). The symbol $X \hookrightarrow Y$ ($Y \hookleftarrow X$) means that $X \subset Y$ and the natural embedding I of X in Y is continuous, that is, there exists a constant $c > 0$ such that $\|z\|_Y \leqslant c\|z\|_X$ for all $z \in X$.

The weighted Cesàro and Copson function spaces are defined as follows:

Definition 1.2. Let $0 < p, q \leqslant \infty$, $u \in \mathfrak{M}^+(0,\infty)$, $v \in \mathcal{W}(0,\infty)$. The weighted Cesàro and Copson spaces are defined by

$$\operatorname{Ces}_{p,q}(u,v) := \{f \in \mathfrak{M}^+(0,\infty)\colon \|f\|_{\operatorname{Ces}_{p,q}(u,v)} := \|\|f\|_{p,v,(0,\cdot)}\|_{q,u,(0,\infty)} < \infty\},$$

and

$$\operatorname{Cop}_{p,q}(u,v) := \{f \in \mathfrak{M}^+(0,\infty)\colon \|f\|_{\operatorname{Cop}_{p,q}(u,v)} := \|\|f\|_{p,v,(\cdot,\infty)}\|_{q,u,(0,\infty)} < \infty\},$$

respectively.

Definition 1.3. Let $0 < q \leqslant \infty$. We denote by Ω_q the set of all functions $u \in \mathfrak{M}^+(0,\infty)$ such that

$$0 < \|u\|_{q,(t,\infty)} < \infty, \quad t > 0,$$

and by $\mathring{\Omega}_q$ the set of all functions $u \in \mathfrak{M}^+(0,\infty)$ such that

$$0 < \|u\|_{q,(0,t)} < \infty, \quad t > 0.$$

Let $v \in \mathcal{W}(0,\infty)$. It is easy to see that $\operatorname{Ces}_{p,q}(u,v)$ and $\operatorname{Cop}_{p,q}(u,v)$ are quasi-normed vector spaces when $u \in \Omega_q$ and $u \in \mathring{\Omega}_q$, respectively.

Many function spaces from the literature, in particular from harmonic analysis, are covered by the spaces $\operatorname{Ces}_{p,q}(u,v)$ and $\operatorname{Cop}_{p,q}(u,v)$. Let us only mention the Beurling algebras A^p and A^*, see [23], [32], [11].

Note that the function spaces C and D defined by Grosse-Erdmann in [29] are related with our definition in the following way:

$$\operatorname{Ces}_{p,q}(u,v) = C(p,q,u)_v \quad \text{and} \quad \operatorname{Cop}_{p,q}(u,v) = D(p,q,u)_v.$$

We use the notations $\mathrm{Ces}_p(u) := \mathrm{Ces}_{1,p}(u,\mathbf{1})$ and $\mathrm{Cop}_p(u) := \mathrm{Cop}_{1,p}(u,\mathbf{1})$. Obviously, $\mathrm{Ces}(p) = \mathrm{Ces}_p(x^{-1})$ and $\mathrm{Cop}(p) = \mathrm{Cop}_p(x^{-1})$. In [33], Kamińska and Kubiak computed the dual norm of the Cesàro function space $\mathrm{Ces}_p(u)$ generated by $1 < p < \infty$ and an arbitrary positive weight u. A description presented in [33] resembles the approach of Jagers in [31] for sequence spaces.

Our principal goal in this paper is to investigate the embeddings between weighted Copson and Cesàro function spaces and vice versa, that is, the embeddings

$$\mathrm{Cop}_{p_1,q_1}(u_1,v_1) \hookrightarrow \mathrm{Ces}_{p_2,q_2}(u_2,v_2), \tag{1.1}$$

$$\mathrm{Ces}_{p_1,q_1}(u_1,v_1) \hookrightarrow \mathrm{Cop}_{p_2,q_2}(u_2,v_2). \tag{1.2}$$

We develop an approach consisting of a duality argument combined with estimates of optimal constants of the embeddings between weighted Cesàro and Copson spaces and weighted Lebesgue spaces, that is,

$$L_s(w) \hookrightarrow \mathrm{Ces}_{p,q}(u,v), \tag{1.3}$$

$$L_s(w) \hookrightarrow \mathrm{Cop}_{p,q}(u,v), \tag{1.4}$$

$$L_s(w) \hookleftarrow \mathrm{Ces}_{p,q}(u,v), \tag{1.5}$$

$$L_s(w) \hookleftarrow \mathrm{Cop}_{p,q}(u,v), \tag{1.6}$$

which reduce the problem to the solutions of the iterated Hardy-type inequalities

$$\left\| \left\| \int_t^\infty f \right\|_{p,u,(0,\cdot)} \right\|_{q,w,(0,\infty)} \leqslant c\|f\|_{\theta,v,(0,\infty)}, \quad f \in \mathfrak{M}^+(0,\infty), \tag{1.7}$$

where $0 < p,q \leqslant \infty$, $1 < \theta < \infty$. There exist different solutions of these inequalities (for more detailed information see, for instance, [25] and [24]). We will use characterizations from [26] and [27].

In order to characterize embeddings (1.3)–(1.6), we are going to use the direct and reverse Hardy-type inequalities. Note that embeddings (1.1)–(1.2) contain embeddings (1.3)–(1.6) as a special case. Indeed, for instance, if $p = q$ and $v(x) = w(x)/\|u\|_{p,(x,\infty)}$, then $\mathrm{Ces}_{p,q}(u,v) = L_p(w)$. Similarly, if $p = q$ and $v(x) = w(x)/\|u\|_{p,(0,x)}$, then $\mathrm{Cop}_{p,q}(u,v) = L_p(w)$. Moreover, by the change of variables $x = 1/t$ it is easy to see that (1.2) is equivalent to the embedding

$$\mathrm{Cop}_{p_1,q_1}(\tilde{u}_1,\tilde{v}_1) \hookrightarrow \mathrm{Ces}_{p_2,q_2}(\tilde{u}_2,\tilde{v}_2),$$

where $\tilde{u}_i(t) = t^{-2/q_i}u_i\big(1/t\big)$, $\tilde{v}_i(t) = t^{-2/p_i}v_i\big(1/t\big)$, $i = 1,2$, $t > 0$. This note allows us to concentrate our attention on the characterization of (1.1). Uniformly, we have to admit that the duality approach works only in the case when, in (1.1)–(1.2), one has $p_2 \leqslant q_2$. In the case when $p_2 > q_2$ the characterization of these embeddings remains open.

It should be noted that none of the above would ever have existed if it wasn't for the (now classical) well-known characterizations of weights for which the Hardy inequality holds. This subject, which is, incidentally, exactly one hundred years old, is absolutely indispensable in this part of mathematics (cf. [37], [35], [34]). In our proof below such results will be heavily used, as well as some more recent characterizations of the weighted reverse inequalities (cf. [22] and [36]).

It is mentioned in [29], page 30, that multipliers between Cesàro and Copson spaces are more difficult to treat. It is worth mentioning that by using characterizations of (1.1)–(1.2) it is possible to give the solution to the multiplier problem between weighted Cesàro and Copson function spaces, and we are going to present it in a future paper.

In particular, we obtain two-sided estimates of the optimal constant c in the inequality

$$(1.8)\qquad \left(\int_0^\infty \left(\int_0^t f(\tau)^{p_2} v_2(\tau)\,\mathrm{d}\tau\right)^{q_2/p_2} u_2(t)\,\mathrm{d}t\right)^{1/q_2} \leqslant c\left(\int_0^\infty \left(\int_t^\infty f(\tau)^{p_1} v_1(\tau)\,\mathrm{d}\tau\right)^{q_1/p_1} u_1(t)\,\mathrm{d}t\right)^{1/q_1},$$

where $p_1, p_2, q_1, q_2 \in (0,\infty)$, $p_2 \leqslant \min\{p_1, q_2\}$ and u_1, u_2, v_1, v_2 are weights on $(0,\infty)$ (it is shown in Lemma 4.1 that inequality (1.8) holds true only for trivial functions f when $p_1 < p_2$ for any $q_1, q_2 \in (0,\infty]$). The most innovative part consists of the fact that possibly different parameters p_1 and p_2 and possibly different inner weights v_1 and v_2 are allowed. Note that (1.8) was characterized in the particular cases when $p_1 = p_2 = 1$, $q_1 = q_2 = p > 1$, $u_1(t) = t^{\beta p-1}$, $u_2(t) = t^{-\alpha p-1}$, $v_1(t) = t^{-\beta-1}$, $v_2(t) = t^{\alpha-1}$, $t > 0$, where $\alpha > 0$ and $\beta > 0$, in [14], page 61, and when $p_1 = p_2 = 1$, $q_1 = p$, $q_2 = q$, $u_1(t) = v(t)$, $u_2(t) = t^{-q}w(t)$, $v_1(t) = t^{-1}$, $v_2(t) = 1$, $t > 0$, where $0 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$ and v, w are weight functions on $(0,\infty)$, in [15], Theorem 2.3.

The paper is organized as follows. We start with formulations of our main results in Section 2. Some properties of weighted Cesàro and Copson spaces are proved in Section 3. Finally, the proofs of the main results are presented in Section 4.

2. Statement of the main results

Our main results are the following theorems.

Throughout the paper we will denote

$$\widetilde{V}(x) := \|v_1^{-1}v_2\|_{p_1\to p_2,(0,x)} \quad \text{and} \quad \mathcal{V}(t,x) := \frac{\widetilde{V}(t)}{\widetilde{V}(t)+\widetilde{V}(x)}, \quad t>0,\ x>0.$$

Theorem 2.1. *Let $0<q_2=p_2\leqslant p_1=q_1<\infty$, $v_1,v_2\in\mathcal{W}(0,\infty)$, $u_1\in {}^{\mathbf{c}}\Omega_{q_1}$ and $u_2\in\Omega_{q_2}$. Then*

$$\|\mathrm{I}\|_{\mathrm{Cop}_{p_1,q_1}(u_1,v_1)\to\mathrm{Ces}_{p_2,q_2}(u_2,v_2)} \approx \big\|\|u_1\|_{p_1,(0,\cdot)}^{-1}\|u_2\|_{p_2,(\cdot,\infty)}\big\|_{p_1\to p_2,v_1^{-1}v_2,(0,\infty)}.$$

Theorem 2.2. *Let $0<p_1,p_2,q_1,q_2<\infty$ and $q_2\neq p_2\leqslant p_1=q_1$. Let $v_1,v_2\in\mathcal{W}(0,\infty)$, $u_1\in {}^{\mathbf{c}}\Omega_{q_1}$ and $u_2\in\Omega_{q_2}$.*

(i) *If $p_1\leqslant q_2$, then*

$$\|\mathrm{I}\|_{\mathrm{Cop}_{p_1,q_1}(u_1,v_1)\to\mathrm{Ces}_{p_2,q_2}(u_2,v_2)} \approx \sup_{t\in(0,\infty)} \big\|\|u_1\|_{p_1,(0,\cdot)}^{-1}\big\|_{p_1\to p_2,v_1^{-1}v_2,(0,t)}\|u_2\|_{q_2,(t,\infty)}.$$

(ii) *If $q_2<p_1$, then*

$$\|\mathrm{I}\|_{\mathrm{Cop}_{p_1,q_1}(u_1,v_1)\to\mathrm{Ces}_{p_2,q_2}(u_2,v_2)} \approx \bigg(\int_0^\infty \big\|\|u_1\|_{p_1,(0,\cdot)}^{-1}\big\|_{p_1\to p_2,v_1^{-1}v_2,(0,t)}^{p_1\to q_2}\,\mathrm{d}\big(-\|u_2\|_{q_2,(t,\infty)}^{p_1\to q_2}\big)\bigg)^{1/(p_1\to q_2)}.$$

Theorem 2.3. *Let $0<p_1,p_2,q_1,q_2<\infty$ and $q_2=p_2\leqslant p_1\neq q_1$. Let $v_1,v_2\in\mathcal{W}(0,\infty)$, $u_1\in {}^{\mathbf{c}}\Omega_{q_1}$ and $u_2\in\Omega_{q_2}$.*

(i) *If $q_1\leqslant p_2$, then*

$$\|\mathrm{I}\|_{\mathrm{Cop}_{p_1,q_1}(u_1,v_1)\to\mathrm{Ces}_{p_2,q_2}(u_2,v_2)} \approx \sup_{t\in(0,\infty)} \|u_1\|_{q_1,(0,t)}^{-1}\big\|\|u_2\|_{p_2,(\cdot,\infty)}\big\|_{p_1\to p_2,v_1^{-1}v_2,(0,t)}.$$

(ii) *If $p_2<q_1$, then*

$$\|\mathrm{I}\|_{\mathrm{Cop}_{p_1,q_1}(u_1,v_1)\to\mathrm{Ces}_{p_2,q_2}(u_2,v_2)} \approx \bigg(\int_0^\infty \big\|\|u_2\|_{p_2,(\cdot,\infty)}\big\|_{p_1\to p_2,v_1^{-1}v_2,(0,t)}^{q_1\to p_2}\,\mathrm{d}\big(-\|u_1\|_{q_1,(0,t)}^{-q_1\to p_2}\big)\bigg)^{1/(q_1\to p_2)} + \|u_1\|_{q_1,(0,\infty)}^{-1}\big\|\|u_2\|_{p_2,(\cdot,\infty)}\big\|_{p_1\to p_2,v_1^{-1}v_2,(0,\infty)}.$$

To state further results we need the following definitions.

Definition 2.4. Let U be a continuous, strictly increasing function on $[0,\infty)$ such that $U(0)=0$ and $\lim\limits_{t\to\infty} U(t)=\infty$. Then we say that U is admissible.

Let U be an admissible function. We say that a function φ is U-quasi-concave if φ is equivalent to an increasing function on $(0,\infty)$ and φ/U is equivalent to a decreasing function on $(0,\infty)$. We say that a U-quasi-concave function φ is nondegenerate if

$$\lim_{t\to 0+}\varphi(t)=\lim_{t\to\infty}\frac{1}{\varphi(t)}=\lim_{t\to\infty}\frac{\varphi(t)}{U(t)}=\lim_{t\to 0+}\frac{U(t)}{\varphi(t)}=0.$$

The family of nondegenerate U-quasi-concave functions is denoted by Q_U.

Definition 2.5. Let U be an admissible function, and let w be a non-negative measurable function on $(0,\infty)$. We say that the function φ, defined by

$$\varphi(t)=U(t)\int_0^\infty \frac{w(\tau)\,\mathrm{d}\tau}{U(\tau)+U(t)},\quad t\in(0,\infty),$$

is a fundamental function of w with respect to U. We also say that $w(\tau)\,\mathrm{d}\tau$ is a representation measure of φ with respect to U.

Remark 2.6. Let φ be the fundamental function of w with respect to U. Assume that

$$\int_0^\infty \frac{w(\tau)\,\mathrm{d}\tau}{U(\tau)+U(t)}<\infty,\quad t>0,\qquad \int_0^1 \frac{w(\tau)\,\mathrm{d}\tau}{U(\tau)}=\int_1^\infty w(\tau)\,\mathrm{d}\tau=\infty.$$

Then $\varphi\in Q_U$.

Remark 2.7. Suppose that $\varphi(x)<\infty$ for all $x\in(0,\infty)$, where φ is defined by

$$\varphi(x)=\operatorname*{ess\,sup}_{t\in(0,x)} U(t)\operatorname*{ess\,sup}_{\tau\in(t,\infty)}\frac{w(\tau)}{U(\tau)}.$$

If

$$\limsup_{t\to 0+} w(t)=\limsup_{t\to\infty}\frac{1}{w(t)}=\limsup_{t\to 0+}\frac{U(t)}{w(t)}=\limsup_{t\to\infty}\frac{w(t)}{U(t)}=0,$$

then $\varphi\in Q_U$.

Theorem 2.8. *Let* $0<p_1,p_2,q_1,q_2<\infty$, $p_2<p_1$, $q_1\leqslant p_2<q_2$. *Let* $v_1,v_2\in\mathcal{W}(0,\infty)$, $u_1\in {}^{\mathbf{c}}\Omega_{q_1}$ *and* $u_2\in\Omega_{q_2}$. *Assume that* $\widetilde{V}$ *is admissible and*

$$\varphi_1(x):=\operatorname*{ess\,sup}_{t\in(0,\infty)}\widetilde{V}(t)\mathcal{V}(x,t)\|u_1\|^{-1}_{q_1,(0,t)}\in Q_{\widetilde{V}^{1/(p_1\to p_2)}}.$$

(i) *If* $p_1 \leqslant q_2$, *then*

$$\|\mathrm{I}\|_{\mathrm{Cop}_{p_1,q_1}(u_1,v_1)\to\mathrm{Ces}_{p_2,q_2}(u_2,v_2)} \approx \sup_{x\in(0,\infty)} \varphi_1(x) \sup_{t\in(0,\infty)} \mathcal{V}(t,x)\|u_2\|_{q_2,(t,\infty)}.$$

(ii) *If* $q_2 < p_1$, *then*

$$\|\mathrm{I}\|_{\mathrm{Cop}_{p_1,q_1}(u_1,v_1)\to\mathrm{Ces}_{p_2,q_2}(u_2,v_2)} \approx \sup_{x\in(0,\infty)} \varphi_1(x)\bigg(\int_0^\infty \mathcal{V}(t,x)^{p_1\to q_2}\,\mathrm{d}(-\|u_2\|_{q_2,(t,\infty)}^{p_1\to q_2})\bigg)^{1/(p_1\to q_2)}.$$

Theorem 2.9. *Let* $0 < p_1,p_2,q_1,q_2 < \infty$, $p_2 < p_1$ *and* $p_2 < \min\{q_1,q_2\}$. *Let* $v_1,v_2 \in \mathcal{W}(0,\infty)$, $u_1 \in {}^{\mathfrak{c}}\Omega_{q_1}$ *and* $u_2 \in \Omega_{q_2}$. *Assume that* $\widetilde{V}$ *is admissible and*

$$\varphi_2(x) := \bigg(\int_0^\infty [\mathcal{V}(x,t)\widetilde{V}(t)]^{q_1\to p_2}\,\mathrm{d}(-\|u_1\|_{q_1,(0,t)}^{-q_1\to p_2})\bigg)^{1/(q_1\to p_2)} \in Q_{\widetilde{V}^{1/(p_1\to p_2)}}.$$

(i) *If* $\max\{p_1,q_1\} \leqslant q_2$, *then*

$$\|\mathrm{I}\|_{\mathrm{Cop}_{p_1,q_1}(u_1,v_1)\to\mathrm{Ces}_{p_2,q_2}(u_2,v_2)} \approx \sup_{x\in(0,\infty)} \varphi_2(x) \sup_{t\in(0,\infty)} \mathcal{V}(t,x)\|u_2\|_{q_2,(t,\infty)} + \|u_1\|_{q_1,(0,\infty)}^{-1} \sup_{t\in(0,\infty)} \widetilde{V}(t)\|u_2\|_{q_2,(t,\infty)}.$$

(ii) *If* $p_1 \leqslant q_2 < q_1$, *then*

$$\begin{aligned}\|\mathrm{I}\|_{\mathrm{Cop}_{p_1,q_1}(u_1,v_1)\to\mathrm{Ces}_{p_2,q_2}(u_2,v_2)} &\approx \bigg(\int_0^\infty \varphi_2(x)^{(q_1\to q_2 q_1\to p_2)/(q_2\to p_2)}\widetilde{V}(x)^{q_1\to p_2} \\ &\qquad\times\bigg(\sup_{t\in(0,\infty)} \mathcal{V}(t,x)\|u_2\|_{q_2,(t,\infty)}\bigg)^{q_1\to q_2}\,\mathrm{d}(-\|u_1\|_{q_1,(0,x)}^{-q_1\to p_2})\bigg)^{1/(q_1\to q_2)} \\ &\quad + \|u_1\|_{q_1,(0,\infty)}^{-1} \sup_{t\in(0,\infty)} \widetilde{V}(t)\|u_2\|_{q_2,(t,\infty)}.\end{aligned}$$

(iii) *If* $q_1 \leqslant q_2 < p_1$, *then*

$$\begin{aligned}\|\mathrm{I}\|_{\mathrm{Cop}_{p_1,q_1}(u_1,v_1)\to\mathrm{Ces}_{p_2,q_2}(u_2,v_2)} &\approx \sup_{x\in(0,\infty)} \varphi_2(x)\bigg(\int_0^\infty \mathcal{V}(t,x)^{p_1\to q_2}\,\mathrm{d}(-\|u_2\|_{q_2,(t,\infty)}^{p_1\to q_2})\bigg)^{1/(p_1\to q_2)} \\ &\quad + \|u_1\|_{q_1,(0,\infty)}^{-1}\bigg(\int_0^\infty \widetilde{V}(t)^{p_1\to q_2}\,\mathrm{d}(-\|u_2\|_{q_2,(t,\infty)}^{p_1\to q_2})\bigg)^{1/(p_1\to q_2)}.\end{aligned}$$

(iv) *If* $q_2 < \min\{p_1, q_1\}$, *then*

$$\begin{aligned}
&\|\mathrm{I}\|_{\mathrm{Cop}_{p_1,q_1}(u_1,v_1)\to \mathrm{Ces}_{p_2,q_2}(u_2,v_2)} \\
&\quad\approx \bigg(\int_0^\infty \varphi_2(x)^{(q_1\to q_2 q_1\to p_2)/(q_2\to p_2)} \widetilde{V}(x)^{q_1\to p_2} \\
&\qquad\quad \times \bigg(\int_0^\infty \mathcal{V}(t,x)^{p_1\to q_2}\, \mathrm{d}(-\|u_2\|_{q_2,(t,\infty)}^{p_1\to q_2})\bigg)^{(q_1\to q_2)/(p_1\to q_2)} \\
&\qquad\qquad \times \mathrm{d}(-\|u_1\|_{q_1,(0,x)}^{-q_1\to p_2})\bigg)^{1/(q_1\to q_2)} \\
&\qquad + \|u_1\|_{q_1,(0,\infty)}^{-1} \bigg(\int_0^\infty \widetilde{V}(t)^{p_1\to q_2}\, \mathrm{d}(-\|u_2\|_{q_2,(t,\infty)}^{p_1\to q_2})\bigg)^{1/(p_1\to q_2)}.
\end{aligned}$$

Theorem 2.10. *Let* $0 < q_1 < p = p_1 = p_2 < q_2 < \infty$. *Assume that* $v_1, v_2 \in \mathcal{W}(0,\infty)$, $u_1 \in {}^{\mathfrak{c}}\Omega_{q_1}$ *and* $u_2 \in \Omega_{q_2}$. *Then*

$$\|\mathrm{I}\|_{\mathrm{Cop}_{p_1,q_1}(u_1,v_1)\to \mathrm{Ces}_{p_2,q_2}(u_2,v_2)} \approx \sup_{t\in(0,\infty)} \big\|\|u_1\|_{q_1,(0,\cdot)}^{-1}\big\|_{\infty, v_1^{-1}v_2,(0,t)} \|u_2\|_{q_2,(t,\infty)}.$$

Theorem 2.11. *Let* $0 < q_1, q_2 < \infty$, *and* $0 < p = p_1 = p_2 < \min\{q_1, q_2\}$. *Let* $v_1, v_2 \in \mathcal{W}(0,\infty)$, $u_1 \in {}^{\mathfrak{c}}\Omega_{q_1}$ *and* $u_2 \in \Omega_{q_2}$. *Assume that* $v_1^{-1} v_2 \in \mathcal{W}(0,\infty) \cap C(0,\infty)$ *and* $0 < \|u_2^{-1}\|_{q_2\to p,(x,\infty)} < \infty,\ x > 0$.

(i) *If* $q_1 \leqslant q_2$, *then*

$$\begin{aligned}
\|\mathrm{I}\|_{\mathrm{Cop}_{p_1,q_1}(u_1,v_1)\to \mathrm{Ces}_{p_2,q_2}(u_2,v_2)} &\approx \sup_{x\in(0,\infty)} \varphi_2(x) \sup_{t\in(0,\infty)} \mathcal{V}(t,x)\, \|u_2\|_{q_2,(t,\infty)} \\
&\quad + \|u_1\|_{q_1,(0,\infty)}^{-1} \sup_{t\in(0,\infty)} \widetilde{V}(t) \|u_2\|_{q_2,(t,\infty)}.
\end{aligned}$$

(ii) *If* $q_2 < q_1$, *then*

$$\begin{aligned}
&\|\mathrm{I}\|_{\mathrm{Cop}_{p_1,q_1}(u_1,v_1)\to \mathrm{Ces}_{p_2,q_2}(u_2,v_2)} \\
&\quad\approx \bigg(\int_0^\infty \varphi_2(x)^{(q_1\to q_2 q_1\to p)/(q_2\to p)} \widetilde{V}(x)^{q_1\to p} \\
&\qquad\quad \times \bigg(\sup_{t\in(0,\infty)} \mathcal{V}(t,x) \|u_2\|_{q_2,(t,\infty)}\bigg)^{q_1\to q_2} \mathrm{d}(-\|u_1\|_{q_1,(0,x)}^{-q_1\to p})\bigg)^{1/(q_1\to q_2)} \\
&\qquad + \|u_1\|_{q_1,(0,\infty)}^{-1} \sup_{t\in(0,\infty)} \widetilde{V}(t) \|u_2\|_{q_2,(t,\infty)}.
\end{aligned}$$

3. Some auxiliary results

We need the following auxiliary results.

Lemma 3.1. *Let $0 < p, q \leqslant \infty$, $v \in \mathcal{W}(0,\infty)$ and let $u \in \mathfrak{M}^+(0,\infty)$. $\operatorname{Ces}_{p,q}(u,v)$ and $\operatorname{Cop}_{p,q}(u,v)$ are nontrivial, i.e., they consist not only of functions equivalent to 0 on $(0,\infty)$, if and only if*

$$\|u\|_{q,(t,\infty)} < \infty \quad \text{for some} \ \ t > 0,$$

and

$$\|u\|_{q,(0,t)} < \infty \quad \text{for some} \ \ t > 0,$$

respectively.

Proof. *Sufficiency.* Let $u \in \mathfrak{M}^+(0,\infty)$ be such that $\|u\|_{q,(t,\infty)} = \infty$ for all $t > 0$. Assume that $f \neq 0$ a.e. Then $\|f\|_{p,v,(0,t_0)} > 0$ for some $t_0 > 0$

$$\begin{aligned}\|f\|_{\operatorname{Ces}_{p,q}(u,v)} &\geqslant \|\|f\|_{p,v,(0,\cdot)}\|_{q,u,(t_0,\infty)} \geqslant \|\mathbf{1}\|_{q,u,(t_0,\infty)}\|f\|_{p,v,(0,t_0)}\\ &= \|u\|_{q,(t_0,\infty)}\|f\|_{p,v,(0,t_0)}.\end{aligned}$$

Hence $\|f\|_{\operatorname{Ces}_{p,q}(u,v)} = \infty$. Consequently, if $\|f\|_{\operatorname{Ces}_{p,q}(u,v)} < \infty$, then $f = 0$ a.e., that is, $\operatorname{Ces}_{p,q}(u,v) = \{0\}$.

Necessity. Assume that $\|u\|_{q,(t,\infty)} < \infty$ for some $t > 0$. If $f \in L_p(v)$ such that $\operatorname{supp} f \subset (\tau,\infty)$ for some $\tau \geqslant t$, then $f \in \operatorname{Ces}_{p,q}(u,v)$. Indeed:

$$\begin{aligned}\|f\|_{\operatorname{Ces}_{p,q}(u,v)} &= \|\|f\|_{p,v,(0,\cdot)}\|_{q,u,(\tau,\infty)} \leqslant \|\mathbf{1}\|_{q,u,(\tau,\infty)}\|f\|_{p,v,(0,\infty)}\\ &= \|u\|_{q,(\tau,\infty)}\|f\|_{p,v,(0,\infty)} < \infty.\end{aligned}$$

The same conclusion can be deduced for the Copson spaces. □

Lemma 3.2. *If $\|u\|_{q,(t_1,\infty)} = \infty$ for some $t_1 > 0$, then*

$$f \in \operatorname{Ces}_{p,q}(u,v) \Rightarrow f = 0 \quad \text{a.e. on} \ \ (0,t_1).$$

If $\|u\|_{q,(0,t_2)} = \infty$ for some $t_2 > 0$, then

$$f \in \operatorname{Cop}_{p,q}(u,v) \Rightarrow f = 0 \quad \text{a.e. on} \ \ (t_2,\infty).$$

P r o o f. Assume that $\|u\|_{q,(t_1,\infty)} = \infty$ for some $t_1 > 0$ and let $f \in \mathrm{Ces}_{p,q}(u,v)$. Then

$$\|f\|_{\mathrm{Ces}_{p,q}(u,v)} \geqslant \|\|f\|_{p,v,(0,\cdot)}\|_{q,u,(t_1,\infty)} \geqslant \|u\|_{q,(t_1,\infty)}\|f\|_{p,v,(0,t_1)}.$$

Therefore, $\|f\|_{p,v,(0,t_1)} = 0$. Hence, $f = 0$ a.e. on $(0, t_1)$.

Assume now that $\|u\|_{q,(0,t_2)} = \infty$ for some $t_2 > 0$ and let $f \in \mathrm{Cop}_{p,q}(u,v)$. Then

$$\|f\|_{\mathrm{Cop}_{p,q}(u,v)} \geqslant \|\|f\|_{p,v,(\cdot,\infty)}\|_{q,u,(0,t_2)} \geqslant \|u\|_{q,(0,t_2)}\|f\|_{p,v,(t_2,\infty)}.$$

Consequently, $\|f\|_{p,v,(t_2,\infty)} = 0$. This yields that $f = 0$ a.e. on (t_2, ∞). □

Remark 3.3. In view of Lemmas 3.1 and 3.2, it is enough to take $u \in \mathfrak{M}^+(0,\infty)$ such that $\|u\|_{q,(t,\infty)} < \infty$ for all $t > 0$, when considering $\mathrm{Ces}_{p,q}(u,v)$ spaces. Similarly, it is enough to take $u \in \mathfrak{M}^+(0,\infty)$ such that $\|u\|_{q,(0,t)} < \infty$ for all $t > 0$, when considering $\mathrm{Cop}_{p,q}(u,v)$ spaces.

Note that $\mathrm{Ces}_{p,p}(u,v)$ and $\mathrm{Cop}_{p,p}(u,v)$ coincide with some weighted Lebesgue spaces.

Lemma 3.4. *Let $0 < p \leqslant \infty$, $u \in \Omega_p$ and $v \in \mathcal{W}(0,\infty)$. Then $\mathrm{Ces}_{p,p}(u,v) = L_p(w)$, where*

$$w(x) := v(x)\|u\|_{p,(x,\infty)}, \quad x > 0. \tag{3.1}$$

P r o o f. Assume first that $p < \infty$. Applying Fubini's theorem, we have

$$\begin{aligned}
\|f\|_{\mathrm{Ces}_{p,p}(u,v)} &= \left(\int_0^\infty u^p(t) \int_0^t f(\tau)^p v(\tau)^p \,\mathrm{d}\tau \,\mathrm{d}t\right)^{1/p} \\
&= \left(\int_0^\infty f(\tau)^p v(\tau)^p \int_\tau^\infty u(t)^p \,\mathrm{d}t \,\mathrm{d}\tau\right)^{1/p} \\
&= \|f\|_{p,w,(0,\infty)},
\end{aligned}$$

where w is defined by (3.1). If $p = \infty$, by exchanging suprema, we have

$$\begin{aligned}
\|f\|_{\mathrm{Ces}_{\infty,\infty}(u,v)} &= \operatorname*{ess\,sup}_{t\in(0,\infty)} u(t) \operatorname*{ess\,sup}_{\tau\in(0,t)} f(\tau)v(\tau) \\
&= \operatorname*{ess\,sup}_{t\in(0,\infty)} f(t)v(t) \operatorname*{ess\,sup}_{\tau\in(t,\infty)} u(\tau) \\
&= \|f\|_{\infty,w,(0,\infty)}.
\end{aligned}$$

□

Lemma 3.5. *Let* $0 < p \leqslant \infty$, $u \in {}^{\mathbf{c}}\Omega_p$ *and* $v \in \mathcal{W}(0,\infty)$. *Then* $\operatorname{Cop}_{p,p}(u,v) = L_p(w)$, *where*

$$w(x) := v(x)\|u\|_{p,(0,x)}, \quad x > 0. \tag{3.2}$$

P r o o f. This follows by the same method as in Lemma 3.4. □

Denote

$$(Hf)(t) := \int_0^t f(x)\,\mathrm{d}x, \quad (H^*f)(t) := \int_t^\infty f(x)\,\mathrm{d}x, \quad f \in \mathfrak{M}^+(0,\infty),\ t \geqslant 0,$$

and

$$(Sf)(t) := \operatorname*{ess\,sup}_{x\in(0,t)} f(x), \quad (S^*f)(t) := \operatorname*{ess\,sup}_{x\in(t,\infty)} f(x), \quad f \in \mathfrak{M}^+(0,\infty),\ t \geqslant 0.$$

Let A, B be some sets and φ, ψ be non-negative functions defined on $A \times B$ (it may happen that $\varphi(\alpha,\beta) = \infty$ or $\psi(\alpha,\beta) = \infty$ for some $\alpha \in A$, $\beta \in B$). We say that φ is dominated by ψ (or ψ dominates φ) on $A \times B$ uniformly in $\alpha \in A$ and write

$$\varphi(\alpha,\beta) \lesssim \psi(\alpha,\beta) \quad \text{uniformly in } \alpha \in A,$$

or

$$\psi(\alpha,\beta) \gtrsim \varphi(\alpha,\beta) \quad \text{uniformly in } \alpha \in A,$$

if for each $\beta \in B$ there exists $C(\beta) > 0$ such that

$$\varphi(\alpha,\beta) \leqslant C(\beta)\psi(\alpha,\beta)$$

for all $\alpha \in A$. We also say that φ is equivalent to ψ on $A \times B$ uniformly in $\alpha \in A$ and write

$$\varphi(\alpha,\beta) \approx \psi(\alpha,\beta) \quad \text{uniformly in } \alpha \in A,$$

if φ and ψ dominate each other on $A \times B$ uniformly in $\alpha \in A$.

Now we characterize (1.3) and (1.4).

Theorem 3.6. *Let* $0 < p_2 \leqslant p_1 \leqslant \infty$, $0 < q \leqslant \infty$, $v_1, v_2 \in \mathcal{W}(0,\infty)$ *and* $u \in \Omega_q$.

(i) *If* $p_1 \leqslant q$, *then*

$$\|\mathrm{I}\|_{L_{p_1}(v_1)\to\operatorname{Ces}_{p_2,q}(u,v_2)} \approx \sup_{t\in(0,\infty)} \|v_1^{-1}v_2\|_{p_1\to p_2,(0,t)}\|u\|_{q,(t,\infty)}$$

uniformly in $u \in \Omega_q$.

(ii) *If* $q < p_1$, *then*

$$\|\mathrm{I}\|_{L_{p_1}(v_1)\to\operatorname{Ces}_{p_2,q}(u,v_2)} \approx \left(\int_0^\infty \|v_1^{-1}v_2\|_{p_1\to p_2,(0,t)}^{p_1\to q}\,\mathrm{d}(-\|u\|_{q,(t,\infty)}^{p_1\to q})\right)^{1/(p_1\to q)}$$

uniformly in $u \in \Omega_q$.

P r o o f. Since

$$\begin{aligned}\|\mathrm{I}\|_{L_{p_1}(v_1)\to \mathrm{Ces}_{p_2,q}(u,v_2)} &= \sup_{f\in\mathfrak{M}^+(0,\infty)} \frac{\|\|f\|_{p_2,v_2,(0,\cdot)}\|_{q,u,(0,\infty)}}{\|f\|_{p_1,v_1,(0,\infty)}} \\ &= \biggl(\sup_{g\in\mathfrak{M}^+(0,\infty)} \frac{\|H(|g|)\|_{q/p_2,u^{p_2},(0,\infty)}}{\|g\|_{p_1/p_2,[v_1^{-1}v_2]^{-p_2},(0,\infty)}} \biggr)^{1/p_2} \\ &= \Bigl(\|H\|_{L_{p_1/p_2}([v_1^{-1}v_2]^{-p_2})\to L_{q/p_2}(u^{p_2})} \Bigr)^{1/p_2},\end{aligned}$$

when $p_2 < \infty$, the statement follows by using the characterization of the boundedness of the operator H in weighted Lebesgue spaces (see, for instance, [37], Section 1).

If $p_2 = p_1 = \infty$, it is easy to see that

$$\begin{aligned}\|\mathrm{I}\|_{L_\infty(v_1)\to \mathrm{Ces}_{\infty,q}(u,v_2)} &= \sup_{f\in\mathfrak{M}^+(0,\infty)} \frac{\|\|f\|_{\infty,v_2,(0,\cdot)}\|_{q,u,(0,\infty)}}{\|f\|_{\infty,v_1,(0,\infty)}} \\ &= \|\|v_1^{-1}\|_{\infty,v_2,(0,\cdot)}\|_{q,u,(0,\infty)}.\end{aligned}$$

□

The following statement can be proved analogously.

Theorem 3.7. *Let* $0 < p_2 \leqslant p_1 \leqslant \infty$, $0 < q \leqslant \infty$, $v_1, v_2 \in \mathcal{W}(0,\infty)$ *and* $u \in \overset{\mathrm{c}}{\Omega}_q$.

(i) *If* $p_1 \leqslant q$, *then*

$$\|\mathrm{I}\|_{L_{p_1}(v_1)\to \mathrm{Cop}_{p_2,q}(u,v_2)} \approx \sup_{t\in(0,\infty)} \|v_1^{-1}v_2\|_{p_1\to p_2,(t,\infty)} \|u\|_{q,(0,t)}$$

uniformly in $u \in \overset{\mathrm{c}}{\Omega}_q$.

(ii) *If* $q < p_1$, *then*

$$\|\mathrm{I}\|_{L_{p_1}(v_1)\to \mathrm{Cop}_{p_2,q}(u,v_2)} \approx \biggl(\int_0^\infty \|v_1^{-1}v_2\|_{p_1\to p_2,(t,\infty)}^{p_1\to q} \,\mathrm{d}(\|u\|_{q,(0,t)}^{p_1\to q}) \biggr)^{1/(p_1\to q)}$$

uniformly in $u \in \overset{\mathrm{c}}{\Omega}_q$.

The embeddings (1.5) and (1.6) are characterized in the following statements.

Theorem 3.8. *Let* $0 < p_1 \leqslant p_2 \leqslant \infty$, $0 < q \leqslant \infty$, $v_1, v_2 \in \mathcal{W}(0,\infty)$ *and* $u \in \Omega_q$.

(i) *If* $q \leqslant p_1$, *then*

$$\|\mathrm{I}\|_{\mathrm{Ces}_{p_2,q}(u,v_2)\to L_{p_1}(v_1)} \approx \sup_{t\in(0,\infty)} \bigl\|v_1 v_2^{-1}\bigr\|_{p_2\to p_1,(t,\infty)} \|u\|_{q,(t,\infty)}^{-1}$$

uniformly in $u \in \Omega_q$.

(ii) *If* $p_1 < q$, *then*

$$\|\mathrm{I}\|_{\mathrm{Ces}_{p_2,q}(u,v_2)\to L_{p_1}(v_1)} \approx \left(\int_0^\infty \|v_1 v_2^{-1}\|_{p_2\to p_1,(t,\infty)}^{q\to p_1}\,\mathrm{d}(\|u\|_{q,(t-,\infty)}^{-q\to p_1})\right)^{1/(q\to p_1)} + \frac{\|v_1 v_2^{-1}\|_{p_2\to p_1,(0,\infty)}}{\|u\|_{q,(0,\infty)}}$$

uniformly in $u \in \Omega_q$.

Proof. Since

$$\|\mathrm{I}\|_{\mathrm{Ces}_{p_2,q}(u,v_2)\to L_{p_1}(v_1)} = \sup_{f\in\mathfrak{M}^+(0,\infty)} \frac{\|f\|_{p_1,v_1,(0,\infty)}}{\|\|f\|_{p_2,v_2,(0,t)}\|_{q,u,(0,\infty)}} = \left(\sup_{g\in\mathfrak{M}^+(0,\infty)} \frac{\|g(v_1 v_2^{-1})^{p_2}\|_{p_1/p_2,(0,\infty)}}{\|H(|g|)u^{p_2}\|_{q/p_2,(0,\infty)}}\right)^{1/p_2},$$

when $p_2 < \infty$, and

$$\|\mathrm{I}\|_{\mathrm{Ces}_{p_2,q}(u,v_2)\to L_{p_1}(v_1)} = \sup_{f\in\mathfrak{M}^+(0,\infty)} \frac{\|f\|_{p_1,v_1,(0,\infty)}}{\|\|f\|_{p_2,v_2,(0,t)}\|_{q,u,(0,\infty)}} = \sup_{g\in\mathfrak{M}^+(0,\infty)} \frac{\|g v_1 v_2^{-1}\|_{p_1,(0,\infty)}}{\|S(|g|)u\|_{q,(0,\infty)}},$$

when $p_2 = \infty$, it remains to apply [22], Theorems 5.1 and 5.4, and [36], Theorem 4.1. $\square$

The following statement can be proved analogously.

Theorem 3.9. *Let* $0 < p_1 \leqslant p_2 \leqslant \infty$, $0 < q \leqslant \infty$, $v_1, v_2 \in \mathcal{W}(0,\infty)$ *and* $u \in {}^{\mathfrak{c}}\Omega_q$.

(i) *If* $q \leqslant p_1$, *then*

$$\|\mathrm{I}\|_{\mathrm{Cop}_{p_2,q}(u,v_2)\to L_{p_1}(v_1)} \approx \sup_{t\in(0,\infty)} \|v_1 v_2^{-1}\|_{p_2\to p_1,(0,t)} \|u\|_{q,(0,t)}^{-1}$$

uniformly in $u \in {}^{\mathfrak{c}}\Omega_q$.

(ii) *If* $p_1 < q$ *then*

$$\|\mathrm{I}\|_{\mathrm{Cop}_{p_2,q}(u,v_2)\to L_{p_1}(v_1)} \approx \left(\int_0^\infty \|v_1 v_2^{-1}\|_{p_2\to p_1,(0,t)}^{q\to p_1}\,\mathrm{d}(-\|u\|_{q,(0,t+)}^{-q\to p_1})\right)^{1/(q\to p_1)} + \frac{\|v_1 v_2^{-1}\|_{p_2\to p_1,(0,\infty)}}{\|u\|_{q,(0,\infty)}}$$

uniformly in $u \in {}^{\mathfrak{c}}\Omega_q$.

In particular, Theorems 3.3 and 3.4 allow us to give a characterization of the associate spaces of weighted Cesàro and Copson function spaces.

Definition 3.10. Let X be a set of functions from $\mathfrak{M}(0,\infty)$, endowed with a positively homogeneous functional $\|\cdot\|_X$, defined for every $f\in\mathfrak{M}(0,\infty)$ and such that $f\in X$ if and only if $\|f\|_X<\infty$. We define the associate space X' of X as the set of all functions $f\in\mathfrak{M}(0,\infty)$ such that $\|f\|_{X'}<\infty$, where

$$\|f\|_{X'}=\sup\biggl\{\int_0^\infty |f(x)g(x)|\,\mathrm{d}x\colon\ \|g\|_X\leqslant 1\biggr\}.$$

Theorem 3.11. *Assume $1\leqslant p<\infty$, $0<q\leqslant\infty$. Let $u\in\Omega_q$ and $v\in\mathcal{W}(0,\infty)$. Set*

$$X=\mathrm{Ces}_{p,q}(u,v).$$

(i) *Let $0<q\leqslant 1$. Then*

$$\|f\|_{X'}\approx\sup_{t\in(0,\infty)}\|f\|_{p',v^{-1},(t,\infty)}\|u\|_{q,(t,\infty)}^{-1},$$

with the positive constants in the equivalence independent of f.

(ii) *Let $1<q\leqslant\infty$. Then*

$$\|f\|_{X'}\approx\biggl(\int_0^\infty\|f\|_{p',v^{-1},(t,\infty)}^{q'}\,\mathrm{d}(\|u\|_{q,(t-,\infty)}^{-q'})\biggr)^{1/q'}+\|f\|_{p',v^{-1},(0,\infty)}\|u\|_{q,(0,\infty)}^{-1},$$

with the positive constants in the equivalence independent of f.

Theorem 3.12. *Assume $1\leqslant p<\infty$, $0<q\leqslant\infty$. Let $u\in{}^{\mathrm{c}}\Omega_q$ and $v\in\mathcal{W}(0,\infty)$. Set*

$$X=\mathrm{Cop}_{p,q}(u,v).$$

(i) *Let $0<q\leqslant 1$. Then*

$$\|f\|_{X'}\approx\sup_{t\in(0,\infty)}\|f\|_{p',v^{-1},(0,t)}\|u\|_{q,(0,t)}^{-1},$$

with the positive constants in the equivalence independent of f.

(ii) *Let $1<q\leqslant\infty$. Then*

$$\|f\|_{X'}\approx\biggl(\int_0^\infty\|f\|_{p',v^{-1},(0,t)}^{q'}\,\mathrm{d}(-\|u\|_{q,(0,t+)}^{-q'})\biggr)^{1/q'}+\|f\|_{p',v^{-1},(0,\infty)}\|u\|_{q,(0,\infty)}^{-1},$$

with the positive constants in the equivalence independent of f.

Finally, we prove another variant of "gluing" lemma, which is very useful and has independent interest (cf. [28], Theorem 3.1).

Lemma 3.13. *Let β be a positive number, $a \in \mathcal{W}(0,\infty)$ is non-decreasing and $g, h \in \mathfrak{M}^+(0,\infty)$. Denote*

$$\mathcal{A}(x,t) := \frac{a(x)}{a(x)+a(t)}, \quad x>0,\ t>0.$$

Then

$$\int_0^\infty \bigg(\int_0^\infty \mathcal{A}(x,t)g(t)\,\mathrm{d}t\bigg)^{\beta-1} \bigg(\operatorname*{ess\,sup}_{t\in(0,\infty)} \mathcal{A}(t,x)h(t)\bigg)^{\beta} g(x)\,\mathrm{d}x$$
$$\approx \int_0^\infty \bigg(\int_0^x g(t)\,\mathrm{d}t\bigg)^{\beta-1} \bigg(\operatorname*{ess\,sup}_{t\in(x,\infty)} h(t)\bigg)^{\beta} g(x)\,\mathrm{d}x$$
$$+ \int_0^\infty \bigg(\int_x^\infty a(\tau)^{-1}g(\tau)\,\mathrm{d}\tau\bigg)^{\beta-1} \bigg(\operatorname*{ess\,sup}_{t\in(0,x)} a(t)h(t)\bigg)^{\beta} a(x)^{-1}g(x)\,\mathrm{d}x.$$

P r o o f. Denote

$$A_1 : = \int_0^\infty \bigg(\int_0^x g(t)\,\mathrm{d}t\bigg)^{\beta-1} \bigg(\operatorname*{ess\,sup}_{t\in(x,\infty)} h(t)\bigg)^{\beta} g(x)\,\mathrm{d}x,$$
$$A_2 : = \int_0^\infty \bigg(\int_x^\infty a(\tau)^{-1}g(\tau)\,\mathrm{d}\tau\bigg)^{\beta-1} \bigg(\operatorname*{ess\,sup}_{t\in(0,x)} a(t)h(t)\bigg)^{\beta} a(x)^{-1}g(x)\,\mathrm{d}x.$$

Obviously,

$$\int_0^\infty \bigg(\int_0^\infty \mathcal{A}(x,t)g(t)\,\mathrm{d}t\bigg)^{\beta-1} \bigg(\operatorname*{ess\,sup}_{t\in(0,\infty)} \mathcal{A}(t,x)h(t)\bigg)^{\beta} g(x)\,\mathrm{d}x$$
$$\approx \int_0^\infty \bigg(\int_0^x g(t)\,\mathrm{d}t + a(x)\int_x^\infty a(t)^{-1}g(t)\,\mathrm{d}t\bigg)^{\beta-1}$$
$$\times \bigg(a(x)^{-1}\operatorname*{ess\,sup}_{t\in(0,x)} a(t)h(t) + \operatorname*{ess\,sup}_{t\in(x,\infty)} h(t)\bigg)^{\beta} g(x)\,\mathrm{d}x$$
$$\approx A_1 + A_2 + B_1 + B_2,$$

where

$$B_1 := \int_0^\infty \bigg(\int_0^x g(t)\,\mathrm{d}t\bigg)^{\beta-1} \bigg(a(x)^{-1}\operatorname*{ess\,sup}_{t\in(0,x)} a(t)h(t)\bigg)^{\beta} g(x)\,\mathrm{d}x,$$
$$B_2 := \int_0^\infty \bigg(\int_x^\infty a(t)^{-1}g(t)\,\mathrm{d}t\bigg)^{\beta-1} \bigg(a(x)\operatorname*{ess\,sup}_{t\in(x,\infty)} h(t)\bigg)^{\beta} a(x)^{-1}g(x)\,\mathrm{d}x.$$

It is enough to show that $B_i \lesssim A_1 + A_2$, $i=1,2$.

Let us show that $B_1 \lesssim A_1 + A_2$. We will consider the case when $\int_0^\infty g(t)\,\mathrm{d}t < \infty$ (the case when $\int_0^\infty g(t)\,\mathrm{d}t = \infty$ is much simpler to treat). Define a sequence $\{x_m\}_{m=-\infty}^{M}$ such that $\int_0^{x_m} g(t)\,\mathrm{d}t = 2^m$ if $-\infty < m \leqslant M$ and $2^M \leqslant \int_0^\infty g(t)\,\mathrm{d}t < 2^{M+1}$. Then, applying [22], Lemma 2.5, we have that

$$\begin{aligned} B_1 &\leqslant \int_0^\infty \biggl(\int_0^x g(t)\,\mathrm{d}t\biggr)^{\beta-1} \Bigl(\operatorname*{ess\,sup}_{y\in(x,\infty)} a(y)^{-1} \operatorname*{ess\,sup}_{t\in(0,y)} a(t)h(t)\Bigr)^{\beta} g(x)\,\mathrm{d}x \\ &\approx \sum_{m=-\infty}^{M} 2^{m\beta}\Bigl(\operatorname*{ess\,sup}_{y\in(x_m,\infty)} a(y)^{-1} \operatorname*{ess\,sup}_{t\in(0,y)} a(t)h(t)\Bigr)^{\beta} \\ &\approx \sum_{m=-\infty}^{M} 2^{m\beta}\Bigl(\operatorname*{ess\,sup}_{y\in(x_m,x_{m+1})} a(y)^{-1} \operatorname*{ess\,sup}_{t\in(0,y)} a(t)h(t)\Bigr)^{\beta}. \end{aligned}$$

For every $-\infty < m \leqslant M$ there exists $y_m \in (x_m, x_{m+1})$ such that

$$\operatorname*{ess\,sup}_{y\in(x_m,x_{m+1})} a(y)^{-1} \operatorname*{ess\,sup}_{t\in(0,y)} a(t)h(t) \leqslant 2a(y_m)^{-1} \operatorname*{ess\,sup}_{t\in(0,y_m)} a(t)h(t).$$

Therefore,

$$\begin{aligned} B_1 &\lesssim \sum_{m=-\infty}^{M} 2^{m\beta}\Bigl(a(y_m)^{-1} \operatorname*{ess\,sup}_{t\in(0,y_m)} a(t)h(t)\Bigr)^{\beta} \\ &\approx \sum_{m=-\infty}^{M} 2^{m\beta}\Bigl(a(y_m)^{-1} \operatorname*{ess\,sup}_{t\in(0,y_{m-2})} a(t)h(t)\Bigr)^{\beta} \\ &\quad + \sum_{m=-\infty}^{M} 2^{m\beta}\Bigl(a(y_m)^{-1} \operatorname*{ess\,sup}_{t\in(y_{m-2},y_m)} a(t)h(t)\Bigr)^{\beta} =: I + \mathit{II}. \end{aligned}$$

Note that $2^m \leqslant \int_0^{y_m} g(x)\,\mathrm{d}x \leqslant 2^{m+1}$ and $2^{m-1} \leqslant \int_{y_{m-2}}^{y_m} g(x)\,\mathrm{d}x \leqslant 2^{m+1}$, $-\infty < m \leqslant M$. It yields that

$$\begin{aligned} I &\lesssim \sum_{m=-\infty}^{M} \int_{y_{m-2}}^{y_m} \biggl(\int_x^{y_m} g(t)\,\mathrm{d}t\biggr)^{\beta-1} g(x)\,\mathrm{d}x \cdot \Bigl(a(y_m)^{-1} \operatorname*{ess\,sup}_{t\in(0,y_{m-2})} a(t)h(t)\Bigr)^{\beta} \\ &\leqslant \sum_{m=-\infty}^{M} \int_{y_{m-2}}^{y_m} \biggl(\int_x^{y_m} a(t)^{-1}g(t)\,\mathrm{d}t\biggr)^{\beta-1} a(x)^{-1}g(x)\,\mathrm{d}x \cdot \Bigl(\operatorname*{ess\,sup}_{t\in(0,y_{m-2})} a(t)h(t)\Bigr)^{\beta} \\ &\leqslant \sum_{m=-\infty}^{M} \int_{y_{m-2}}^{y_m} \biggl(\int_x^{\infty} a(t)^{-1}g(t)\,\mathrm{d}t\biggr)^{\beta-1} \Bigl(\operatorname*{ess\,sup}_{t\in(0,x)} a(t)h(t)\Bigr)^{\beta} a(x)^{-1}g(x)\,\mathrm{d}x \\ &\lesssim \int_0^\infty \biggl(\int_x^{\infty} a(\tau)^{-1}g(\tau)\,\mathrm{d}\tau\biggr)^{\beta-1} \Bigl(\operatorname*{ess\,sup}_{t\in(0,x)} a(t)h(t)\Bigr)^{\beta} a(x)^{-1}g(x)\,\mathrm{d}x = A_2. \end{aligned}$$

For II we have that

$$
\begin{aligned}
II &\lesssim \sum_{m=-\infty}^{M} \int_{y_{m-4}}^{y_{m-2}} \left(\int_{y_{m-4}}^{x} g(t)\,\mathrm{d}t \right)^{\beta-1} g(x)\,\mathrm{d}x \cdot \left(a(y_m)^{-1} \operatorname*{ess\,sup}_{t\in(y_{m-2},y_m)} a(t)h(t) \right)^{\beta} \\
&\leqslant \sum_{m=-\infty}^{M} \int_{y_{m-4}}^{y_{m-2}} \left(\int_{0}^{x} g(t)\,\mathrm{d}t \right)^{\beta-1} g(x)\,\mathrm{d}x \cdot \left(\operatorname*{ess\,sup}_{t\in(y_{m-2},\infty)} h(t) \right)^{\beta} \\
&\leqslant \sum_{m=-\infty}^{M} \int_{y_{m-4}}^{y_{m-2}} \left(\int_{0}^{x} g(t)\,\mathrm{d}t \right)^{\beta-1} \left(\operatorname*{ess\,sup}_{t\in(x,\infty)} h(t) \right)^{\beta} g(x)\,\mathrm{d}x \\
&\lesssim \int_{0}^{\infty} \left(\int_{0}^{x} g(t)\,\mathrm{d}t \right)^{\beta-1} \left(\operatorname*{ess\,sup}_{t\in(x,\infty)} h(t) \right)^{\beta} g(x)\,\mathrm{d}x = A_1.
\end{aligned}
$$

Combining the previous inequalities, we get that $B_1 \lesssim A_1 + A_2$.

Now we show that $B_2 \lesssim A_1 + A_2$. We will consider the case when $\int_0^\infty a(t)^{-1} \times g(t)\,\mathrm{d}t < \infty$ (it is much simpler to deal with the case when $\int_0^\infty a(t)^{-1} g(t)\,\mathrm{d}t = \infty$). Define a sequence $\{x_m\}_{m=N}^{\infty}$ such that $2^{-m} = \int_{x_m}^\infty a(\tau)^{-1} g(\tau)\,\mathrm{d}\tau$ if $N \leqslant m < \infty$ and $2^{-N} < \int_0^\infty a(\tau)^{-1} g(\tau)\,\mathrm{d}\tau \leqslant 2^{-N+1}$. By using [22], Lemma 2.4, we find that

$$
\begin{aligned}
B_2 &\leqslant \int_0^\infty \left(\int_x^\infty a(\tau)^{-1} g(\tau)\,\mathrm{d}\tau \right)^{\beta-1} \left(\operatorname*{ess\,sup}_{y\in(0,x)} a(y) \sup_{t\in(y,\infty)} h(t) \right)^{\beta} a(x)^{-1} g(x)\,\mathrm{d}x \\
&\approx \sum_{m=N}^{\infty} 2^{-m\beta} \left(\sup_{y\in(0,x_m)} a(y) \operatorname*{ess\,sup}_{t\in(y,\infty)} h(t) \right)^{\beta} \\
&\approx \sum_{m=N}^{\infty} 2^{-m\beta} \left(\operatorname*{ess\,sup}_{y\in(x_{m-1},x_m)} a(y) \operatorname*{ess\,sup}_{t\in(y,\infty)} h(t) \right)^{\beta}.
\end{aligned}
$$

For every $m = N, N+1, \ldots$ there exists $y_m \in (x_{m-1}, x_m)$ such that

$$
\operatorname*{ess\,sup}_{y\in(x_{m-1},x_m)} a(y) \operatorname*{ess\,sup}_{t\in(y,\infty)} h(t) \leqslant 2U(y_m) \operatorname*{ess\,sup}_{t\in(y_m,\infty)} h(t).
$$

Hence

$$
\begin{aligned}
B_2 &\lesssim \sum_{m=N}^{\infty} 2^{-m\beta} \left(a(y_m) \operatorname*{ess\,sup}_{t\in(y_m,\infty)} h(t) \right)^{\beta} \\
&\approx \sum_{m=N}^{\infty} 2^{-m\beta} \left(a(y_m) \operatorname*{ess\,sup}_{t\in(y_m,y_{m+2})} h(t) \right)^{\beta} + \sum_{m=N}^{\infty} 2^{-m\beta} \left(a(y_m) \operatorname*{ess\,sup}_{t\in(y_{m+2},\infty)} h(t) \right)^{\beta} \\
&=: III + IV.
\end{aligned}
$$

Since $2^{-m-1} \leqslant \int_{y_m}^{\infty} a(\tau)^{-1} g(\tau)\,\mathrm{d}\tau \leqslant 2^{-m}$ and $2^{-m-2} \leqslant \int_{y_m}^{y_{m+2}} a(\tau)^{-1} g(\tau)\,\mathrm{d}\tau \leqslant 2^{-m}$, $m = N, N+1, \ldots$, we have that

$$\begin{aligned} III &\lesssim \sum_{m=N}^{\infty} \int_{y_{m+2}}^{y_{m+4}} \biggl(\int_x^{y_{m+4}} a^{-1} g\biggr)^{\beta-1} a(x)^{-1} g(x)\,\mathrm{d}x \cdot \biggl(a(y_m) \operatorname*{ess\,sup}_{t\in(y_m,y_{m+2})} h(t)\biggr)^{\beta} \\ &\leqslant \sum_{m=N}^{\infty} \int_{y_{m+2}}^{y_{m+4}} \biggl(\int_x^{\infty} a^{-1} g\biggr)^{\beta-1} \biggl(\operatorname*{ess\,sup}_{t\in(0,x)} a(t)h(t)\biggr)^{\beta} a(x)^{-1} g(x)\,\mathrm{d}x \\ &\lesssim \int_0^{\infty} \biggl(\int_x^{\infty} a^{-1} g\biggr)^{\beta-1} \biggl(\operatorname*{ess\,sup}_{t\in(0,x)} a(t)h(t)\biggr)^{\beta} a(x)^{-1} g(x)\,\mathrm{d}x \approx A_2. \end{aligned}$$

Moreover,

$$\begin{aligned} IV &\lesssim \sum_{m=N}^{\infty} \int_{y_m}^{y_{m+2}} \biggl(\int_{y_m}^{x} a^{-1} g\biggr)^{\beta-1} a(x)^{-1} g(x)\,\mathrm{d}x \cdot \biggl(a(y_m) \operatorname*{ess\,sup}_{t\in(y_{m+2},\infty)} h(t)\biggr)^{\beta} \\ &\leqslant \sum_{m=N}^{\infty} \int_{y_m}^{y_{m+2}} \biggl(\int_{y_m}^{x} g\biggr)^{\beta-1} g(x)\,\mathrm{d}x \cdot \biggl(\operatorname*{ess\,sup}_{t\in(y_{m+2},\infty)} h(t)\biggr)^{\beta} \\ &\leqslant \sum_{m=N}^{\infty} \int_{y_m}^{y_{m+2}} \biggl(\int_0^{x} g\biggr)^{\beta-1} \biggl(\operatorname*{ess\,sup}_{t\in(x,\infty)} h(t)\biggr)^{\beta} g(x)\,\mathrm{d}x \\ &\lesssim \int_0^{\infty} \biggl(\int_0^{x} g\biggr)^{\beta-1} \biggl(\operatorname*{ess\,sup}_{t\in(x,\infty)} h(t)\biggr)^{\beta} g(x)\,\mathrm{d}x \approx A_1. \end{aligned}$$

Therefore, we obtain $B_2 \lesssim A_1 + A_2$. The proof is complete. $\square$

4. Proofs of main results

In this section we prove our main results.

Lemma 4.1. *Let $0 < p_1, p_2, q_1, q_2 \leqslant \infty$ and $p_1 < p_2$. Assume that $v_1, v_2 \in \mathcal{W}(0,\infty)$, $u_1 \in {}^{\mathbf{c}}\Omega_{q_1}$ and $u_2 \in \Omega_{q_2}$. Then $\mathrm{Cop}_{p_1,q_1}(u_1,v_1) \not\hookrightarrow \mathrm{Ces}_{p_2,q_2}(u_2,v_2)$.*

P r o o f. Assume that $\mathrm{Cop}_{p_1,q_1}(u_1,v_1) \hookrightarrow \mathrm{Ces}_{p_2,q_2}(u_2,v_2)$ holds. Then there exists $c > 0$ such that

$$\|f\|_{\mathrm{Ces}_{p_2,q_2}(u_2,v_2)} \leqslant c\|f\|_{\mathrm{Cop}_{p_1,q_1}(u_1,v_1)}$$

holds for all $f \in \mathfrak{M}^+(0,\infty)$. Let $\tau \in (0,\infty)$ and $f = 0$ on (τ,∞). Thus, we have

$$\begin{aligned} \|f\|_{\mathrm{Ces}_{p_2,q_2}(u_2,v_2)} &= \|\|f\|_{p_2,v_2,(0,\cdot)}\|_{q_2,u_2,(0,\infty)} \\ &\geqslant \|\|f\|_{p_2,v_2,(0,\cdot)}\|_{q_2,u_2,(\tau,\infty)} \\ &\geqslant \|u_2\|_{q_2,(\tau,\infty)}\|f\|_{p_2,v_2,(0,\tau)} \end{aligned} \tag{4.1}$$

and

$$(4.2)\qquad \begin{aligned}\|f\|_{\mathrm{Cop}_{p_1,q_1}(u_1,v_1)} &= \|\|f\|_{p_1,v_1,(\cdot,\infty)}\|_{q_1,u_1,(0,\infty)}\\ &\leqslant \|\|f\|_{p_1,v_1,(\cdot,\infty)}\|_{q_1,u_1,(0,\tau)}\\ &\leqslant \|u_1\|_{q_1,(0,\tau)}\|f\|_{p_1,v_1,(0,\tau)}.\end{aligned}$$

Combining (4.1) with (4.2), we can assert that

$$\|u_2\|_{q_2,(\tau,\infty)}\|f\|_{p_2,v_2,(0,\tau)} \leqslant c\|u_1\|_{q_1,(0,\tau)}\|f\|_{p_1,v_1,(0,\tau)}.$$

Since $u_1 \in {}^{\mathfrak{c}}\Omega_{q_1}$ and $u_2 \in \Omega_{q_2}$, we conclude that $L_{p_1}(v_1) \hookrightarrow L_{p_2}(v_2)$, which is a contradiction. □

P r o o f of Theorem 2.1. In view of Lemmas 3.4 and 3.5, we have that

$$\|\mathrm{I}\|_{\mathrm{Cop}_{p_1,q_1}(u_1,v_1)\to\mathrm{Ces}_{p_2,q_2}(u_2,v_2)} = \|\mathrm{I}\|_{L_{p_1}(w_1)\to L_{p_2}(w_2)}$$

with $w_1(x) = v_1(x)\|u_1\|_{p_1,(0,x)}$ and $w_2(x) = v_2(x)\|u_2\|_{p_2,(x,\infty)}$, $x > 0$. □

P r o o f of Theorem 2.2. In view of Lemma 3.5, we have that

$$\|\mathrm{I}\|_{\mathrm{Cop}_{p_1,q_1}(u_1,v_1)\to\mathrm{Ces}_{p_2,q_2}(u_2,v_2)} = \|\mathrm{I}\|_{L_{p_1}(w_1)\to\mathrm{Ces}_{p_2,q_2}(u_2,v_2)}$$

with $w_1(x) = v_1(x)\|u_1\|_{p_1,(0,x)}$, $x > 0$. Then the result follows from Theorem 3.1. □

P r o o f of Theorem 2.3. In view of Lemma 3.4, we have that

$$\|\mathrm{I}\|_{\mathrm{Cop}_{p_1,q_1}(u_1,v_1)\to\mathrm{Ces}_{p_2,q_2}(u_2,v_2)} = \|\mathrm{I}\|_{\mathrm{Cop}_{p_1,q_1}(u_1,v_1)\to L_{p_2}(w_2)}$$

with $w_2(x) = v_2(x)\|u_2\|_{p_2,(x,\infty)}$, $x > 0$. Then the result follows from Theorem 3.9. □

Lemma 4.2. *Let* $0 < p_1, p_2, q_1, q_2 < \infty$, $p_2 \leqslant p_1$ *and* $p_2 < q_2$. *Let* $v_1, v_2 \in \mathcal{W}(0,\infty)$, $u_1 \in {}^{\mathfrak{c}}\Omega_{q_1}$ *and* $u_2 \in \Omega_{q_2}$. *Then*

$$\begin{aligned}&\|\mathrm{I}\|_{\mathrm{Cop}_{p_1,q_1}(u_1,v_1)\to\mathrm{Ces}_{p_2,q_2}(u_2,v_2)}\\ &\qquad= \bigg(\sup_{g\in\mathfrak{M}^+(0,\infty)} \frac{\|\mathrm{I}\|^{p_2}_{\mathrm{Cop}_{p_1,q_1}(u_1,v_1)\to L_{p_2}(v_2 H^*(g)^{1/p_2})}}{\|g\|_{q_2/(q_2-p_2),u_2^{-p_2},(0,\infty)}}\bigg)^{1/p_2}.\end{aligned}$$

P r o o f. By duality, interchanging suprema, we have that

$$
\begin{aligned}
&\|\mathrm{I}\|_{\mathrm{Cop}_{p_1,q_1}(u_1,v_1)\to \mathrm{Ces}_{p_2,q_2}(u_2,v_2)}\\
&\quad = \sup_{f\in\mathfrak{M}^+(0,\infty)} \frac{\|f\|_{\mathrm{Ces}_{p_2,q_2}(u_2,v_2)}}{\|f\|_{\mathrm{Cop}_{p_1,q_1}(u_1,v_1)}}\\
&\quad = \sup_{f\in\mathfrak{M}^+(0,\infty)} \frac{1}{\|f\|_{\mathrm{Cop}_{p_1,q_1}(u_1,v_1)}} \sup_{g\in\mathfrak{M}^+(0,\infty)} \frac{\big(\int_0^\infty \big(\int_0^\tau f^{p_2} v_2^{p_2}\big) g(\tau)\,\mathrm{d}\tau\big)^{1/p_2}}{\|g\|^{1/p_2}_{q_2/(q_2-p_2),u_2^{-p_2},(0,\infty)}}\\
&\quad = \sup_{g\in\mathfrak{M}^+(0,\infty)} \frac{1}{\|g\|^{1/p_2}_{q_2/(q_2-p_2),u_2^{-p_2},(0,\infty)}} \sup_{f\in\mathfrak{M}^+(0,\infty)} \frac{\big(\int_0^\infty \big(\int_0^\tau f^{p_2} v_2^{p_2}\big) g(\tau)\,\mathrm{d}\tau\big)^{1/p_2}}{\|f\|_{\mathrm{Cop}_{p_1,q_1}(u_1,v_1)}}.
\end{aligned}
$$

Applying Fubini's theorem, we get that

$$
\begin{aligned}
(4.3)\quad &\|\mathrm{I}\|_{\mathrm{Cop}_{p_1,q_1}(u_1,v_1)\to \mathrm{Ces}_{p_2,q_2}(u_2,v_2)}\\
&\quad = \sup_{g\in\mathfrak{M}^+(0,\infty)} \Bigg(\frac{1}{\|g\|^{1/p_2}_{q_2/(q_2-p_2),u_2^{-p_2},(0,\infty)}}\\
&\qquad\quad \times \sup_{f\in\mathfrak{M}^+(0,\infty)} \frac{\big(\int_0^\infty f(x)^{p_2} v_2(x)^{p_2} \big(\int_x^\infty g\big)\,\mathrm{d}x\big)^{1/p_2}}{\|f\|_{\mathrm{Cop}_{p_1,q_1}(u_1,v_1)}}\Bigg)\\
&\quad = \sup_{g\in\mathfrak{M}^+(0,\infty)} \frac{1}{\|g\|^{1/p_2}_{q_2/(q_2-p_2),u_2^{-p_2},(0,\infty)}} \|\mathrm{I}\|_{\mathrm{Cop}_{p_1,q_1}(u_1,v_1)\to L_{p_2}(v_2(\cdot)H^*(g)^{1/p_2})}.
\end{aligned}
$$

□

P r o o f of Theorem 2.8. By Lemma 4.2, we have that

$$
\begin{aligned}
&\|\mathrm{I}\|_{\mathrm{Cop}_{p_1,q_1}(u_1,v_1)\to \mathrm{Ces}_{p_2,q_2}(u_2,v_2)}\\
&\quad = \sup_{g\in\mathfrak{M}^+(0,\infty)} \frac{1}{\|g\|^{1/p_2}_{q_2/(q_2-p_2),u_2^{-p_2},(0,\infty)}} \|\mathrm{I}\|_{\mathrm{Cop}_{p_1,q_1}(u_1,v_1)\to L_{p_2}(v_2 H^*(g)^{1/p_2})}.
\end{aligned}
$$

Since $q_1 \leqslant p_2$, applying Theorem 3.9 (i), we obtain that

$$
\begin{aligned}
&\|\mathrm{I}\|_{\mathrm{Cop}_{p_1,q_1}(u_1,v_1)\to \mathrm{Ces}_{p_2,q_2}(u_2,v_2)}\\
&\quad \approx \Bigg(\sup_{g\in\mathfrak{M}^+(0,\infty)} \frac{\sup_{t\in(0,\infty)} \|u_1\|^{-p_2}_{q_1,(0,t)} \|H^*g\|_{p_1/(p_1-p_2),(v_1^{-1}v_2)^{p_2},(0,t)}}{\|g\|_{q_2/(q_2-p_2),u_2^{-p_2},(0,\infty)}}\Bigg)^{1/p_2}.
\end{aligned}
$$

(i) If $p_1 \leqslant q_2$, then applying [26], Theorem 3.2 (i), we arrive at

$$
\|\mathrm{I}\|_{\mathrm{Cop}_{p_1,q_1}(u_1,v_1)\to \mathrm{Ces}_{p_2,q_2}(u_2,v_2)} \approx \sup_{x\in(0,\infty)} \varphi_1(x) \sup_{t\in(0,\infty)} \mathcal{V}(t,x)\|u_2\|_{q_2,(t,\infty)}.
$$

(ii) If $q_2 < p_1$, then applying [26], Theorem 3.2 (ii), we arrive at

$$\|\mathrm{I}\|_{\mathrm{Cop}_{p_1,q_1}(u_1,v_1)\to\mathrm{Ces}_{p_2,q_2}(u_2,v_2)} \approx \sup_{x\in(0,\infty)} \varphi_1(x)\biggl(\int_0^\infty \mathcal{V}(t,x)^{p_1\to q_2}\,\mathrm{d}(-\|u_2\|_{q_2,(t,\infty)}^{p_1\to q_2})\biggr)^{1/(p_1\to q_2)}.$$

□

Remark 4.3. In view of Remark 2.7, if

$$\limsup_{t\to 0+} \widetilde{V}(t)\|u_1\|_{q_1,(0,t)}^{-1} = \limsup_{t\to\infty} \widetilde{V}(t)\|u_1\|_{q_1,(0,t)} = \limsup_{t\to 0+} \|u_1\|_{q_1,(0,t)} = \limsup_{t\to\infty} \|u_1\|_{q_1,(0,t)}^{-1} = 0,$$

then $\varphi_1 \in Q_{\widetilde{V}^{1/(p_1\to p_2)}}$.

P r o o f of Theorem 2.9. By Lemma 4.2, applying Theorem 3.9 (ii), we have that

$$\begin{aligned}
&\|\mathrm{I}\|_{\mathrm{Cop}_{p_1,q_1}(u_1,v_1)\to\mathrm{Ces}_{p_2,q_2}(u_2,v_2)}\\
&\approx \|u_1\|_{q_1,(0,\infty)}^{-1}\biggl(\sup_{g\in\mathfrak{M}^+(0,\infty)} \frac{\|H^*g\|_{p_1/(p_1-p_2),(v_1^{-1}v_2)^{p_2},(0,\infty)}}{\|g\|_{q_2/(q_2-p_2),u_2^{-p_2},(0,\infty)}}\biggr)^{1/p_2}\\
&\quad+\biggl(\sup_{g\in\mathfrak{M}^+(0,\infty)} \frac{\bigl(\int_0^\infty \|H^*g\|_{p_1/(p_1-p_2),(v_1^{-1}v_2)^{p_2},(0,t)}^{q_1/(q_1-p_2)}\,\mathrm{d}(-\|u_1\|_{q_1,(0,t)}^{-q_1p_2/(q_1-p_2)})\bigr)^{(q_1-p_2)/q_1}}{\|g\|_{q_2/(q_2-p_2),u_2^{-p_2},(0,\infty)}}\biggr)^{1/p_2}\\
&:= C_1 + C_2.
\end{aligned}$$

Note that

$$C_1 = \|u_1\|_{q_1,(0,\infty)}^{-1}\Bigl\{\|\mathrm{I}\|_{L_{q_2/(q_2-p_2)}(u_2^{-p_2})\to\mathrm{Cop}_{1,p_1/(p_1-p_2)}((v_1^{-1}v_2)^{p_2},\mathbf{1})}\Bigr\}^{1/p_2}.$$

Assume first that $p_1 \leqslant q_2$. Applying Theorem 3.7 (i), we arrive at

$$C_1 \approx \|u_1\|_{q_1,(0,\infty)}^{-1} \sup_{t\in(0,\infty)} \widetilde{V}(t)\|u_2\|_{q_2,(t,\infty)}. \tag{4.4}$$

(i) Let $q_1 \leqslant q_2$. Using [26], Theorem 3.1 (i),we obtain that

$$C_2 \approx \sup_{x\in(0,\infty)} \varphi_2(x) \sup_{t\in(0,\infty)} \mathcal{V}(t,x)\|u_2\|_{q_2,(t,\infty)}.$$

Consequently, the proof is completed in this case.

(ii) Let $q_2 < q_1$. Applying [26], Theorem 3.1 (ii), we have that

$$C_2 \approx \biggl(\int_0^\infty \varphi_2(x)^{(q_1\to q_2 q_1\to p_2)/(q_2\to p_2)}\widetilde{V}(x)^{q_1\to p_2} \times \biggl(\sup_{t\in(0,\infty)} \mathcal{V}(t,x)\|u_2\|_{q_2,(t,\infty)}\biggr)^{q_1\to q_2} \,\mathrm{d}(-\|u_1\|_{q_1,(0,x)}^{-q_1\to p_2})\biggr)^{1/(q_1\to q_2)},$$

and the statement follows in this case.

Let us now assume that $q_2 < p_1$. Then using Theorem 3.2 (ii), we have that

$$C_1 \approx \|u_1\|_{q_1,(0,\infty)}^{-1}\biggl(\int_0^\infty \widetilde{V}(t)^{p_1\to q_2}\,\mathrm{d}(-\|u_2\|_{q_2,(t,\infty)}^{p_1\to q_2})\biggr)^{1/(p_1\to q_2)}. \tag{4.5}$$

(iii) Let $q_1 \leqslant q_2$. Then [26], Theorem 3.1 (iii), yields that

$$C_2 \approx \sup_{x\in(0,\infty)} \varphi_2(x)\biggl(\int_0^\infty \mathcal{V}(t,x)^{p_1\to q_2}\,\mathrm{d}(-\|u_2\|_{q_2,(t,\infty)}^{p_1\to q_2})\biggr)^{1/(p_1\to q_2)},$$

which completes the proof in this case.

(iv) If $q_2 < q_1$, then using [26], Theorem 3.1 (iv), we arrive at

$$C_2 \approx \biggl(\int_0^\infty \varphi_2(x)^{(q_1\to q_2 q_1\to p_2)/(q_2\to p_2)}\widetilde{V}(x)^{q_1\to p_2} \times \biggl(\int_0^\infty \mathcal{V}(t,x)^{p_1\to q_2}\,\mathrm{d}(-\|u_2\|_{q_2,(t,\infty)}^{p_1\to q_2})\biggr)^{(q_1\to q_2)/(p_1\to q_2)} \,\mathrm{d}(-\|u_1\|_{q_1,(0,x)}^{-q_1\to p_2})\biggr)^{1/(q_1\to q_2)},$$

and the proof follows. □

Remark 4.4. Assume that $\varphi_2(x) < \infty$, $x > 0$. In view of Remark 2.3, if

$$\int_0^1 \biggl(\int_0^t u_1^{q_1}\biggr)^{-q_1/(q_1-p_2)} u_1^{q_1}(t)\,\mathrm{d}t = \int_1^\infty \widetilde{V}(t)^{(q_1 p_2)/(q_1-p_2)}\biggl(\int_0^t u_1^{q_1}\biggr)^{-q_1/(q_1-p_2)} u_1^{q_1}(t)\,\mathrm{d}t = \infty,$$

then $\varphi_2 \in Q_{\widetilde{V}^{1/(p_1\to p_2)}}$.

P r o o f of Theorem 2.6. By Lemma 4.2, applying Theorem 3.4 (i), we have that

$$\|\mathrm{I}\|_{\mathrm{Cop}_{p_1,q_1}(u_1,v_1)\to\mathrm{Ces}_{p_2,q_2}(u_2,v_2)} \approx \biggl(\sup_{g\in\mathfrak{M}^+(0,\infty)} \frac{\|H^*g\|_{\infty,(v_1^{-1}v_2)^p\|u_1\|_{q_1,(0,\cdot)}^{-p},(0,\infty)}}{\|g\|_{q_2/(q_2-p),u_2^{-p},(0,\infty)}}\biggr)^{1/p} = \Bigl(\|\mathrm{I}\|_{L_{q_2/(q_2-p)}(u_2^{-p})\to\mathrm{Cop}_{1,\infty}((v_1^{-1}v_2)^p\|u_1\|_{q_1,(0,\cdot)}^{-p},\mathbf{1})}\Bigr)^{1/p}.$$

Therefore, by Theorem 3.7 (i),

$$\|\mathrm{I}\|_{\mathrm{Cop}_{p_1,q_1}(u_1,v_1)\to\mathrm{Ces}_{p_2,q_2}(u_2,v_2)} \approx \sup_{t\in(0,\infty)} \big\|\|u_1\|^{-1}_{q_1,(0,\cdot)}\big\|_{\infty,v_1^{-1}v_2,(0,t)}\|u_2\|_{q_2,(t,\infty)}.$$

□

P r o o f of Theorem 2.11. By Lemma 4.2, applying Theorem 3.9 (ii), we get that

$$\begin{aligned}
&\|\mathrm{I}\|_{\mathrm{Cop}_{p_1,q_1}(u_1,v_1)\to\mathrm{Ces}_{p_2,q_2}(u_2,v_2)} \\
&\quad\approx \|u_1\|^{-1}_{q_1,(0,\infty)}\bigg(\sup_{g\in\mathfrak{M}^+(0,\infty)}\frac{\|H^*g\|_{\infty,(v_1^{-1}v_2)^p,(0,\infty)}}{\|g\|_{q_2/(q_2-p),u_2^{-p},(0,\infty)}}\bigg)^{1/p} \\
&\quad+\bigg(\sup_{g\in\mathfrak{M}^+(0,\infty)}\frac{\big(\int_0^\infty\|H^*g\|^{q_1/(q_1-p)}_{\infty,(v_1^{-1}v_2)^p,(0,t)}\,\mathrm{d}\big(-\|u_1\|^{-(q_1p)/(q_1-p)}_{q_1,(0,t)}\big)\big)^{(q_1-p)/q_1}}{\|g\|_{(q_2)/(q_2-p),u_2^{-p},(0,\infty)}}\bigg)^{1/p} \\
&\quad:= C_3 + C_4.
\end{aligned}$$

Note that

$$C_3 = \|u_1\|^{-1}_{q_1,(0,\infty)}\Big(\|\mathrm{I}\|_{L_{q_2/(q_2-p)}(u_2^{-p})\to\mathrm{Cop}_{1,\infty}((v_1^{-1}v_2)^p,\mathbf{1})}\Big)^{1/p}.$$

Using Theorem 3.7 (i), we have that

$$C_3 \approx \|u_1\|^{-1}_{q_1,(0,\infty)}\sup_{t\in(0,\infty)}\widetilde{V}(t)\|u_2\|_{q_2,(t,\infty)}.$$

(i) Let $q_1 \leqslant q_2$. Then [27], Theorem 4.1, yields that

$$C_4 \approx \sup_{x\in(0,\infty)}\bigg(\int_0^\infty(\mathcal{V}(x,t)\widetilde{V}(t))^{q_1\to p}\,\mathrm{d}\big(-\|u_1\|^{-q_1\to p}_{q_1,(0,t)}\big)\bigg)^{1/(q_1\to p)}\|u_2\|_{q_2,(x,\infty)}.$$

Since $\varphi_2/\widetilde{V}$ is equivalent to a decreasing function we have that

$$\begin{aligned}
\sup_{x\in(0,\infty)}\varphi_2(x)\|u_2\|_{q_2,(x,\infty)} &= \sup_{x\in(0,\infty)}\varphi_2(x)\widetilde{V}(x)^{-1}\sup_{t\in(0,x)}\widetilde{V}(t)\|u_2\|_{q_2,(t,\infty)} \\
&= \sup_{x\in(0,\infty)}\varphi_2(x)\sup_{t\in(0,\infty)}\mathcal{V}(t,x)\|u_2\|_{q_2,(t,\infty)}, \quad x>0.
\end{aligned}$$

(ii) Let $q_2 < q_1$. Then [27], Theorem 4.4, yields that

$$C_4 \approx \left(\int_0^\infty \left(\int_x^\infty \mathrm{d}(-\|u_1\|_{q_1,(0,t)}^{-q_1\to p})\right)^{(q_1\to q_2)/(q_2\to p)}\right.$$

$$\left.\times \left(\sup_{0<\tau\leqslant x} \widetilde{V}(\tau)\|u_2\|_{q_2,(\tau,\infty)}\right)^{q_1\to q_2} \mathrm{d}(-\|u_1\|_{q_1,(0,x)}^{-q_1\to p})\right)^{1/(q_1\to q_2)}$$

$$+ \left(\int_0^\infty \left(\int_0^x \widetilde{V}(t)^{q_1\to p}\,\mathrm{d}(-\|u_1\|_{q_1,(0,t)}^{-q_1\to p})\right)^{(q_1\to q_2)/(q_2\to p)}\right.$$

$$\left.\times \widetilde{V}(x)^{q_1\to p}\|u_2\|_{q_2,(x,\infty)}^{q_1\to q_2}\,\mathrm{d}(-\|u_1\|_{q_1,(0,x)}^{-q_1\to p})\right)^{1/(q_1\to q_2)}$$

$$\approx \left(\int_0^\infty \varphi_2(x)^{(q_1\to q_2 q_1\to p)/(q_2\to p)}\widetilde{V}(x)^{q_1\to p}\right.$$

$$\left.\times \left(\sup_{t\in(0,\infty)} \mathcal{V}(t,x)\|u_2\|_{q_2,(t,\infty)}\right)^{q_1\to q_2} \mathrm{d}(-\|u_1\|_{q_1,(0,x)}^{-q_1\to p})\right)^{1/(q_1\to q_2)}.$$

In the last equivalence we have used Lemma 3.7 with $a(x) = \widetilde{V}(x)^{q_1\to p}$, $g(t)\,\mathrm{d}t = \widetilde{V}(t)^{q_1\to p}\,\mathrm{d}(-\|u_1\|_{q_1,(0,t)}^{-q_1\to p})$, $\beta = (q_1\to q_2)/(q_1\to p)$ and $h(t) = \|u_2\|_{q_2,(t,\infty)}^{q_1\to p}$.

It is clear that $\mathcal{A}(x,t) \approx \mathcal{V}(x,t)^{q_1\to p}$. $\square$

Acknowledgement. We thank the anonymous referee for his/her remarks.

References

[1] *R. Askey, R. P. Boas,* Jr.: Some integrability theorems for power series with positive coefficients. Math. Essays Dedicated to A. J. Macintyre. Ohio Univ. Press, Athens, 1970, pp. 23–32.

[2] *S. V. Astashkin*: On the geometric properties of Cesàro spaces. Sb. Math. *203* (2012), 514–533. (In English. Russian original.); translation from Mat. Sb. *203* (2012), 61–80.

[3] *S. V. Astashkin, L. Maligranda*: Cesàro function spaces fail the fixed point property. Proc. Am. Math. Soc. *136* (2008), 4289–4294.

[4] *S. V. Astashkin, L. Maligranda*: Structure of Cesàro function spaces. Indag. Math., New Ser. *20* (2009), 329–379.

[5] *S. V. Astashkin, L. Maligranda*: Rademacher functions in Cesàro type spaces. Stud. Math. *198* (2010), 235–247.

[6] *S. V. Astashkin, L. Maligranda*: Geometry of Cesàro function spaces. Funct. Anal. Appl. *45* (2011), 64–68. (In English. Russian original.); translation from Funkts. Anal. Prilozh. *45* (2011), 79–83.

[7] *S. V. Astashkin, L. Maligranda*: A short proof of some recent results related to Cesàro function spaces. Indag. Math., New Ser. *24* (2013), 589–592.

[8] *S. V. Astashkin, L. Maligranda*: Interpolation of Cesàro sequence and function spaces. Stud. Math. *215* (2013), 39–69.

[9] *S. V. Astashkin, L. Maligranda*: Interpolation of Cesàro and Copson spaces. Proc. Int. Symp. on Banach and Function Spaces IV (ISBFS 2012), Kitakyushu 2012 (M. Kato et al., eds.). Yokohama Publishers, Yokohama, 2014, pp. 123–133.

[10] *S. V. Astashkin, L. Maligranda*: Structure of Cesàro function spaces: a survey. Function Spaces X (H. Hudzik et al., eds.). Proc. Int. Conf., Poznań 2012, Polish Academy of Sciences, Institute of Mathematics, Warszawa; Banach Center Publications 102, 2014, pp. 13–40.

[11] *E. S. Belinskii, E. R. Liflyand, R. M. Trigub*: The Banach algebra A^* and its properties. J. Fourier Anal. Appl. *3* (1997), 103–129.

[12] *G. Bennett*: Factorizing the classical inequalities. Mem. Am. Math. Soc. *120* (1996), 130 pages.

[13] *R. P. Boas,* Jr.: Integrability Theorems for Trigonometric Transforms. Ergebnisse der Mathematik und ihrer Grenzgebiete 38, Springer, New York, 1967.

[14] *R. P. Boas,* Jr.: Some integral inequalities related to Hardy's inequality. J. Anal. Math. *23* (1970), 53–63.

[15] *M. Carro, A. Gogatishvili, J. Martín, L. Pick*: Weighted inequalities involving two Hardy operators with applications to embeddings of function spaces. J. Oper. Theory *59* (2008), 309–332.

[16] *S. Chen, Y. Cui, H. Hudzik, B. Sims*: Geometric properties related to fixed point theory in some Banach function lattices. Handbook of Metric Fixed Point Theory (W. Kirk et al., eds.). Kluwer Academic Publishers, Dordrecht, 2001, pp. 339–389.

[17] *Y. Cui, H. Hudzik*: Some geometric properties related to fixed point theory in Cesàro spaces. Collect. Math. *50* (1999), 277–288.

[18] *Y. Cui, H. Hudzik*: Packing constant for Cesaro sequence spaces. Nonlinear Anal., Theory Methods Appl. *47* (2001), 2695–2702.

[19] *Y. Cui, H. Hudzik, Y. Li*: On the García-Falset coefficient in some Banach sequence spaces. Function Spaces. Proc. Int. Conf. (Poznań 1998) (H. Hudzik et al., eds.). Lect. Notes Pure Appl. Math. 213, Marcel Dekker, New York, 2000, pp. 141–148.

[20] *Y. Cui, C.-H. Meng, R. Płuciennik*: Banach-Saks property and property (β) in Cesàro sequence spaces. Southeast Asian Bull. Math. *24* (2000), 201–210.

[21] *Y. Cui, R. Płuciennik*: Local uniform nonsquareness in Cesàro sequence spaces. Ann. Soc. Math. Pol., Ser. I, Commentat. Math. *37* (1997), 47–58.

[22] *W. D. Evans, A. Gogatishvili, B. Opic*: The reverse Hardy inequality with measures. Math. Inequal. Appl. *11* (2008), 43–74.

[23] *J. E. Gilbert*: Interpolation between weighted L^p-spaces. Ark. Mat. *10* (1972), 235–249.

[24] *A. Gogatishvili, R. Ch. Mustafayev*: Iterated Hardy-type inequalities involving suprema. Math. Inequal. Appl. *20* (2017), 901–927.

[25] *A. Gogatishvili, R. Ch. Mustafayev*: Weighted iterated Hardy-type inequalities. Math. Inequal. Appl. *20* (2017), 683–728.

[26] *A. Gogatishvili, R. Ch. Mustafayev, L.-E. Persson*: Some new iterated Hardy-type inequalities. J. Funct. Spaces Appl. *2012* (2012), Article ID 734194, 30 pages.

[27] *A. Gogatishvili, B. Opic, L. Pick*: Weighted inequalities for Hardy-type operators involving suprema. Collect. Math. *57* (2006), 227–255.

[28] *A. Gogatishvili, L.-E. Persson, V. D. Stepanov, P. Wall*: Some scales of equivalent conditions to characterize the Stieltjes inequality: the case $q < p$. Math. Nachr. *287* (2014), 242–253.

[29] *K.-G. Grosse-Erdmann*: The Blocking Technique, Weighted Mean Operators and Hardy's Inequality. Lecture Notes in Mathematics 1679, Springer, Berlin, 1998.

[30] *B. D. Hassard, D. A. Hussein*: On Cesàro function spaces. Tamkang J. Math. *4* (1973), 19–25.

[31] *A. A. Jagers*: A note on Cesàro sequence spaces. Nieuw Arch. Wiskd., III. Ser. *22* (1974), 113–124.

[32] *R. Johnson*: Lipschitz spaces, Littlewood-Paley spaces, and convoluteurs. Proc. Lond. Math. Soc., III. Ser. *29* (1974), 127–141.
[33] *A. Kamińska, D. Kubiak*: On the dual of Cesàro function space. Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods *75* (2012), 2760–2773.
[34] *A. Kufner, L. Maligranda, L.-E. Persson*: The Hardy Inequality. About Its History and Some Related Results. Vydavatelský Servis, Plzeň, 2007.
[35] *A. Kufner, L.-E. Persson*: Weighted Inequalities of Hardy Type. World Scientific Publishing, Singapore, 2003.
[36] *R. Mustafayev, T. Ünver*: Reverse Hardy-type inequalities for supremal operators with measures. Math. Inequal. Appl. *18* (2015), 1295–1311.
[37] *B. Opic, A. Kufner*: Hardy-Type Inequalities. Pitman Research Notes in Mathematics 219, Longman Scientific & Technical, New York, 1990.
[38] Programma van Jaarlijkse Prijsvragen (Annual Problem Section). Nieuw Arch. Wiskd. *16* (1968), 47–51.
[39] *J.-S. Shiue*: A note on Cesàro function space. Tamkang J. Math. *1* (1970), 91–95.
[40] *P. W. Sy, W. Y. Zhang, P. Y. Lee*: The dual of Cesàro function spaces. Glas. Mat., III Ser. *22(42)* (1987), 103–112.

Authors' addresses: Amiran Gogatishvili, Institute of Mathematics of the Czech Academy of Sciences, Žitná 25, 115 67 Praha 1, Czech Republic, e-mail: gogatish@math.cas.cz; Rza Mustafayev, Institute of Mathematics and Mechanics, Academy of Sciences of Azerbaijan, B. Vahabzade St. 9, Baku, AZ 1141, Azerbaijan, and Department of Mathematics Faculty of Science and Arts, Kırıkkale University, 71450 Yahsihan, Kırıkkale, Turkey, e-mail: rzamustafayev@gmail.com; Tuğçe Ünver, Department of Mathematics, Faculty of Science and Arts, Kırıkkale University, 71450 Yahsihan, Kırıkkale, Turkey, e-mail: tugceunver@gmail.com.