FURTHER DETERMINANT IDENTITIES RELATED TO CLASSICAL ROOT SYSTEMS

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Received May 27, 2016. First published August 15, 2017.

Abstract. By introducing polynomials in matrix entries, six determinants are evaluated which may be considered extensions of Vandermonde-like determinants related to the classical root systems.

Keywords: Vandermonde determinant; symmetric function; classical root system

MSC 2010: 05E05, 15A15

1. Introduction and preliminaries

The Vandermonde determinant

(1)
$$\det_{0\leqslant i,j\leqslant n}[x_i^j] = \prod_{0\leqslant i<j\leqslant n}(x_j - x_i)$$

is well-known for its wide applications in mathematics, in particular, to symmetric functions [1], [2], [9] and constant term identities [5], [7], [8]. It can be derived from the denominator formula associated with the root system of the classical Lie algebra A_n. For the other three classical root systems B_n, C_n and D_n, the corresponding determinant identities (see [4] and [6], Exercises A52, A62 and A66 for example) may be reproduced as

(2)
$$\det_{1\leqslant i,j\leqslant n}[x_i^{j-1} + x_i^{1-j}] = 2V(X),$$

(3)
$$\det_{1\leqslant i,j\leqslant n}[x_i^j - x_i^{-j}] = V(X)\prod_{k=1}^{n}\frac{x_k^2-1}{x_k},$$

(4)
$$\det_{1\leqslant i,j\leqslant n}[x_i^{j-1/2} + x_i^{1/2-j}] = V(X)\prod_{k=1}^{n}\frac{x_k+1}{x_k^{1/2}},$$

where for the sake of brevity, we have adopted the notation

$$V(X) := \prod_{1\leqslant i<j\leqslant n} \frac{(x_i-x_j)(1-x_ix_j)}{x_ix_j} = \prod_{1\leqslant i<j\leqslant n} (x_i-x_j)(1-x_ix_j)\Big/\prod_{k=1}^{n} x_k^{n-1}.$$

By introducing polynomials in the matrix entries, the author in [3] has recently extended the three determinants in (2), (3) and (4). This paper investigates further these determinants with the entries involving polynomials. Two classes of determinant identities will be presented in the rest of the paper.

2. The first class of determinant identities

For the indeterminates $\{y_k\}_{k\geqslant 1}$, define a polynomial sequence by

$$P_m(x) := \prod_{k=1}^{m}(1-xy_k) = \sum_{k=0}^{m}(-x)^k\sigma_k(y|[1,m])$$

where $\sigma_k(y|[m,n])$ stands for the kth elementary symmetric function in $\{y_i\}_{i=m}^{n}$.

This section will present three determinant identities, which contain those displayed in (2), (3) and (4) as particular cases when all the polynomials $P_m(x)$ are identically equal to one (i.e., $y_k=0$ for $k=1,2,\ldots$).

Theorem 1 (Determinant identity)**.**

$$\det_{1\leqslant i,j\leqslant n}[x_i^{j-1}P_{2j-2}(x_i^{-1})+x_i^{1-j}P_{2j-2}(x_i)] = 2V(X)\prod_{k=1}^{n-1}\Big\{1+\prod_{i=1}^{2k}y_i\Big\}.$$

P r o o f. Due to the expression

$$x_i^{j-1}P_{2j-2}(x_i^{-1}) = \sum_{k=2-j}^{j}(-1)^{j-k}x_i^{k-1}\sigma_{j-k}(y|[1,2j-2])$$

we can reformulate the matrix entries as

$$\begin{aligned}
&x_i^{j-1}P_{2j-2}(x_i^{-1})+x_i^{1-j}P_{2j-2}(x_i)\\
&\quad= \sum_{k=2-j}^{j}(-1)^{j-k}\{x_i^{k-1}+x_i^{1-k}\}\sigma_{j-k}(y|[1,2j-2])\\
&\quad= \sum_{k=1}^{j}(-1)^{j-k}\frac{x_i^{k-1}+x_i^{1-k}}{1+\chi_{\{k=1\}}}\{\sigma_{j-k}(y|[1,2j-2])+\sigma_{j+k-2}(y|[1,2j-2])\}
\end{aligned}$$

where we have splitted the bilateral sum $\sum\limits_{k=2-j}^{j}$ into $\sum\limits_{k=1}^{j}$ and $\sum\limits_{k=2-j}^{0}$ and then inverted the summation index $k \to 2-k$ for the second sum.

Then the determinant in question becomes the product

$$\det_{1\leqslant i,k\leqslant n}\left[\frac{x_i^{k-1}+x_i^{1-k}}{1+\chi_{\{k=1\}}}\right] \times \det_{1\leqslant k,j\leqslant n}[(-1)^{j-k}\{\sigma_{j-k}(y|[1,2j-2])+\sigma_{j+k-2}(y|[1,2j-2])\}]$$

where the second matrix is upper triangular with the jth diagonal entry equal to $\left(1+\prod\limits_{i=1}^{2j-2} y_i\right)$. Evaluating the first determinant by (2), we prove the theorem. $\square$

Theorem 2 (Determinant identity)**.**

$$\det_{1\leqslant i,j\leqslant n}[x_i^j P_{2j}(x_i^{-1}) - x_i^{-j} P_{2j}(x_i)] = V(X)\prod_{k=1}^{n}\frac{x_k^2-1}{x_k}\left\{1-\prod_{i=1}^{2k} y_i\right\}.$$

Proof. Analogously, we have the expression

$$x_i^j P_{2j}(x_i^{-1}) = \sum_{k=-j}^{j}(-1)^{j-k}x_i^k\sigma_{j-k}(y|[1,2j])$$

from which we can reformulate the matrix entries as

$$\begin{aligned} x_i^j P_{2j}(x_i^{-1}) - x_i^{-j}P_{2j}(x_i) &= \sum_{k=-j}^{j}(-1)^{j-k}(x_i^k - x_i^{-k})\sigma_{j-k}(y|[1,2j]) \\ &= \sum_{k=1}^{j}(-1)^{j-k}(x_i^k - x_i^{-k})\{\sigma_{j-k}(y|[1,2j]) - \sigma_{j+k}(y|[1,2j])\} \end{aligned}$$

where we have splitted the bilateral sum $\sum\limits_{k=-j}^{j}$ into $\sum\limits_{k=1}^{j}$ and $\sum\limits_{k=-j}^{1}$ and then inverted the summation index $k \to -k$ for the second sum.

Then the determinant in question can be factorized into

$$\det_{1\leqslant i,k\leqslant n}[x_i^k - x_i^{-k}] \times \det_{1\leqslant k,j\leqslant n}[(-1)^{j-k}\{\sigma_{j-k}(y|[1,2j]) - \sigma_{j+k}(y|[1,2j])\}]$$

where the second matrix is upper triangular with the jth diagonal entry equal to $\left(1+\prod\limits_{i=1}^{2j} y_i\right)$. Evaluating the first determinant by (3), we prove the theorem. $\square$

Theorem 3 (Determinant identity)**.**

$$\det_{1\leqslant i,j\leqslant n}[x_i^{j-1/2}P_{2j-1}(x_i^{-1})+x_i^{1/2-j}P_{2j-1}(x_i)] = V(X)\prod_{k=1}^{n}\frac{x_k+1}{x_k^{1/2}}\Big\{1-\prod_{i=1}^{2k-1}y_i\Big\}.$$

P r o o f. Writing similarly the expression

$$x_i^{j-1/2}P_{2j-1}(x_i^{-1}) = \sum_{k=1-j}^{j}(-1)^{j-k}x_i^{k-1/2}\sigma_{j-k}(y|[1,2j-1])$$

we can restate the matrix entries as

$$\begin{aligned}x_i^{j-1/2}&P_{2j-1}(x_i^{-1})+x_i^{1/2-j}P_{2j-1}(x_i)\\ &=\sum_{k=1-j}^{j}(-1)^{j-k}(x_i^{k-1/2}+x_i^{1/2-k})\sigma_{j-k}(y|[1,2j-1])\\ &=\sum_{k=1}^{j}(-1)^{j-k}(x_i^{k-1/2}+x_i^{1/2-k})\{\sigma_{j-k}(y|[1,2j-1])-\sigma_{j+k-1}(y|[1,2j-1])\}\end{aligned}$$

where we have splitted the bilateral sum $\sum\limits_{k=1-j}^{j}$ into $\sum\limits_{k=1}^{j}$ and $\sum\limits_{k=1-j}^{0}$ and then inverted the summation index $k\to 1-k$ for the second sum.

Therefore the determinant in question admits the decomposition

$$\det_{1\leqslant i,k\leqslant n}[x_i^{k-1/2}-x_i^{1/2-k}]\times\det_{1\leqslant k,j\leqslant n}[(-1)^{j-k}\{\sigma_{j-k}(y|[1,2j-1])-\sigma_{j+k-1}(y|[1,2j-1])\}]$$

where the second matrix is upper triangular with the jth diagonal entry equal to $\Big(1-\prod\limits_{i=1}^{2j-1}y_i\Big)$. Evaluating the first determinant by (4), we prove the theorem. $\square$

3. The second class of determinant identities

In a recent paper, the author in [3] extended (2), (3) and (4), respectively, to the following three determinant identities.

Lemma 4 (Chu [3], Theorem 25: $m<n$)**.**

$$\det_{1\leqslant i,j\leqslant n}[x_i^{j-1}P_m(x_i^{-1})+x_i^{1-j}P_m(x_i)] = 2V(X)\prod_{1\leqslant \imath\leqslant \jmath\leqslant m}(1-y_\imath y_\jmath).$$

Lemma 5 (Chu [3], Theorem 19: $m \leqslant n+1$).

$$\det_{1\leqslant i,j\leqslant n}[x_i^j P_m(x_i^{-1}) - x_i^{-j} P_m(x_i)] = V(X)\prod_{k=1}^{n}\frac{x_k^2-1}{x_k}\prod_{1\leqslant \imath<\jmath\leqslant m}(1-y_\imath y_\jmath).$$

Lemma 6 (Chu [3], Theorem 22: $m \leqslant n$).

$$\det_{1\leqslant i,j\leqslant n}[x_i^{j-1/2}P_m(x_i^{-1}) + x_i^{1/2-j}P_m(x_i)]$$
$$= V(X)\prod_{k=1}^{n}\frac{x_k+1}{x_k^{1/2}}\prod_{1\leqslant \imath<\jmath\leqslant m}(1-y_\imath y_\jmath)\prod_{\kappa=1}^{m}(1-y_\kappa).$$

It should be pointed out that Theorem 16 in [3] is, in fact, the limiting case $y_m \to \infty$ of Lemma 5 (Theorem 19 in the same paper). The last three determinants will be generalized further in this section by increasing polynomial degrees.

Theorem 7 (Determinant identity: $m < n$).

$$\det_{1\leqslant i,j\leqslant n}[x_i^{j-1}P_{m+j-1}(x_i^{-1}) + x_i^{1-j}P_{m+j-1}(x_i)] = 2V(X)\prod_{1\leqslant \imath\leqslant\jmath\leqslant m}(1-y_\imath y_\jmath).$$

P r o o f. Using the expression

$$x_i^{j-1}P_{m+j-1}(x_i^{-1}) = \sum_{k=1}^{j}(-1)^{j-k}x_i^{k-1}\sigma_{j-k}(y|[m+1,m+j-1])P_m(x_i^{-1})$$

we can reformulate the matrix entries as

$$x_i^{j-1}P_{m+j-1}(x_i^{-1}) + x_i^{1-j}P_{m+j-1}(x_i)$$
$$= \sum_{k=1}^{j}(-1)^{j-k}\sigma_{j-k}(y|[m+1,m+j-1])\{x_i^{k-1}P_m(x_i^{-1}) + x_i^{1-k}P_m(x_i)\}.$$

Then the determinant in question becomes the product

$$\det_{1\leqslant i,k\leqslant n}[x_i^{k-1}P_m(x_i^{-1}) + x_i^{1-k}P_m(x_i)] \times \det_{1\leqslant k,j\leqslant n}[(-1)^{j-k}\sigma_{j-k}(y|[m+1,m+j-1])]$$

where the second matrix is upper triangular with the diagonal entries equal to one. Evaluating the first determinant by Lemma 4, we prove the theorem. $\square$

Theorem 8 (Determinant identity: $m \leqslant n+1$).

$$\det_{1\leqslant i,j\leqslant n}[x_i^j P_{m+j-1}(x_i^{-1}) - x_i^{-j} P_{m+j-1}(x_i)] = V(X) \prod_{1\leqslant \imath<\jmath\leqslant m}(1-y_\imath y_\jmath) \prod_{k=1}^{n} \frac{x_k^2-1}{x_k}.$$

P r o o f. Analogously, we have the expression

$$\begin{aligned} &x_i^j P_{m+j-1}(x_i^{-1}) - x_i^{-j} P_{m+j-1}(x_i) \\ &\quad= \sum_{k=1}^{j} (-1)^{j-k} \sigma_{j-k}(y|[m+1, m+j-1])\{x_i^k P_m(x_i^{-1}) - x_i^{-k} P_m(x_i)\} \end{aligned}$$

which enables us to factorize the determinant in question into

$$\det_{1\leqslant i,k\leqslant n}[x_i^k P_m(x_i^{-1}) - x_i^{-k} P_m(x_i)] \times \det_{1\leqslant k,j\leqslant n}[(-1)^{j-k}\sigma_{j-k}(y|[m+1, m+j-1])].$$

Then the determinant identity displayed in the theorem follows from Lemma 5. □

Theorem 9 (Determinant identity: $m \leqslant n$).

$$\begin{aligned} &\det_{1\leqslant i,j\leqslant n}[x_i^{j-1/2} P_{m+j-1}(x_i^{-1}) + x_i^{1/2-j} P_{m+j-1}(x_i)] \\ &\qquad =: \; V(X) \prod_{1\leqslant \imath<\jmath\leqslant m}(1-y_\imath y_\jmath) \prod_{\kappa=1}^{m}(1-y_\kappa) \prod_{k=1}^{n} \frac{x_k+1}{x_k^{1/2}}. \end{aligned}$$

P r o o f. Writing similarly the matrix entries as

$$\begin{aligned} &x_i^{j-1/2} P_{m+j-1}(x_i^{-1}) + x_i^{1/2-j} P_{m+j-1}(x_i) \\ &\quad= \sum_{k=1}^{j} (-1)^{j-k} \sigma_{j-k}(y|[m+1, m+j-1])\{x_i^{k-1/2} P_m(x_i^{-1}) + x_i^{1/2-k} P_m(x_i)\} \end{aligned}$$

which leads the determinant in question to the decomposition

$$\det_{1\leqslant i,k\leqslant n}[x_i^{k-1/2} P_m(x_i^{-1}) + x_i^{1/2-k} P_m(x_i)] \times \det_{1\leqslant k,j\leqslant n}[(-1)^{j-k}\sigma_{j-k}(y|[m+1, m+j-1])].$$

Recalling Lemma 6, we get the determinant identity in the theorem. □

It is clear that when $y_k = 0$ for $k > m$, the determinant identities displayed in the last three theorems reduce respectively to those in Lemmas 4–6. Instead, specifying $y_1 = y_2 = \ldots = y_m = 0$ in the last theorems, we deduce the following three interesting determinant identities, which can also serve as extensions of (2), (3) and (4), resembling those displayed in Theorems 1–3.

Corollary 10 (Determinant identity).

$$\det_{1\leqslant i,j\leqslant n}[x_i^{j-1}P_{j-1}(x_i^{-1})+x_i^{1-j}P_{j-1}(x_i)]=2V(X).$$

Corollary 11 (Determinant identity).

$$\det_{1\leqslant i,j\leqslant n}[x_i^{j}P_{j-1}(x_i^{-1})-x_i^{-j}P_{j-1}(x_i)]=V(X)\prod_{k=1}^{n}\frac{x_k^2-1}{x_k}.$$

Corollary 12 (Determinant identity).

$$\det_{1\leqslant i,j\leqslant n}[x_i^{j-1/2}P_{j-1}(x_i^{-1})+x_i^{1/2-j}P_{j-1}(x_i)]=V(X)\prod_{k=1}^{n}\frac{x_k+1}{x_k^{1/2}}.$$

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