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# ON BUCHSBAUM TYPE MODULES AND THE ANNIHILATOR OF CERTAIN LOCAL COHOMOLOGY MODULES

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*Abstract.* We consider the annihilator of certain local cohomology modules. Moreover, some results on vanishing of these modules will be considered.

*Keywords*: annihilator of local cohomology; non-Artinian local cohomology; Buchsbaum type module

*MSC 2010*: 13D45

## 1. Introduction

Let $R$ be a commutative Noetherian ring and $M$ an $R$-module. There is a natural map $\mu_M\colon R \to \mathrm{Hom}_R(M, M)$ of $R$ to the endomorphism ring of $M$ that maps $r \in R$ to multiplication by $r$ on $M$. First and foremost, note that $\mu_M$ is a homomorphism of $R$-algebras. In general, it is neither injective nor surjective.

Let $\mathfrak{a} \subset R$ denote an ideal of $R$ and let $H^i_{\mathfrak{a}}(M)$ denote the $i$th local cohomology module of $M$ with respect to $\mathfrak{a}$, where $i$ is an integer. We refer to [1] for the definitions and basic results about local cohomology. For the local cohomology module $H^{d-1}_{\mathfrak{a}}(R)$ with $d = \dim R$, we try to examine the injectivity of $\mu_{H^{d-1}_{\mathfrak{a}}(R)}$. To be more precise, one has the injection

$$\frac{R}{\mathrm{Ann}_R(H^{d-1}_{\mathfrak{a}}(R))} \hookrightarrow \mathrm{Hom}_R(H^{d-1}_{\mathfrak{a}}(R), H^{d-1}_{\mathfrak{a}}(R)).$$

It is interesting to see when does the injectivity of $\mu_{H^{d-1}_{\mathfrak{a}}(R)}$ occur? One way to consider this question is to examine $\mathrm{Ann}_R(H^{d-1}_{\mathfrak{a}}(R))$.

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From the other point of view and in the light of [12], Theorem 2.9, vanishing of the local cohomology modules $H^i_{\mathfrak{a}}(R)$ for $i = d$, $d-1$ paves the ground for connectedness results. The vanishing of $H^d_{\mathfrak{a}}(R)$ is well understood by the Hartshorne-Lichtenbaum vanishing theorem. However, the vanishing of $H^{d-1}_{\mathfrak{a}}(R)$ is still mysterious. In this direction, we consider the question whether non-vanishing of $H^{d-1}_{\mathfrak{a}}(R)$ is equivalent to the vanishing of its annihilator. This kind of consideration is the aim of the present paper.

In the case of $(R, \mathfrak{m})$ being a regular local ring containing a field and $H^i_{\mathfrak{a}}(R) \neq 0$ for a given integer $i$, then in characteristic zero Lyubeznik, see [14], and in characteristic $p > 0$ Huneke and Koh in [11] showed that $\operatorname{Ann}_R(H^i_{\mathfrak{a}}(R)) = 0$.

There were many attempts to compute $\operatorname{Ann}_R(H^i_{\mathfrak{a}}(R))$ with some affirmative answers collected below:

(1-1) If $\mathfrak{a}$ is an ideal of a local complete ring $R$ with $H^i_{\mathfrak{a}}(R) = 0$ for every $i \neq \operatorname{ht}(\mathfrak{a})$, height of $\mathfrak{a}$, then $\operatorname{Ann}_R(H^{\operatorname{ht}(\mathfrak{a})}_{\mathfrak{a}}(R)) = 0$ (see [9]).

(1-2) If $R$ is a complete Gorenstein local domain, then under some mild assumptions $\operatorname{Ann}_R(H^i_I(R)) = 0$, where $i = \operatorname{grade}(\mathfrak{a}, R)$ (see [15]).

(1-3) If $\mathfrak{a}$ is an arbitrary ideal in a complete local ring, then

$$\operatorname{Ann}_R(H^{\dim R}_{\mathfrak{a}}(R)) = \bigcap \mathfrak{q},$$

where the intersection is taken over all primary components of (0) such that $\dim(R/\mathfrak{q}) = \dim R$ and $\operatorname{rad}(\mathfrak{a} + \mathfrak{q}) = \mathfrak{m}$ (see [5] or [13]).

In continuation of the above attempts, we prove Proposition 2.1.

Recall that the cohomological dimension of an ideal $\mathfrak{a}$, denoted by $\operatorname{cd}(\mathfrak{a}, R)$, is defined as

$$\operatorname{cd}(\mathfrak{a}, R) = \sup\{i \in \mathbb{Z} \colon \ H^i_{\mathfrak{a}}(R) \neq 0\}.$$

In the light of (1-1), the equality $\operatorname{ht}(\mathfrak{a}) = \operatorname{cd}(\mathfrak{a}, R)$ is vital for the vanishing of the annihilator of the local cohomology modules $H^{\operatorname{ht}(\mathfrak{a})}_{\mathfrak{a}}(R)$, whenever $R$ is a complete local ring. In Section 3, we consider conditions for the mentioned equality. Among other results in Section 3, see Theorem 3.1 and Theorem 3.2.

## 2. Results

Throughout, $R$ is a $d$-dimensional ring and $\mathfrak{a}$ is an ideal of $R$. Assume that $H^i_{\mathfrak{a}}(R) \neq 0$ for some $i \in \mathbb{Z}$. We examine the annihilator of these local cohomology modules when its Artinian structure fails.

**Proposition 2.1.** *Let $(R, \mathfrak{m})$ be a local domain. Suppose $H^d_{\mathfrak{a}}(R) = 0$ and $H^{d-1}_{\mathfrak{a}}(R)$ is not Artinian. Then $\operatorname{Ann}_R(H^{d-1}_{\mathfrak{a}}(R)) = 0$.*

P r o o f. On the contrary, assume that $\operatorname{Ann}_R(H^{d-1}_{\mathfrak{a}}(R)) \neq 0$, i.e. there is a nonzero element $0 \neq r \in \operatorname{Ann}_R(H^{d-1}_{\mathfrak{a}}(R))$. As $r$ is a nonzero divisor of $R$ so it implies the short exact sequence

$$0 \longrightarrow R \xrightarrow{r} R \longrightarrow \frac{R}{Rr} \longrightarrow 0. \tag{2.1}$$

Applying $H^i_{\mathfrak{a}}(-)$ to the short exact sequence (2.1) and thinking of the fact that, by [1], Lemma 8.1.7, $H^{d-1}_{\mathfrak{a}}(-)$ is a right exact functor, we obtain the following isomorphism of $R$-modules

$$H^{d-1}_{\mathfrak{a}}(R) \cong H^{d-1}_{\mathfrak{a}}\Big(\frac{R}{Rr}\Big),$$

where, by [1], Theorem 7.1.6, the latter module is Artinian as $\dim R/Rr = d-1$. This contradicts our assumption that $H^{d-1}_{\mathfrak{a}}(R)$ is not Artinian. □

Recall that the arithmetic rank of the ideal $\mathfrak{a}$, denoted by $\operatorname{ara}(\mathfrak{a})$, is the least number of elements of $R$ required to generate an ideal whose radical is $\operatorname{rad}(\mathfrak{a})$. Thus $\operatorname{ara}(\mathfrak{a})$ equals to the integer

$$\operatorname{ara}(\mathfrak{a}) = \inf\{i \in \mathbb{N}_0 \colon \exists\, a_1, \ldots, a_i \in R \ \text{ such that } \operatorname{rad}(a_1, \ldots, a_i) = \operatorname{rad}(\mathfrak{a})\}.$$

**Example 2.1.** Let $k$ be a field. Put $R = A/J$, where $A = k[x,y][|u,v|]$ is the formal power series ring in two variables over a polynomial ring in two variables and $J = (xu+yv)$. Then, by [2], Proposition 2.2.4, $R$ is a non-regular 3-dimensional local complete domain. Put $\mathfrak{a} = (u,v)$. Since $\operatorname{ara}(\mathfrak{a}) = 2$ hence, by [1], Corollary 3.3.3, we have $H^i_{\mathfrak{a}}(R) = 0$ for all $i \geqslant 3$. Also, by [6], Section 3, we see that $H^2_{\mathfrak{a}}(R)$ has a submodule isomorphic to the direct sum of infinitely many copies of $k$. Hence, $H^2_{\mathfrak{a}}(R)$ is not Artinian and so, by Proposition 2.1, we deduce that $\operatorname{Ann}_R(H^2_{\mathfrak{a}}(R)) = 0$.

By Proposition 2.1, non-Artinianness of $H^{d-1}_{\mathfrak{a}}(R)$ leads to the vanishing of $\operatorname{Ann}_R(H^{d-1}_{\mathfrak{a}}(R))$. In the sequel, we give conditions under which $H^{d-1}_{\mathfrak{a}}(R)$ is not an Artinian $R$-module. Recall that by $\operatorname{grade}(\mathfrak{a}, R)$ we mean the maximal length of an $R$-regular sequence in $\mathfrak{a}$.

**Proposition 2.2.** *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring and $\mathfrak{a}$ a one-dimensional ideal. Then $H^{d-1}_{\mathfrak{a}}(R)$ is not Artinian. In particular, if the ideal $\mathfrak{a}$ is generated by an $R$-regular sequence of length $d-1$, then $\operatorname{Ann}_R(H^{d-1}_{\mathfrak{a}}(R)) = 0$.*

P r o o f. Since $\mathfrak{a}$ is a one-dimensional ideal, there exists an element $y \in \mathfrak{m} \setminus \mathfrak{a}$ such that $\operatorname{rad}(\mathfrak{a} + Ry) = \mathfrak{m}$. Now, by [1], Proposition 8.1.2, we have the following long exact sequence:

$$\ldots \longrightarrow H^{d-1}_{\mathfrak{a}+Ry}(R) \longrightarrow H^{d-1}_{\mathfrak{a}}(R) \longrightarrow (H^{d-1}_{\mathfrak{a}}(R))_y \longrightarrow H^{d}_{\mathfrak{a}+Ry}(R) \longrightarrow \ldots.$$

As $R$ is Cohen-Macaulay, by [1], Theorem 6.2.7, and Grothendieck's vanishing theorem (see [1], Theorem 6.1.2,) we have the following short exact sequence:

$$0 \longrightarrow H_{\mathfrak{a}}^{d-1}(R) \longrightarrow (H_{\mathfrak{a}}^{d-1}(R))_y \longrightarrow H_{\mathfrak{m}}^{d}(R) \longrightarrow 0. \tag{2.2}$$

By [1], Corollary 2.2.21, we have $H_{Ry}^{0}(H_{\mathfrak{a}}^{d-1}(R)) = 0$ and $H_{Ry}^{1}(H_{\mathfrak{a}}^{d-1}(R)) = H_{\mathfrak{m}}^{d}(R)$. Thinking of the fact that over a local ring $(R, \mathfrak{m})$ an Artinian $R$-module $M$ is $\mathfrak{m}$-torsion, i.e. $H_{\mathfrak{m}}^{0}(M) = M$, in case that $H_{\mathfrak{a}}^{d-1}(R)$ is Artinian, from the short exact sequence (2.2) we deduce that $H_{\mathfrak{m}}^{d}(R) = 0$, which contradicts Grothendieck's non-vanishing theorem (see [1], Theorem 6.1.4).

In case that the ideal $\mathfrak{a}$ is generated by an $R$-regular sequence of length $d-1$, then $\dim(R/\mathfrak{a}) = 1$ and so by what we have proved earlier $H_{\mathfrak{a}}^{d-1}(R)$ is not Artinian. Now, by [1], Theorem 3.3.1, we have $H_{\mathfrak{a}}^{i}(R) = 0$ for all $i > d-1$ and so the result follows by Proposition 2.1, as desired. □

**Proposition 2.3.** *Let $\mathfrak{a}$ be a one-dimensional ideal of a $d$-dimensional complete local domain $(R, \mathfrak{m})$. If $\operatorname{rad}(\mathfrak{a})$ is not a prime ideal, then there is an epimorphism $H_{\mathfrak{a}}^{d-1}(R) \longrightarrow H_{\mathfrak{m}}^{d}(R) \longrightarrow 0$. In particular, $\operatorname{Ann}_R(H_{\mathfrak{a}}^{d-1}(R)) = 0$.*

Proof. Put $\operatorname{rad}(\mathfrak{a}) = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_n$, where the $\mathfrak{p}_i$ are distinct minimal prime ideals of $\mathfrak{a}$ for $i = 1, \ldots, n$. As $\operatorname{rad}(\mathfrak{a})$ is not prime, so none of the $\mathfrak{p}_i$ is an $\mathfrak{m}$-primary ideal. Since $\mathfrak{a}$ is a one-dimensional ideal of $R$, there exists an integer $t \in \{1, \ldots, n\}$ such that $\operatorname{rad}\Big(\mathfrak{p}_t + \bigcap\limits_{j=1, j\neq t}^{n} \mathfrak{p}_j\Big) = \mathfrak{m}$.

Set $\mathfrak{a}_1 := \mathfrak{p}_t$ and $\mathfrak{a}_2 := \bigcap\limits_{j=1, j\neq t}^{n} \mathfrak{p}_j$. By Mayer-Vietoris sequence we have the long exact sequence

$$\ldots \longrightarrow H_{\mathfrak{a}_1 \cap \mathfrak{a}_2}^{d-1}(R) \longrightarrow H_{\mathfrak{a}_1 + \mathfrak{a}_2}^{d}(R) \longrightarrow H_{\mathfrak{a}_1}^{d}(R) \oplus H_{\mathfrak{a}_2}^{d}(R) \longrightarrow \ldots. \tag{2.3}$$

As $R$ is a complete local domain, hence by Hartshorne-Lichtenbaum vanishing theorem (see [1], Theorem 8.2.1) we conclude that $H_{\mathfrak{a}_1}^{d}(R) = 0 = H_{\mathfrak{a}_2}^{d}(R)$, which in turn, due to the long exact sequence (2.3), implies the epimorphism

$$H_{\mathfrak{a}}^{d-1}(R) \longrightarrow H_{\mathfrak{m}}^{d}(R) \longrightarrow 0. \tag{2.4}$$

Now, by [5], Theorem 4.2 (i), we have $\operatorname{Ann}_R(H_{\mathfrak{m}}^{d}(R)) = 0$, wherefrom by using the epimorphism (2.4), it is concluded that $\operatorname{Ann}_R(H_{\mathfrak{a}}^{d-1}(R)) = 0$, and we are done. □

Let $(R, \mathfrak{m})$ be a local ring. A Noetherian $R$-module $M$ is called a Buchsbaum module if every system of parameters of $M$ is a weak $M$-sequence (cf. [17]). Note that every Cohen-Macaulay module is Buchsbaum (see [2], Theorem 2.1.2 (d)). By a result of Stückrad-Vogel (cf. [17], Corollary 2.4) if $M$ is a Buchsbaum module then $\mathfrak{m}H^i_{\mathfrak{m}}(M) = 0$ for all $i < \dim M$. This implies that $\mathfrak{m}^u H^i_{\mathfrak{m}}(M) = 0$ for all $i < \dim M$ and for some integer $u$, which is equivalent to saying that $\mathfrak{m} \subseteq \operatorname{rad}(\operatorname{Ann}_R(H^i_{\mathfrak{m}}(M)))$ for all $i < \dim M$. Note that $D(M)$, for an $R$-module $M$, stands for its Matlis dual, i.e. $D(M) = \operatorname{Hom}_R(M, E(R/\mathfrak{m}))$.

**Proposition 3.1.** *Let $(R, \mathfrak{m})$ be a Gorenstein local ring and $M$ a finitely generated faithful $R$-module. Suppose that $i$ and $t$ are positive integers. Then the following statements are equivalent:*

(1) $\mathfrak{m}^u H^i_{\mathfrak{m}}(M) = 0$ *for all $i < t$ and for some integer $u$.*
(2) $\operatorname{Supp}_R(H^{i-\dim(R/\mathfrak{p})}_{\mathfrak{p}}(M)) \subset \{\mathfrak{m}\}$ *for all $\mathfrak{p} \in \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$ and all $i < t$.*

Proof. Assume that there exists a natural integer $u$ such that for all integers $i < t$ we have $\mathfrak{m}^u H^i_{\mathfrak{m}}(M) = 0$, i.e. $\mathfrak{m}^u \subseteq \operatorname{Ann}_R(H^i_{\mathfrak{m}}(M))$ for all $i < t$. Now, by Grothendieck's local duality theorem (see [1], Theorem 11.2.5 and Remark 10.2.2 (ii)), we have $\mathfrak{m}^u \subseteq \operatorname{Ann}_R(H^i_{\mathfrak{m}}(M))$ if and only if $\mathfrak{m}^u \subseteq \operatorname{Ann}_R(\operatorname{Ext}^{\dim R - i}_R(M, R))$ if and only if $(\operatorname{Ext}^{\dim R-i}_R(M, R))_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \neq \mathfrak{m}$ if and only if $H^{i-\dim(R/\mathfrak{p})}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = 0$ for all $\mathfrak{p} \neq \mathfrak{m}$ and all $i < t$. □

**Proposition 3.2.** *Let $(R, \mathfrak{m})$ be a regular local ring containing a field and let $\mathfrak{a}$ be an ideal of $R$. Suppose that $i$ and $t$ are positive integers. Then the following statements are equivalent:*

(1) $\mathfrak{m}^u H^i_{\mathfrak{a}}(R) = 0$ *for all $i < t$ and for some integer $u$.*
(2) $H^i_{\mathfrak{a}}(R) = 0$ *for all $i < t$.*

Proof. Suppose the contrary and assume that there exists an integer $i < t$ such that $H^i_{\mathfrak{a}}(R) \neq 0$. Fix this $i$. By [7], page 407, $H^i_{\mathfrak{a}}(R)$ is not finitely generated, hence [1], Proposition 9.1.2, implies that $\mathfrak{a}^u H^i_{\mathfrak{a}}(R) \neq 0$ for all $u \in \mathbb{N}$. It follows that $\mathfrak{m}^u H^i_{\mathfrak{a}}(R) \neq 0$ for all $u \in \mathbb{N}$, which contradicts our assumption. □

**Proposition 3.3.** *Let $(R, \mathfrak{m})$ be a Gorenstein local ring and $\mathfrak{a}$ an ideal of $R$. Suppose that $t$ is an integer. Then the following statements are equivalent:*

(1) $\mathfrak{a}^u H^i_{\mathfrak{a}}(R) = 0$ *for all $i > t$ and for some integer $u$.*
(2) $H^i_{\mathfrak{a}}(R) = 0$ *for all $i > t$.*

P r o o f. Suppose that there exist natural integers $u$ and $t$ such that for all integers $i > t$ we have $\mathfrak{a}^u H^i_{\mathfrak{a}}(R) = 0$. As mentioned above, by [1], Remark 10.2.2 (ii), we have $\operatorname{Ann}_R(H^i_{\mathfrak{a}}(R)) = \operatorname{Ann}_R(D(H^i_{\mathfrak{a}}(R)))$, which, by [16], Remark 3.6, is equivalent to saying that $\mathfrak{a}^u \subseteq \operatorname{Ann}_R(D(H^i_{\mathfrak{a}}(R))) = \operatorname{Ann}_R(\varprojlim_n H^i_{\mathfrak{m}}(R/\mathfrak{a}^n))$ for all $i > t$ and for some integer $u$. It follows from [4], Theorem 1.1, that the latter is equivalent to the vanishing of $H^i_{\mathfrak{a}}(R)$ for all $i > t$. □

**3.1. One non-vanishing spot.** Suppose that $R$ is a ring and let $\mathfrak{a}$ be an ideal of $R$. It is well known that $\operatorname{ht}(\mathfrak{a}) \leqslant \operatorname{cd}(\mathfrak{a}, R)$. In case that the equality holds, $\mathfrak{a}$ is said to be a *cohomologically complete intersection ideal.* Hellus and Stückrad in [9], Corollary 2.4, have shown that $\operatorname{Ann}_R(H^c_{\mathfrak{a}}(R)) = 0$, whenever $c = \operatorname{ht}(\mathfrak{a}) = \operatorname{cd}(\mathfrak{a}, R)$ and $(R, \mathfrak{m})$ is a complete local ring. In what follows we are going to concentrate on this kind of ideals and give some characterizations of them.

**Proposition 3.4.** *Let $(R, \mathfrak{m})$ be a Gorenstein local ring and $\mathfrak{a}$ an ideal of $R$. Suppose that $R/\mathfrak{a}^n$ is a Cohen-Macaulay ring for all $n \geqslant 1$. Then $\mathfrak{a}$ is a cohomologically complete intersection ideal.*

P r o o f. Since $R/\mathfrak{a}^n$ is a Cohen-Macaulay ring for all $n \geqslant 1$, by [1], Theorem 6.2.7, $H^i_{\mathfrak{m}}(R/\mathfrak{a}^n) = 0$ for all $i < \dim(R/\mathfrak{a})$. So, it follows that $\varprojlim_l H^i_{\mathfrak{m}}(R/\mathfrak{a}^l) = 0$ for all integers $i < \dim(R/\mathfrak{a})$. Hence, by [16], Remark 3.6, we have $H^j_{\mathfrak{a}}(R) = 0$ for all $j > \operatorname{ht}(\mathfrak{a})$, i.e. $\mathfrak{a}$ is a cohomologically complete intersection. □

**Theorem 3.1.** *Let $\mathfrak{a}$ be an ideal of a local ring $R$, $M$ a finite $R$-module, and suppose that there exists an integer $t$ such that $\mathfrak{a} \subseteq \operatorname{rad}(\operatorname{Ann}_R(H^i_{\mathfrak{a}}(M)))$ for all $i > t$. Then $H^i_{\mathfrak{a}}(M) = 0$ for all $i > t$.*

P r o o f. The theorem will be proved by induction on $n := \dim M$. In case that $d = 0$ it is easily seen that $M$ is an Artinian $R$-module and so $\mathfrak{a}$-torsion. Now, by [1], Corollary 2.1.7 (i), we deduce that $H^i_{\mathfrak{a}}(M) = 0$ for all $i > 0$. Now assume that $n > 0$, $\mathfrak{a} \subseteq \operatorname{rad}(\operatorname{Ann}_R(H^i_{\mathfrak{a}}(M)))$ for all $i > t$ and the claim is true for all $i = t+2, t+3, \ldots$. We want to show that $H^{t+1}_{\mathfrak{a}}(M) = 0$.

By [1], Corollary 2.1.3 (ii) and Corollary 2.1.7 (iii), we may assume that $\mathfrak{a}$ contains an $M$-regular element $r$ and so $\dim(M/rM) = \dim M - 1$.

On the other hand, as $\mathfrak{a} \subseteq \operatorname{rad}(\operatorname{Ann}_R(H^{t+1}_{\mathfrak{a}}(M)))$, there exists an integer $u$ such that $r^u H^{t+1}_{\mathfrak{a}}(M) = 0$. Now, from the short exact sequence

$$0 \longrightarrow M \xrightarrow{r^u} M \longrightarrow \frac{M}{r^u M} \longrightarrow 0,$$

we get the following long exact sequence:

$$(3.1)\qquad \ldots \longrightarrow H^{t+1}_{\mathfrak{a}}(M) \xrightarrow{r^u} H^{t+1}_{\mathfrak{a}}(M) \longrightarrow H^{t+1}_{\mathfrak{a}}\Big(\frac{M}{r^u M}\Big) \longrightarrow H^{t+2}_{\mathfrak{a}}(M) \longrightarrow \ldots.$$

By [1], Lemma 9.1.1, $\mathfrak{a} \subseteq \operatorname{rad}(\operatorname{Ann}_R(H^{t+1}_{\mathfrak{a}}(M/r^u M)))$, so our induction hypothesis implies that $H^{t+1}_{\mathfrak{a}}(M/r^u M) = 0$. Hence, the long exact sequence (3.1) implies that $H^{t+1}_{\mathfrak{a}}(M) = 0$, as desired. □

Recall that an $R$-module $M$ is said to be $\mathfrak{a}$-cofinite if $\operatorname{Supp}(M) \subset V(\mathfrak{a})$ and $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ is finitely generated for every $i \in \mathbb{N}_0$.

**Corollary 3.1.** *Let $(R, \mathfrak{m})$ be a local ring and $\mathfrak{a}$ an ideal. Then:*

(1) *If $\mathfrak{a} \subseteq \operatorname{rad}(\operatorname{Ann}_R(H^i_{\mathfrak{a}}(M)))$ for all $i > \operatorname{ht}(\mathfrak{a})$, then $\mathfrak{a}$ is a cohomologically complete intersection ideal.*

(2) *If $R$ is a Cohen-Macaulay ring and $\mathfrak{a} \subseteq \operatorname{rad}(\operatorname{Ann}_R(H^i_{\mathfrak{a}}(R)))$ for all $i > \operatorname{ht}(\mathfrak{a})$ then $\dim_R(H^i_{\mathfrak{a}}(R)) \leqslant \operatorname{injdim}_R(H^i_{\mathfrak{a}}(R))$ and $H^i_{\mathfrak{a}}(R)$ is $\mathfrak{a}$-cofinite for every $i \in \mathbb{N}$.*

P r o o f. Part (1) is an immediate consequence of Theorem 3.1 and the fact that $\operatorname{ht}(\mathfrak{a}) \leqslant \operatorname{cd}(\mathfrak{a}, R)$. Part (2) is clear by Theorem 3.1 and [8], Corollary 2.4. □

Now, we are going to examine perfect ideals. Recall that an ideal $\mathfrak{a}$ is said to be perfect if $\operatorname{grade}(\mathfrak{a}, R)$ equals the projective dimension of $R/\mathfrak{a}$.

**Remark 3.1.** Note that, by [2], Theorem 2.1.5, over a Cohen-Macaulay ring $S$ (not necessarily local) perfectness of an ideal $\mathfrak{b}$ of finite projective dimension implies that $S/\mathfrak{b}$ is Cohen-Macaulay. Therefore, by [2], Theorem 2.2.7, we conclude that if $R$ is a regular local ring, then $\mathfrak{a}$ is a perfect ideal if and only if $R/\mathfrak{a}$ is Cohen-Macaulay. In particular, if $(R, \mathfrak{m})$ is a regular local ring and $\mathfrak{a}$ is generated by an $R$-regular sequence (e.g., $\mathfrak{a}$ could be generated by part of a regular system of parameters of $R$) then by [2], Exercise 1.4.27, for any natural integer $n$, $\mathfrak{a}^n$ is perfect. Therefore, by Proposition 3.4, we see that $\mathfrak{a}^n$ is a cohomologically complete intersection ideal.

**Theorem 3.2.** *Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring in $n$ variables $x_1, \ldots, x_n$ over a field $k$ and let $\mathfrak{a}$ be a square free monomial ideal which is a perfect ideal. Then $\mathfrak{a}$ is a cohomologically complete intersection ideal.*

P r o o f. Let $\varphi\colon R \to R$ be the $k$-linear endomorphism with $\varphi(x_i) = x_i^2$ for $1 \leqslant i \leqslant n$. Then it is clear that $\varphi^j(\mathfrak{a}) \subseteq \mathfrak{a}^2$ for $j > 0$, and $\varphi^0 = \mathrm{id}_R$. It follows from the definition of $\varphi$ that $\varphi^r(\mathfrak{a}) \subseteq \varphi^s(\mathfrak{a})$ for all $r \geqslant s$. We claim that $\{\varphi^n(\mathfrak{a})R\}$ and $\{\mathfrak{a}^n\}$ are cofinal.

P r o o f of the claim. (1) Let $n$ be an arbitrary positive integer. In case $n = 2k$, $k \in \mathbb{Z}$, we have

$$\mathfrak{a}^n = (\mathfrak{a}^2)^k \supseteq (\varphi^l(\mathfrak{a})R)^k = \varphi^{lk}(\mathfrak{a})R$$

for some $l \in \mathbb{Z}$. Similarly, the case $n = 2k + 1$ is verified.

(2) Let $m$ be an arbitrary positive integer. As $\mathfrak{a}$ is a square-free monomial ideal, there exists an integer $s$ such that $\mathfrak{a}^s \subseteq \varphi(\mathfrak{a})R$. So, by iteration we get the claim. □

Now, by [18], Lemma 2.1, we conclude that $\operatorname{ht}(\mathfrak{a}) = \operatorname{cd}(\mathfrak{a}, R)$, i.e. $\mathfrak{a}$ is a cohomologically complete intersection ideal. □

**Example 3.1.** Let $k$ be a field of characteristic 0, $(x_{ij})$ an $m \times n$ matrix of indeterminates over $k$, and $R = k[x_{ij}]$. If $\mathfrak{a}$ is the ideal generated by the $t$-minors of the matrix $(x_{ij})$ then by [10], Corollary 4, for $t \leqslant \min\{m, n\}$ we deduce that $\mathfrak{a}$ is a perfect ideal and $\operatorname{ht}(\mathfrak{a}) = (m - t + 1)(n - t + 1)$. However, by [3], we have $\operatorname{cd}(\mathfrak{a}, R) = mn - t^2 + 1$. Then, in case that $m = n = t$ or $t = 1$, the equality $\operatorname{ht}(\mathfrak{a}) = \operatorname{cd}(\mathfrak{a}, R)$ holds.

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