Li Zu; Daqing Jiang; Donal O'Regan

Population dynamical behavior of a single-species nonlinear diffusion system with random perturbation

Czechoslovak Mathematical Journal, Vol. 67 (2017), No. 4, 867–890

Persistent URL: `http://dml.cz/dmlcz/146954`

Czechoslovak Mathematical Journal, 67 (142) (2017), 867–890

POPULATION DYNAMICAL BEHAVIOR OF A SINGLE-SPECIES NONLINEAR DIFFUSION SYSTEM WITH RANDOM PERTURBATION

Li Zu, Haikou, Daqing Jiang, Qingdao, Jeddah,
Donal O'Regan, Galway

Received June 28, 2015. First published October 24, 2017.

Abstract. We consider a single-species stochastic logistic model with the population's nonlinear diffusion between two patches. We prove the system is stochastically permanent and persistent in mean, and then we obtain sufficient conditions for stationary distribution and extinction. Finally, we illustrate our conclusions through numerical simulation.

Keywords: stochastic permanence; persistent in mean; extinction; stationary distribution

MSC 2010: 34F05, 92D25

1. Introduction

Dispersal is a life history trait that has profound effects on both the species persistence and evolution. There are two typical equations to model the diffusion process. One is semilinear parabolic equations (i.e., reaction-diffusion systems) where the populations are continuously spread out in space, like oceanic plankton (see Okubo [20]). The other is discrete diffusion systems where several species are distributed over an interconnected network of multiple patches and there are population migrations among patches. The types of discrete diffusion mechanisms are referred to as linear diffusion, biased diffusion and directed diffusion. Allen in [1] studied the effects of the three different dispersal mechanisms on species survival, and investigated the

The work was supported by Program for NSFC of China (No. 11371085), the Fundamental Research Funds for the Central Universities (No. 15CX08011A), the Foundation of Hainan Provincial Science and Technology Department (No. 20161006), and Hainan Province Colleges and Universities Scientific Research Projects (No. Hnky2017ZD-14).

DOI: 10.21136/CMJ.2017.0350-15

logistic nonlinear directed diffusion model

$$
(1.1) \qquad \begin{cases} \dot{x}_1(t) = x_1(t)(a_1 - b_1 x_1(t)) + d_{12}(x_2^2(t) - \alpha_{12} x_1^2(t)), \\ \dot{x}_2(t) = x_2(t)(a_2 - b_2 x_2(t)) + d_{21}(x_1^2(t) - \alpha_{21} x_2^2(t)), \end{cases}
$$

where $x_i(t)$, $i = 1, 2$ denotes the density dependent growth rate in patch i at time t. The constant d_{ij}, $i, j = 1, 2$, $j \neq i$ is the dispersal rate from the jth patch to the ith patch, and the nonnegative constant α_{ij} can be selected to represent different boundary conditions in the continuous diffusion case, see [16]. Allen proved that initial value problems (1.1) have unique positive solutions. In [16], the authors extended Allen's results and obtained the following necessary and sufficient conditions:

(i) The system (1.1) possesses a globally stable positive equilibrium point (x_1^*, x_2^*), if the largest eigenvalue of matrix A is less than 0.

(ii) Every solution of the system is unbounded, if the above condition does not hold. Here $A = (a_{ij})_{2\times 2}$, and $a_{ij} = d_{ij}$ for $i \neq j$, $a_{11} = -b_1 - d_{12}\alpha_{12}$, $a_{22} = -b_2 - d_{21}\alpha_{21}$. That is to say, if $(b_1 + d_{12}\alpha_{12})(b_2 + d_{21}\alpha_{21}) > d_{12}d_{21}$, then system (1.1) has a globally stable positive equilibrium point.

However, system (1.1) is a deterministic model, which has some limitations in mathematical modeling of ecological systems and does not incorporate the effect of a fluctuating environment. In fact, a real system will not persist at such steady-state values, since population dynamics is inevitably affected by environmental white noise which is an important factor in an ecosystem. Therefore, the deterministic models are often subject to stochastic perturbations, and it is useful to reveal how the noise affects the population system. There are many papers which study differential equations with stochastic perturbations (see [8], [10], [11], [15], [13], [19]). Liu and Wang in [15] studied the stochastic non-autonomous logistic equation. Li et al. in [13] investigated the stochastic logistic populations under regime switching. Mao et al. in [19] revealed the effects of environmental noise on the delay Lotka-Volterra model. Jiang et al. in [10], [11] investigated the logistic equation with random perturbation and obtained many results, for example on global stability and stochastic permanence. Ji et al. in [8] studied the Lotka-Volterra mutualism system for two species and established that if the strength of the white noise is small, the system has a stationary distribution and is ergodic. More investigations and improvements of these stochastic models can be found in [8], [9], [18] and the references therein. There is very little known on the dynamic behavior in the single-species dispersal system with stochastic perturbation and the study of diffusion phenomena and the white noise impact on population is of significance.

Now we consider system (1.1) and we take into account the effect of randomly fluctuating, i.e., we stochastically perturb the intrinsic growth rate a_i. Suppose

$$a_1 \to a_1 + \sigma_1 \dot{B}_1(t), \quad a_2 \to a_2 + \sigma_2 \dot{B}_2(t),$$

where $B_i(t)$ are mutually independent Brownian motions, σ_i are positive constants and σ_i^2 represent the intensity of the white noise. Then the stochastic system takes the form

$$(1.2) \quad \begin{cases} \mathrm{d}x_1(t) = (x_1(t)(a_1 - b_1 x_1(t)) + d_{12}(x_2^2(t) - \alpha_{12} x_1^2(t)))\,\mathrm{d}t + \sigma_1 x_1(t)\,\mathrm{d}B_1(t), \\ \mathrm{d}x_2(t) = (x_2(t)(a_2 - b_2 x_2(t)) + d_{21}(x_1^2(t) - \alpha_{21} x_2^2(t)))\,\mathrm{d}t + \sigma_2 x_2(t)\,\mathrm{d}B_2(t). \end{cases}$$

For convenience, let $\bar{b}_1 = b_1 + d_{12}\alpha_{12}$, $\bar{b}_2 = b_2 + d_{21}\alpha_{21}$, and we have

$$(1.3) \quad \begin{cases} \mathrm{d}x_1(t) = (x_1(t)(a_1 - \bar{b}_1 x_1(t)) + d_{12} x_2^2(t))\,\mathrm{d}t + \sigma_1 x_1(t)\,\mathrm{d}B_1(t), \\ \mathrm{d}x_2(t) = (x_2(t)(a_2 - \bar{b}_2 x_2(t)) + d_{21} x_1^2(t))\,\mathrm{d}t + \sigma_2 x_2(t)\,\mathrm{d}B_2(t). \end{cases}$$

In this paper, we assume d_{12}, d_{21} and α_{ij} are nonnegative constants, the parameters a_i, b_i are positive constants and so $\bar{b}_1 > 0$, $\bar{b}_2 > 0$.

The rest of the paper is arranged as follows. In Section 2, we show that there exists a unique positive global solution with any positive initial condition. In the study of a population system, permanence is a very important and interesting topic regarding the survival of populations in an ecological system. In Section 3, we investigate sufficient conditions for stochastic permanence and persistence in mean. In a deterministic system, the global attractivity of the positive equilibrium is studied, but, as mentioned above, it is impossible to expect system (1.3) to tend to a steady state. So we attempt to investigate the stationary distribution of this system by the Lyapunov functional technique. This can be viewed as weak stability, which appears as the solution is fluctuating in a neighborhood of the point. In Section 4, we will show if the intensity of the white noise is small, there is a stationary distribution of (1.3) and it has an ergodic property. Existing results on dynamics in a patchy environment have largely been restricted to extinction analysis, which means that the population system will survive or die out in the future due to the increased complexity of global analysis. In Section 5, we give sufficient conditions for extinction. In Sections 6 and 7, we make numerical simulation to confirm our analytical results and give a conclusion. Finally, for the completeness of the article, we give an Appendix containing some results which will be used in other sections. Note the key method used in this paper is the analysis of Lyapunov functions [8], [9], [10], [11], [19].

Throughout this paper, unless otherwise specified, let $(\Omega, \{\mathscr{F}_t\}_{t\geqslant 0}, \mathbf{P})$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathscr{F}_0$ contains all P-null sets). Let $\mathbb{R}_+^2$ denote the positive cone of $\mathbb{R}^2$, namely $\mathbb{R}_+^2 = \{(x_1, x_2) \in \mathbb{R}^2\colon\ x_i > 0,\ i = 1, 2\}$. For convenience and simplicity in the forthcoming discussion, denote $x(t) = (x_1(t), x_2(t))$. If A is a vector or matrix, its transpose is denoted by A^{T}. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\mathrm{trace}(A^{\mathrm{T}}A)}$. We impose the following assumptions:

Assumption 1. $\bar{b}_1\bar{b}_2 > d_{12}d_{21}$.

Assumption 2. $a_i - \frac{1}{2}\sigma_i^2 > 0$, $i = 1, 2$.

2. Positive and global solutions

Since $x_1(t)$, $x_2(t)$ in stochastic differential equation (SDE) (1.3) are population densities at time t, we are only interested in positive solutions. Moreover, in order for a SDE to have a unique global (i.e. no explosion in a finite time) solution for any given initial value, the coefficients of the equation are generally required to satisfy a linear growth condition and a local Lipschitz condition (cf. Mao [17]). However, the coefficients of SDE (1.3) do not satisfy the linear growth condition, though they are locally Lipschitz continuous. In this section, we will use a method similar to Theorem 2.1 in Mao [19] to prove the solution of (1.3) is nonnegative and global.

Theorem 2.1. *Let Assumption 1 hold. For any given initial value $x(0) \in \mathbb{R}_+^2$, there is a unique positive solution $x(t)$ of system (1.3), and the solution will remain in $\mathbb{R}_+^2$ with probability 1.*

Proof. Define a C^2-function $V\colon\ R_+^2 \to R_+$ by

$$V(x(t)) = c_1(x_1(t) - 1 - \log x_1(t)) + c_2(x_2(t) - 1 - \log x_2(t)), \tag{2.1}$$

where c_1 and c_2 are positive constants to be determined. The nonnegativity of this function can be observed from $a - 1 - \log a \geqslant 0$ on $a > 0$ with equality holding if an only if $a = 1$. For $x \in \mathbb{R}_+^2$, applying Itô's formula, we have

$$\begin{aligned}
(2.2)\quad & \mathrm{d}V(x(t)) \\
&= c_1\Big(\mathrm{d}x_1(t) - \frac{\mathrm{d}x_1(t)}{x_1(t)} + \frac{(\mathrm{d}x_1(t))^2}{2x_1^2(t)}\Big) + c_2\Big(\mathrm{d}x_2(t) - \frac{\mathrm{d}x_2}{x_2(t)} + \frac{(\mathrm{d}x_2(t))^2}{2x_2^2(t)}\Big) \\
&\leqslant \Big((-c_1\bar{b}_1 + c_2 d_{21})x_1^2(t) + (-c_2\bar{b}_2 + c_1 d_{12})x_2^2(t) \\
&\qquad + c_1(a_1 + \bar{b}_1)x_1(t) + c_2(a_2 + \bar{b}_2)x_2(t) + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2}\Big)\,\mathrm{d}t \\
&\qquad + c_1\sigma_1(x_1(t) - 1)\,\mathrm{d}B_1(t) + c_2\sigma_2(x_2(t) - 1)\,\mathrm{d}B_2(t)) \\
&:= LV(x(t))\,\mathrm{d}t + c_1\sigma_1(x_1(t) - 1)\,\mathrm{d}B_1(t) + c_2\sigma_2(x_2(t) - 1)\,\mathrm{d}B_2(t),
\end{aligned}$$

where

$$LV(x(t)) = (-c_1\bar{b}_1 + c_2 d_{21})x_1^2(t) + (-c_2\bar{b}_2 + c_1 d_{12})x_2^2(t) + c_1(a_1 + \bar{b}_1)x_1(t) + c_2(a_2 + \bar{b}_2)x_2(t) + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2}.$$

In fact, in order to ensure LV is bounded, we only need

$$-c_1\bar{b}_1 + c_2 d_{21} < 0, \quad -c_2\bar{b}_2 + c_1 d_{12} < 0, \tag{2.3}$$

that is

$$\frac{d_{21}}{\bar{b}_1} < \frac{c_1}{c_2} < \frac{\bar{b}_2}{d_{12}}, \tag{2.4}$$

and by Assumption 1 we are able to find positive constants c_1, c_2 satisfying the inequality (2.4). The coefficient of the quadratic term of LV is negative, so we can find a positive constant number K satisfying

$$LV \leqslant K$$

and K is independent of $x_1(t)$, $x_2(t)$ and t. The rest of the proof is similar to Theorem 2.1 in [19] so we omit it. □

Remark 2.2. Theorem 2.1 shows that there exists a unique positive solution $x(t)$ of SDE (1.3) and a positive constant K independent of $x_1(t)$, $x_2(t)$ and t, such that

$$LV \leqslant K.$$

Now let $\overline{V} = V + K$, so

$$L\overline{V} \leqslant \overline{V},$$

and then

$$\overline{V}_R = \inf_{x\in\mathbb{R}_+^2\setminus D_m} \overline{V}(x) \to \infty \qquad \text{as } m \to \infty,$$

where $D_m = (1/m, m) \times (1/m, m)$. From $\overline{V} = V + K$ we have that

$$V_R = \inf_{x\in\mathbb{R}_+^2\setminus D_m} V(x) \to \infty \qquad \text{as } m \to \infty.$$

Hence from Remark 2 of [12], Theorem 4.1, page 86, one obtains that the solution $x(t)$ is a time-homogeneous Markov process in $\mathbb{R}_+^2$ (see Remark 8.1 for the definition of time-homogeneous Markov process).

3. Stochastic permanence and persistence in mean

In this section, we will investigate the persistence under two different meanings: stochastic permanence and persistence in mean.

3.1. Stochastic permanence. Theorem 2.1 shows that the solution of SDE (1.3) will remain in the positive cone $\mathbb{R}^2_+$ with probability 1. This nice property provides us with a great opportunity to discuss how the solution varies in $\mathbb{R}^2_+$ in detail. We will first give the definitions of the stochastically ultimate boundedness and the stochastic permanence.

Definition 3.1. The SDE (1.3) is said to be stochastically ultimately bounded, if for any $\varepsilon \in (0,1)$ there exist positive constants $\chi_1(=\chi_1(\varepsilon))$, $\chi_2(=\chi_2(\varepsilon))$ such that for any initial value $x(0) \in \mathbb{R}^2_+$, the solution of the SDE (1.3) has the property that

$$\limsup_{t\to\infty} P\{x_1(t) > \chi_1\} < \varepsilon, \quad \limsup_{t\to\infty} P\{x_2(t) > \chi_2\} < \varepsilon,$$

where $(x_1(t), x_2(t))$ is the solution of SDE (1.3) with any initial value $x(0) \in \mathbb{R}^2_+$.

Definition 3.2. The SDE (1.3) is said to be stochastically permanent, if for any $\varepsilon \in (0,1)$ there are positive constants $\chi_1(=\chi_1(\varepsilon))$, $\chi_2(=\chi_2(\varepsilon))$ and $\delta_1(=\delta_1(\varepsilon))$, $\delta_1'(=\delta_1'(\varepsilon))$ such that

$$\liminf_{t\to\infty} P\{x_1(t) \leqslant \chi_1\} \geqslant 1-\varepsilon, \quad \liminf_{t\to\infty} P\{x_1(t) \geqslant \delta_1\} \geqslant 1-\varepsilon,$$

and

$$\liminf_{t\to\infty} P\{x_2(t) \leqslant \chi_2\} \geqslant 1-\varepsilon, \quad \liminf_{t\to\infty} P\{x_2(t) \geqslant \delta_1'\} \geqslant 1-\varepsilon.$$

It is clear that if the system is stochastically permanent, it must be stochastically ultimately bounded.

Lemma 3.3. *Under Assumption* 1, *for any given initial value* $x(0) \in \mathbb{R}^2_+$ *there exist positive constants* c_1, c_2 *and* $\kappa(p)$ *such that the solution* $x(t)$ *of SDE* (1.3) *has the following property:*

$$E(c_1 x_1^p(t) + c_2 x_2^p(t)) \leqslant \kappa(p), \quad t \geqslant 0,\ p > 1. \tag{3.1}$$

P r o o f. From Theorem 2.1 we know that the solution $x(t)$ with initial value $x(0) \in \mathbb{R}^2_+$ will remain in $\mathbb{R}^2_+$ with probability 1. For any given positive constant $p > 1$ and positive constants c_1, c_2 to be determined, define

$$V(x(t)) = c_1 x_1^p(t) + c_2 x_2^p(t). \tag{3.2}$$

By Itô's formula and the Young inequality, we compute

$$
\begin{aligned}
\mathrm{d}\Big(\frac{1}{p}x_1^p(t)\Big) &= x_1^{p-1}(t)\,\mathrm{d}x_1(t) + \frac{p-1}{2}x_1^{p-2}(t)(\mathrm{d}x_1(t))^2 \\
&= \Big(-\bar{b}_1 x_1^{p+1}(t) + d_{12}x_1^{p-1}(t)x_2^2(t) + \Big(a_1 + \frac{p-1}{2}\sigma_1^2\Big)x_1^p(t)\Big)\,\mathrm{d}t \\
&\quad + \sigma_1 x_1^p(t)\,\mathrm{d}B_1(t) \\
&\leqslant \Big(-\bar{b}_1 x_1^{p+1}(t) + d_{12}\Big(\frac{p-1}{p+1}\varepsilon_1 x_1^{p+1}(t) + \frac{2}{p+1}\varepsilon_1^{-(p-1)/2}x_2^{p+1}(t)\Big) \\
&\quad + \Big(a_1 + \frac{p-1}{2}\sigma_1^2\Big)x_1^p(t)\Big)\,\mathrm{d}t + \sigma_1 x_1^p(t)\,\mathrm{d}B_1(t) \\
&= \Big(\Big(-\bar{b}_1 + d_{12}\frac{p-1}{p+1}\varepsilon_1\Big)x_1^{p+1}(t) + d_{12}\frac{2}{p+1}\varepsilon_1^{-(p-1)/2}x_2^{p+1}(t) \\
&\quad + \Big(a_1 + \frac{p-1}{2}\sigma_1^2\Big)x_1^p(t)\Big)\,\mathrm{d}t + \sigma_1 x_1^p(t)\,\mathrm{d}B_1(t)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{d}\Big(\frac{1}{p}x_2^p(t)\Big) &= x_2^{p-1}(t)\,\mathrm{d}x_2(t) + \frac{p-1}{2}x_2^{p-2}(t)(\mathrm{d}x_2(t))^2 \\
&= \Big(-\bar{b}_2 x_2^{p+1}(t) + d_{21}x_2^{p-1}(t)x_1^2(t) + \Big(a_2 + \frac{p-1}{2}\sigma_2^2\Big)x_2^p(t)\Big)\,\mathrm{d}t \\
&\quad + \sigma_2 x_2^p(t)\,\mathrm{d}B_2(t) \\
&\leqslant \Big(-\bar{b}_2 x_2^{p+1}(t) + d_{21}\Big(\frac{p-1}{p+1}\varepsilon_2 x_2^{p+1} + \frac{2}{p+1}\varepsilon_2^{-(p-1)/2}x_1^{p+1}(t)\Big) \\
&\quad + \Big(a_2 + \frac{p-1}{2}\sigma_2^2\Big)x_2^p(t)\Big)\,\mathrm{d}t + \sigma_2 x_2^p(t)\,\mathrm{d}B_2(t) \\
&= \Big(\Big(-\bar{b}_2 + d_{21}\frac{p-1}{p+1}\varepsilon_2\Big)x_2^{p+1}(t) + d_{21}\frac{2}{p+1}\varepsilon_2^{-(p-1)/2}x_1^{p+1}(t) \\
&\quad + \Big(a_2 + \frac{p-1}{2}\sigma_2^2\Big)x_2^p(t)\Big)\,\mathrm{d}t + \sigma_2 x_2^p(t)\,\mathrm{d}B_2(t),
\end{aligned}
$$

so we have

$$
\begin{aligned}
LV(x(t)) \leqslant p\Big\{&\Big(c_1\Big(-\bar{b}_1 + d_{12}\frac{p-1}{p+1}\varepsilon_1\Big) + c_2 d_{21}\frac{2}{p+1}\varepsilon_2^{-(p-1)/2}\Big)x_1^{p+1}(t) \\
&+ \Big(c_2\Big(-\bar{b}_2 + d_{21}\frac{p-1}{p+1}\varepsilon_2\Big) + c_1 d_{12}\frac{2}{p+1}\varepsilon_1^{-(p-1)/2}\Big)x_2^{p+1}(t) \\
&+ c_1\Big(a_1 + \frac{p-1}{2}\sigma_1^2\Big)x_1^p(t) + c_2\Big(a_2 + \frac{p-1}{2}\sigma_2^2\Big)x_2^p(t)\Big\}.
\end{aligned}
$$

Now, we can find ε_1, ε_2 and c_1, c_2 such that

$$
\begin{cases}
c_1\Big(-\bar{b}_1 + d_{12}\dfrac{p-1}{p+1}\varepsilon_1\Big) + c_2 d_{21}\dfrac{2}{p+1}\varepsilon_2^{-(p-1)/2} < 0, \\
c_2\Big(-\bar{b}_2 + d_{21}\dfrac{p-1}{p+1}\varepsilon_2\Big) + c_1 d_{12}\dfrac{2}{p+1}\varepsilon_1^{-(p-1)/2} < 0,
\end{cases}
$$

and noting the inequalities can be turned into

$$\frac{d_{21}2(p+1)^{-1}\varepsilon_2^{-(p-1)/2}}{\bar{b}_1 - d_{12}(p-1)(p+1)^{-1}\varepsilon_1} < \frac{c_1}{c_2} < \frac{\bar{b}_2 - d_{21}(p-1)(p+1)^{-1}\varepsilon_2}{d_{12}2(p+1)^{-1}\varepsilon_1^{-(p-1)/2}},$$

namely

$$\bar{b}_1\bar{b}_2\Big(1 - \frac{d_{21}}{\bar{b}_2}\frac{p-1}{p+1}\varepsilon_2\Big)\Big(1 - \frac{d_{12}}{\bar{b}_1}\frac{p-1}{p+1}\varepsilon_1\Big) > d_{12}d_{21}\Big(\frac{2}{p+1}\Big)^2(\varepsilon_1\varepsilon_2)^{-(p-1)/2},$$

so taking $\varepsilon_1 = \bar{b}_1/d_{12}$, $\varepsilon_2 = \bar{b}_2/d_{21}$, by Assumption 1 the above inequality holds. Let

$$\begin{aligned}
\breve{\alpha} &=: \max\Big\{pa_1 + \frac{p(p-1)}{2}\sigma_1^2, pa_2 + \frac{p(p-1)}{2}\sigma_2^2\Big\},\\
\beta_1 &= c_1^{-1/p}p\Big(\Big(\bar{b}_1 - d_{12}\frac{p-1}{p+1}\frac{\bar{b}_1}{d_{12}}\Big) - \frac{c_2}{c_1}d_{21}\frac{2}{p+1}\Big(\frac{\bar{b}_2}{d_{21}}\Big)^{-(p-1)/2}\Big),\\
\beta_2 &= c_2^{-1/p}p\Big(\Big(\bar{b}_2 - d_{21}\frac{p-1}{p+1}\frac{\bar{b}_2}{d_{21}}\Big) - \frac{c_1}{c_2}d_{12}\frac{2}{p+1}\Big(\frac{\bar{b}_1}{d_{12}}\Big)^{-(p-1)/2}\Big),\\
\hat{\beta} &=: \min\{\beta_1, \beta_2\}.
\end{aligned}$$

It is clear that $\breve{\alpha} > 0$ and $\hat{\beta} > 0$. Hence we can get

$$\begin{aligned}
\mathrm{d}V(x(t)) \leqslant{}& \breve{\alpha}(c_1x_1^p(t) + c_2x_2^p(t))\,\mathrm{d}t - \hat{\beta}(c_1^{(p+1)/p}x_1^{p+1}(t) + c_2^{(p+1)/p}x_2^{p+1}(t))\,\mathrm{d}t\\
&+ pc_1\sigma_1x_1^p(t)\,\mathrm{d}B_1(t) + pc_2\sigma_2x_2^p(t)\,\mathrm{d}B_2(t),
\end{aligned}$$

and then we have

$$\begin{aligned}
&\frac{\mathrm{d}E(c_1x_1^p(t) + c_2x_2^p(t))}{\mathrm{d}t}\\
&\quad\leqslant \breve{\alpha}E(c_1x_1^p(t) + c_2x_2^p(t)) - \hat{\beta}E(c_1^{(p+1)/p}x_1^{p+1}(t) + c_2^{(p+1)/p}x_2^{p+1}(t))\\
&\quad\leqslant \breve{\alpha}E(c_1x_1^p(t) + c_2x_2^p(t)) - 2^{-1/p}\hat{\beta}(E(c_1x_1^p(t) + c_2x_2^p(t)))^{(p+1)/p}\\
&\quad= E(c_1x_1^p(t) + c_2x_2^p(t))\{\breve{\alpha} - 2^{-1/p}\hat{\beta}(E(c_1x_1^p(t) + c_2x_2^p(t)))^{1/p}\}.
\end{aligned}$$

Therefore, letting $z(t) = E(c_1x_1^p(t) + c_2x_2^p(t))$, then

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} \leqslant z(t)(\breve{\alpha} - 2^{-1/p}\hat{\beta}z^{1/p}(t)).$$

Consider the equation

$$\frac{\mathrm{d}\overline{z}(t)}{\mathrm{d}t} = \overline{z}(t)(\breve{\alpha} - 2^{-1/p}\hat{\beta}\overline{z}^{\,1/p}(t)).$$

Let $y(t) = \overline{z}(t)^{-1/p}$. Then

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = -\frac{\breve{\alpha}}{p}y(t) + \frac{\hat{\beta}}{p}2^{-1/p},$$

and

$$y(t) = \mathrm{e}^{-\int \breve{\alpha}p^{-1}\,\mathrm{d}t}\Big(\int \frac{\hat{\beta}}{p}2^{-1/p}\mathrm{e}^{\int \breve{\alpha}p^{-1}\,\mathrm{d}t}\,\mathrm{d}t + C\Big) = 2^{-1/p}\frac{\hat{\beta}}{\breve{\alpha}} + C\mathrm{e}^{-\breve{\alpha}p^{-1}t},$$

where C is an arbitrary constant. Letting $t \to \infty$ on both sides of the above equation, we have

$$y(t) \to 2^{-1/p}\frac{\hat{\beta}}{\breve{\alpha}},$$

and therefore,

$$\overline{z}(t) \to 2\Big(\frac{\breve{\alpha}}{\hat{\beta}}\Big)^p \quad \text{as } t \to \infty.$$

Thus by the comparison argument we get

$$\limsup_{t\to\infty} z(t) \leqslant 2\Big(\frac{\breve{\alpha}}{\hat{\beta}}\Big)^p.$$

Then we have

$$\limsup_{t\to\infty} E(c_1x_1^p(t) + c_2x_2^p(t)) \leqslant 2\Big(\frac{\breve{\alpha}}{\hat{\beta}}\Big)^p =: L(p),$$

which implies that there is a $T > 0$ such that

$$E(c_1x_1^p(t) + c_2x_2^p(t)) \leqslant 2L(p), \quad t > T.$$

In addition, $E(c_1x_1^p(t) + c_2x_2^p(t))$ is continuous, so we have

$$E(c_1x_1^p(t) + c_2x_2^p(t)) \leqslant C(p), \quad t \in [0, T].$$

Let $\kappa(p) = \max\{2L(p), C(p)\}$, then

$$E(c_1x_1^p(t) + c_2x_2^p(t)) \leqslant \kappa(p), \quad t \geqslant 0,\ p > 1.$$

This completes the proof. $\square$

Theorem 3.4. *Under Assumption* 1, *the solutions of SDE* (1.3) *are stochastically ultimately bounded.*

The proof of Theorem 3.4 is a simple application of the Chebyshev inequality and Lemma 3.3.

Since the solution of SDE (1.3) is positive, by the classical comparison theorem of stochastic differential equations [7] we have the following result.

Lemma 3.5. *Let Assumptions* 1 *and* 2 *hold, let* $x(t) \in \mathbb{R}^2_+$ *be the solution of SDE* (1.3) *with initial value* $x(0) \in \mathbb{R}^2_+$*. Then* $x(t)$ *satisfies*

$$x_1(t) \geqslant \varphi_1(t), \quad x_2(t) \geqslant \varphi_2(t), \tag{3.3}$$

where $\varphi_1(t)$ *and* $\varphi_2(t)$ *are the solutions of equations:*

$$\begin{cases} \mathrm{d}\varphi_1(t) = \varphi_1(t)[(a_1 - \bar{b}_1\varphi_1(t))\,\mathrm{d}t + \sigma_1\,\mathrm{d}B_1(t)], & \varphi_1(0) = x_1(0), \\ \mathrm{d}\varphi_2(t) = \varphi_2(t)[(a_2 - \bar{b}_2\varphi_2(t))\,\mathrm{d}t + \sigma_2\,\mathrm{d}B_2(t)], & \varphi_2(0) = x_2(0). \end{cases}$$

In view of Lemma 3.6 in [13], one sees that, if Assumption 2 holds, there exist positive constants H_1, H_2 and θ such that $a_i - \frac{1}{2}(\theta+1)\sigma_i^2 > 0$, $i = 1, 2$ satisfying the inequalities

$$\limsup_{t\to\infty} E\Big(\frac{1}{(\varphi_1(t))^\theta}\Big) \leqslant H_1, \quad \limsup_{t\to\infty} E\Big(\frac{1}{(\varphi_2(t))^\theta}\Big) \leqslant H_2.$$

This, together with Lemma 3.5, gives

Lemma 3.6. *Under Assumptions* 1 *and* 2*, the solution* $x(t)$ *of SDE* (1.3) *with any initial value* $x(0) \in \mathbb{R}^2_+$ *satisfies*

$$\limsup_{t\to\infty} E\Big(\frac{1}{(x_1(t))^\theta}\Big) \leqslant H_1, \quad \limsup_{t\to\infty} E\Big(\frac{1}{(x_2(t))^\theta}\Big) \leqslant H_2, \tag{3.4}$$

where H_1, H_2 *are positive constants and* $\theta > 0$ *is such that* $a_i - \frac{1}{2}(\theta+1)\sigma_i^2 > 0$, $i = 1, 2$.

Theorem 3.7. *Under Assumptions* 1 *and* 2*, SDE* (1.3) *is stochastically permanent.*

P r o o f. Let $x(t)$ be the solution of SDE (1.3) with any given positive initial value $x(0) \in \mathbb{R}^2_+$. From Lemma 3.6, we have

$$\limsup_{t\to\infty} E\Big(\frac{1}{(x_1(t))^\theta}\Big) \leqslant H_1, \quad \limsup_{t\to\infty} E\Big(\frac{1}{(x_2(t))^\theta}\Big) \leqslant H_2.$$

For $x(t) \in \mathbb{R}_+^2$ and for any $\varepsilon > 0$, let $\delta_1 = (\varepsilon/H_1)^{1/\theta}$, $\delta_1' = (\varepsilon/H_2)^{1/\theta}$. Then we derive that

$$P\{x_1(t) < \delta_1\} = P\Big\{\frac{1}{(x_1(t))^\theta} > \frac{1}{\delta_1^\theta}\Big\} \leqslant \frac{E((x_1(t))^{-\theta})}{\delta_1^{-\theta}} \leqslant \delta_1^\theta H_1 = \varepsilon,$$

and

$$P\{x_2(t) < \delta_1'\} = P\Big\{\frac{1}{(x_2(t))^\theta} > \frac{1}{(\delta_1')^\theta}\Big\} \leqslant \frac{E((x_2(t))^{-\theta})}{(\delta_1')^{-\theta}} \leqslant (\delta_1')^\theta H_2 = \varepsilon.$$

Hence

$$\limsup_{t\to\infty} P\{x_1(t) < \delta_1\} \leqslant \varepsilon, \quad \limsup_{t\to\infty} P\{x_2(t) < \delta_1'\} \leqslant \varepsilon,$$

and this implies

$$\liminf_{t\to\infty} P\{x_1(t) \geqslant \delta_1\} \geqslant 1-\varepsilon, \quad \liminf_{t\to\infty} P\{x_2(t) \geqslant \delta_1'\} \geqslant 1-\varepsilon.$$

The other condition of Definition 3.2 follows from Theorem 3.4. $\square$

3.2. Persistence in mean. Chen et al. in [3] proposed the definition of persistence in mean for the deterministic system. Here, we also use this definition for the stochastic system.

Definition 3.8. SDE (1.3) is said to be persistent in mean, if

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t x_i(s)\,\mathrm{d}s > 0 \quad \text{a.s. } i = 1,2. \tag{3.5}$$

From the result in [9] we know that

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t \varphi_i(s)\,\mathrm{d}s = \frac{a_i - \frac{1}{2}\sigma_i^2}{\bar{b}_i}, \quad \lim_{t\to\infty} \frac{\log \varphi_i(t)}{t} = 0 \quad \text{a.s. } i = 1,2.$$

Using the above conclusions, we can get the following theorem.

Theorem 3.9. *Suppose Assumptions* 1 *and* 2 *are satisfied, then the solution* $x(t)$ *of SDE* (1.3) *with any initial value* $x(0) \in \mathbb{R}^2$ *has the following properties:*

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t x_i(s)\,\mathrm{d}s \geqslant \frac{a_i - \frac{1}{2}\sigma_i^2}{\bar{b}_i}, \quad \liminf_{t\to\infty} \frac{\log x_i(t)}{t} \geqslant 0 \quad \text{a.s. } i = 1,2,$$

and so system (1.3) *is persistent in mean.*

4. Stationary distribution

In this section, we investigate if there is a stationary distribution for SDE (1.3) instead of asymptotically stable equilibria. System (1.1), if Assumption 1 holds, has a globally stable positive equilibrium point $x^* = (x_1^*, x_2^*)$ satisfying the equations

$$\begin{cases} a_1 x_1^* - \bar{b}_1 {x_1^*}^2 + d_{12} {x_2^*}^2 = 0, \\ a_2 x_2^* - \bar{b}_2 {x_2^*}^2 + d_{21} {x_1^*}^2 = 0, \end{cases} \tag{4.1}$$

where x_i^*, $i = 1, 2$ are positive constants. We will prove SDE (1.3) is ergodic under Assumption 1 (see Remark 8.1 for the definition of the ergodic property).

Theorem 4.1. *Let Assumptions* 1 *hold. Let* $\delta_2 = \frac{1}{2} c_1 \sigma_1^2 x_1^* + \frac{1}{2} c_2 \sigma_2^2 x_2^*$ *and let the positive constants* c_1, c_2 *satisfy the inequality* (2.4). *Assume* $\delta_2 < \min\{(c_1 \bar{b}_1 - c_2 d_{21}) {x_1^*}^2, (c_2 \bar{b}_2 - c_1 d_{12}) {x_2^*}^2\}$. *Then there is a stationary distribution* $\mu(\cdot)$ *for SDE* (1.3) *and it has the ergodic property.*

Proof. Define $V\colon \mathbb{E}_l = \mathbb{R}_+^2 \to \mathbb{R}_+$ by

$$\begin{aligned} V(x(t)) = {} & c_1 \Big(x_1(t) - x_1^* - x_1^* \log \frac{x_1(t)}{x_1^*} \Big) \\ & + c_2 \Big(x_2(t) - x_2^* - x_2^* \log \frac{x_2(t)}{x_2^*} \Big). \end{aligned} \tag{4.2}$$

By Itô's formula, we compute

$$\begin{aligned} LV(x(t)) = {} & c_1 (x_1(t) - x_1^*) \Big(a_1 - \bar{b}_1 x_1(t) + \frac{d_{12} x_2^2(t)}{x_1(t)} \Big) \\ & + c_2 \Big(x_2(t) - x_2^*)(a_2 - \bar{b}_2 x_2(t) + \frac{d_{21} x_1^2(t)}{x_2(t)} \Big) \\ & + \frac{1}{2} c_1 \sigma_1^2 x_1^* + \frac{1}{2} c_2 \sigma_2^2 x_2^*. \end{aligned} \tag{4.3}$$

By (4.1), we have

$$\begin{cases} a_1 = \bar{b}_1 x_1^* - d_{12} \dfrac{{x_2^*}^2}{x_1^*}, \\ a_2 = \bar{b}_2 x_2^* - d_{21} \dfrac{{x_1^*}^2}{x_2^*}. \end{cases} \tag{4.4}$$

Substituting (4.4) into (4.3) one sees that

$$
\begin{aligned}
(4.5)\qquad & LV(x(t)) \\
=\ & -c_1\bar{b}_1(x_1(t)-x_1^*)^2 - c_2\bar{b}_2(x_2(t)-x_2^*)^2 \\
& + c_1 d_{12} {x_2^*}^2\Big(\frac{x_2^2(t)}{{x_2^*}^2} + 1 - \frac{x_1(t)}{x_1^*} - \frac{x_1^* x_2^2(t)}{x_1(t){x_2^*}^2}\Big) \\
& + c_2 d_{21} {x_1^*}^2\Big(\frac{x_1^2(t)}{{x_1^*}^2} + 1 - \frac{x_2(t)}{x_2^*} - \frac{x_2^* x_1^2(t)}{x_2(t){x_1^*}^2}\Big) \\
& + \frac{1}{2}c_1\sigma_1^2 x_1^* + \frac{1}{2}c_2\sigma_2^2 x_2^* \\
\leqslant\ & -c_1\bar{b}_1(x_1(t)-x_1^*)^2 - c_2\bar{b}_2(x_2(t)-x_2^*)^2 \\
& + c_1 d_{12} {x_2^*}^2\Big(\frac{x_2^2(t)}{{x_2^*}^2} + 1 - 2\frac{x_2(t)}{x_2^*}\Big) \\
& + c_2 d_{21} {x_1^*}^2\Big(\frac{x_1^2(t)}{{x_1^*}^2} + 1 - 2\frac{x_1(t)}{x_1^*}\Big) \\
& + \frac{1}{2}c_1\sigma_1^2 x_1^* + \frac{1}{2}c_2\sigma_2^2 x_2^* \\
=\ & -(c_1\bar{b}_1 - c_2 d_{21})(x_1(t)-x_1^*)^2 - (c_2\bar{b}_2 - c_1 d_{12})(x_2(t)-x_2^*)^2 \\
& + \frac{1}{2}c_1\sigma_1^2 x_1^* + \frac{1}{2}c_2\sigma_2^2 x_2^* \\
=:\ & -(c_1\bar{b}_1 - c_2 d_{21})(x_1(t)-x_1^*)^2 - (c_2\bar{b}_2 - c_1 d_{12})(x_2(t)-x_2^*)^2 + \delta_2,
\end{aligned}
$$

where $\delta_2 = \frac{1}{2}c_1\sigma_1^2 x_1^* + \frac{1}{2}c_2\sigma_2^2 x_2^*$ (we use the fact $a^2+b^2 \geqslant 2ab$ in the first inequality). Then

$$LV(x(t)) \leqslant -(c_1\bar{b}_1 - c_2 d_{21})(x_1(t)-x_1^*)^2 - (c_2\bar{b}_2 - c_1 d_{12})(x_2(t)-x_2^*)^2 + \delta_2.$$

As c_1, c_2 satisfy the inequality (2.4), the quadratic coefficients are less than zero. The following proof of ergodicity is similar to that of Theorem 3.2 in [8]. Note that $\delta_2 < \min\{(c_1\bar{b}_1 - c_2 d_{21}){x_1^*}^2, (c_2\bar{b}_2 - c_1 d_{12}){x_2^*}^2\}$, so the ellipse

$$-(c_1\bar{b}_1 - c_2 d_{21})(x_1(t)-x_1^*)^2 - (c_2\bar{b}_2 - c_1 d_{12})(x_2(t)-x_2^*)^2 + \delta_2 = 0$$

lies entirely in $\mathbb{R}^2_+$. We can take U to be a neighborhood of the ellipsoid with $\overline{U} \subset \mathbb{E}_l = \mathbb{R}^2_+$, so for $x \in \mathbb{R}^2_+ \setminus U$, $LV \leqslant -K$ (K is a positive constant), which implies the condition (B.2) in Lemma 8.2 (see the Appendix) is satisfied. Therefore, the solution $x(t)$ is recurrent in the domain U. This together with Lemma 8.4 implies $x(t)$ is recurrent in any bounded domain $D \subset \mathbb{R}^2_+$. In addition, for any D we have

$$M = \min\{\sigma_1^2 x_1^2,\ \sigma_2^2 x_2^2\colon\ (x_1,x_2) \in \overline{D}\} > 0$$

so that

$$\sum_{i,j=1}^{2}\Big(\sum_{k=1}^{2}g_{ik}(x)g_{jk}(x)\Big)\xi_i\xi_j=\sum_{i=1}^{2}(\sigma_i^2x_i^2\xi_i^2)\geqslant M|\xi|^2$$

for all $(x_1,x_2)\in\overline{D}$, $\xi\in\mathbb{R}^2$. From Remark 8.3 we know that condition (B.1) of Lemma 8.2 is also satisfied (see [21], page 349). Therefore, the stochastic system (1.3) has a stationary distribution $\mu(\cdot)$, satisfies the strong law of large numbers, and it is ergodic (see e.g. [14], Theorem 4.2 and Corollary 1 and [12], Theorem 4.2 on page 110). □

5. Extinction

We know that, if Assumptions 1 holds, the solution of ODE (1.1) converges to a positive equilibrium point or is unbounded, so the population will not become extinct, and by Theorem 3.7, we note that if the condition $a_i>\frac{1}{2}\sigma_i^2$, $i=1,2$, is also satisfied, then the small white noise intensity makes both species stochastically permanent and persistent in mean. We will show in this section that if the noise is sufficiently large, the solution to the associated SDE (1.3) will become extinct with probability 1.

Theorem 5.1. *Let Assumptions* 1 *hold. Let* $\check{a}=\max\{a_1,a_2\}$, $\frac{1}{2}\hat{\sigma}^2=\frac{1}{2}(\sigma_1^{-2}+\sigma_2^{-2})^{-1}$ *and let the positive constants* c_1,c_2 *satisfy the inequality* (2.4). *For any given initial value* $x(0)\in\mathbb{R}_+^2$, *the solution of the SDE* (1.3) *satisfies*

$$\limsup_{t\to\infty}\frac{\log(c_1x_1(t)+c_2x_2(t))}{t}\leqslant\check{a}-\frac{\hat{\sigma}^2}{2}\quad\text{a.s.}$$

Particularly, if $\check{a}-\frac{1}{2}\hat{\sigma}^2<0$, *then* $\lim\limits_{t\to\infty}x(t)=0$ *a.s.*

P r o o f. Define

$$V(x(t))=c_1x_1(t)+c_2x_2(t),\quad t\geqslant 0. \tag{5.1}$$

Using Itô's formula, one can derive that

$$\begin{aligned}\mathrm{d}V(x(t))=(-(c_1\overline{b}_1-c_2d_{21})x_1^2(t)-(c_2\overline{b}_2-c_1d_{12})x_2^2(t)+c_1a_1x_1(t)\\+c_2a_2x_2(t))\,\mathrm{d}t+c_1\sigma_1x_1(t)\,\mathrm{d}B_1(t)+c_2\sigma_2x_2(t)\,\mathrm{d}B_2(t).\end{aligned} \tag{5.2}$$

Let $\breve{a} = \max\{a_1, a_2\}$, $\frac{1}{2}\hat{\sigma}^2 = \frac{1}{2}(\sigma_1^{-2} + \sigma_2^{-2})^{-1}$. Applying the Cauchy inequality and Assumptions 1, we compute

$$
\begin{aligned}
\mathrm{d}\log V(x(t)) &= \frac{1}{V(x(t))}(c_1a_1x_1(t) + c_2a_2x_2(t) - (c_1\bar{b}_1 - c_2d_{21})x_1^2(t) \\
&\quad - (c_2\bar{b}_2 - c_1d_{12})x_2^2(t))\,\mathrm{d}t \\
&\quad - \frac{1}{2V^2(x(t))}(c_1^2\sigma_1^2x_1^2(t) + c_2^2\sigma_2^2x_2^2(t))\,\mathrm{d}t \\
&\quad + \frac{1}{V(x(t))}(c_1\sigma_1x_1(t)\,\mathrm{d}B_1(t) + c_2\sigma_2x_2(t)\,\mathrm{d}B_2(t)) \\
&\leqslant \frac{1}{c_1x_1(t) + c_2x_2(t)}\max\{a_1, a_2\}(c_1x_1(t) + c_2x_2(t))\,\mathrm{d}t \\
&\quad - \frac{(c_1x_1(t) + c_2x_2(t))^2}{2(\sigma_1^{-2} + \sigma_2^{-2})(c_1x_1(t) + c_2x_2(t))^2}\,\mathrm{d}t \\
&\quad + \frac{c_1\sigma_1x_1(t)\,\mathrm{d}B_1(t) + c_2\sigma_2x_2(t)\,\mathrm{d}B_2(t)}{c_1x_1(t) + c_2x_2(t)} \\
&\leqslant \Big(\breve{a} - \frac{\hat{\sigma}^2}{2}\Big)\,\mathrm{d}t + \frac{c_1\sigma_1x_1(t)\,\mathrm{d}B_1(t) + c_2\sigma_2x_2(t)\,\mathrm{d}B_2(t)}{c_1x_1(t) + c_2x_2(t)}.
\end{aligned}
\tag{5.3}
$$

Integrating both sides of inequality (5.3) from 0 to t gives

$$
\log V(x(t)) \leqslant \log V(x(0)) + \int_0^t \Big(\breve{a} - \frac{\hat{\sigma}^2}{2}\Big)\,\mathrm{d}s + M(t), \tag{5.4}
$$

where $M(t)$ is the martingale defined by

$$
M(t) = \int_0^t \frac{c_1\sigma_1x_1(s)\,\mathrm{d}B_1(s) + c_2\sigma_2x_2(s)\,\mathrm{d}B_2(s)}{c_1x_1(s) + c_2x_2(s)}
$$

with $M(0) = 0$. The quadratic variation of this martingale is

$$
\langle M, M\rangle_t = \int_0^t \frac{c_1^2\sigma_1^2x_1^2(s) + c_2^2\sigma_2^2x_2^2(s)}{(c_1x_1(s) + c_2x_2(s))^2}\,\mathrm{d}s \leqslant \max\{\sigma_1^2, \sigma_2^2\}t.
$$

By the strong law of large numbers for martingales (see [9]), we have

$$
\lim_{t\to\infty} \frac{M(t)}{t} = 0 \quad \text{a.s.}
$$

It finally follows from (5.4) by dividing by t on both sides and then letting $t \to \infty$ that

$$\limsup_{t\to\infty} \frac{\log V(x(t))}{t} \leqslant \lim_{t\to\infty} \frac{\log V(x(0))}{t} + \limsup_{t\to\infty} \frac{1}{t}\int_0^t \Big(\check{a} - \frac{\hat{\sigma}^2}{2}\Big)\,\mathrm{d}s + \lim_{t\to\infty}\frac{M(t)}{t} = \check{a} - \frac{\hat{\sigma}^2}{2} \quad \text{a.s.} \tag{5.5}$$

By the definition of $V(x(t)) = c_1x_1(t) + c_2x_2(t)$, $t \geqslant 0$, we have that

$$\limsup_{t\to\infty} \frac{\log(c_1x_1(t) + c_2x_2(t))}{t} \leqslant \check{a} - \frac{\hat{\sigma}^2}{2} \quad \text{a.s.}$$

Thus the required assertion follows. $\square$

6. Numerical simulation

We assume $\alpha_{ij} = 1$, and then $\bar{b}_1 = b_1 + d_{12}$, $\bar{b}_2 = b_2 + d_{21}$, so the SDE (1.3) can be rewritten in the form

$$\begin{cases} \mathrm{d}x_1(t) = (x_1(t)(a_1 - \bar{b}_1x_1(t)) + d_{12}x_2^2(t))\,\mathrm{d}t + \sigma_1x_1(t)\,\mathrm{d}B_1(t), \\ \mathrm{d}x_2(t) = (x_2(t)(a_2 - \bar{b}_2x_2(t)) + d_{21}x_1^2(t))\,\mathrm{d}t + \sigma_2x_2(t)\,\mathrm{d}B_2(t). \end{cases} \tag{6.1}$$

We numerically simulate the solution of (6.1). Using Milstein's higher order method in [6], we get the discretization equation

$$\begin{cases} x_{1,k+1} = x_{1,k} + (x_{1,k}(a_1 - \bar{b}_1x_{1,k}) + d_{12}x_{2,k}^2)\Delta t + \sigma_1x_{1,k}\sqrt{\Delta t}\xi_{1,k} \\ \qquad\qquad + \frac{1}{2}\sigma_1^2x_{1,k}(\Delta t\xi_{1,k}^2 - \Delta t), \\ x_{2,k+1} = x_{2,k} + (x_{2,k}(a_2 - \bar{b}_2x_{2,k}) + d_{21}x_{1,k}^2)\Delta t + \sigma_2x_{2,k}\sqrt{\Delta t}\xi_{2,k} \\ \qquad\qquad + \frac{1}{2}\sigma_2^2x_{2,k}(\Delta t\xi_{2,k}^2 - \Delta t), \end{cases} \tag{6.2}$$

where the time increment is $\Delta t > 0$, $\xi_{1,k}$ and $\xi_{2,k}$, $k = 1, 2, \ldots, n$ are independent Gaussian random variables with distribution $N(0,1)$. We choose the initial value $(x_1(0), x_2(0)) = (0.58, 0.60)$ and the parameters $a_1 = 0.3$, $a_2 = 0.4$, $\bar{b}_1 = 1.2$, $\bar{b}_2 = 1.1$, $d_{12} = 0.6$, $d_{21} = 0.5$. We take $\Delta t = 0.01$. From Matlab, we get Figures 1–6 and we will use them to illustrate our results. Obviously, Assumptions 1 is satisfied, so the corresponding deterministic model has a globally stable positive equilibrium point $x^* = (x_1^*, x_2^*) \doteq (0.5734, 0.6090)$. According to Theorem 2.1, system (6.1) has

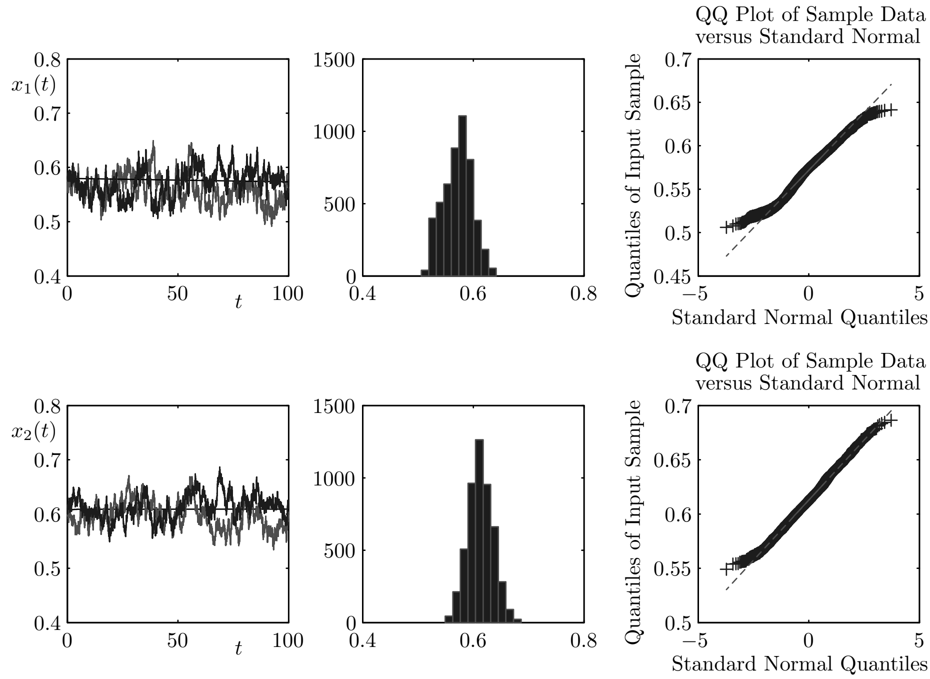

Figure 1. The left subgraphs are the solutions of SDE (6.1) and the corresponding deterministic system. The two curves of represent $x_i(t)$ from the same initial point. The middle subgraphs are the histograms of (6.1) and the right subgraphs are normal quantile-quantile plots of $x_1(t)$ and $x_2(t)$, respectively. The stochastic system is stochastically permanent and has stationary distribution. Here $\sigma_1 = 0.05$, $\sigma_2 = 0.04$.

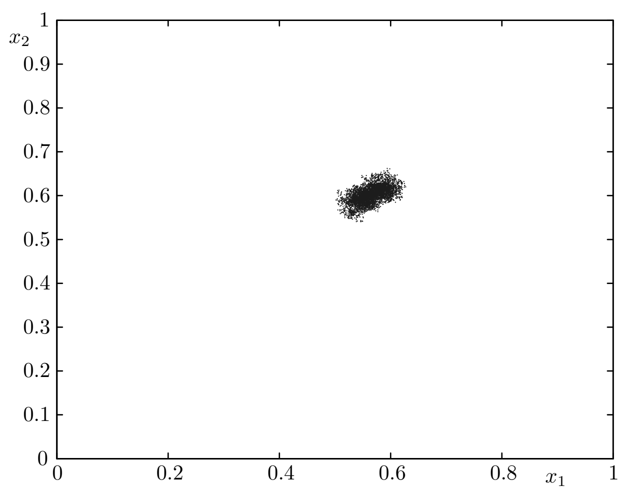

Figure 2. Population distribution of stochastic system (6.1) around the deterministic model's positive equilibrium $x^* \doteq (0.5734, 0.6090)$. Here $\sigma_1 = 0.05$, $\sigma_2 = 0.04$.

a unique positive solution. In Theorem 4.1 and Theorem 5.1, we need c_1 and c_2 to satisfy the inequality (2.4), so we only take both numbers c_1, c_2 equal to 1. We divide the white noise intensity into three cases to study the impact of white noise on the system.

Case I. White noise of small intensity. In Figures 1–2, we choose $\sigma_1 = 0.05$, $\sigma_2 = 0.04$. Obviously Assumption 2 holds and the SDE (6.1) is stochastically permanent and persistent in mean. We compute $\delta_2 = \frac{1}{2}c_1\sigma_1^2 x_1^* + \frac{1}{2}c_2\sigma_2^2 x_2^* \doteq 1.204 * 10^{-3}$, and $\min\{(c_1\bar{b}_1 - c_2 d_{21})(x_1^*)^2, (c_2\bar{b}_2 - c_1 d_{12})(x_2^*)^2\} \doteq 0.18544$, so the condition $\delta_2 < \min\{(c_1\bar{b}_1 - c_2 d_{21})(x_1^*)^2, (c_2\bar{b}_2 - c_1 d_{12})(x_2^*)^2\}$ is satisfied. By virtue of Theorem 4.1, there is a stationary distribution (see the middle histogram in Figure 1). The left pictures in Figure 1 show that the stochastic system imitates the deterministic system and their curves nearly coincide. The right subgraphs are the normal quantile-quantile plots of the values of the paths $x_1(t)$ and $x_2(t)$, and they are quite similar to straight lines. This means that the distribution is an approximately standard normal distribution. Moreover, from Figure 2, we find that almost all population distributions lie in the neighborhood, which can be imagined as a circular or elliptic region centered at (x_1^*, x_2^*) (see the scatter picture in Figure 2). Hence, although there is no equilibrium of the stochastic system (6.1) as a deterministic system, it is stochastically permanent, persistent in mean and has ergodic property by Theorems 3.7, 3.9 and 4.1.

Case II. White noise with relatively large intensity. In Figures 3–4, we choose $\sigma_1 = 0.4, \sigma_2 = 0.3$. The populations of x_1 and x_2 suffer relatively large white noise. Comparing Figures 1 and 3, we see that in Figure 3 the left curves fluctuations are more violent, the histograms distribute in relatively large regions, and the curves of QQ plots slightly deviate from a straight line. Comparing Figures 2 and 4, the points distribute in larger areas in Figure 4, but we can find an ellipse to meet the condition $\delta_2 \doteq 0.073277 < \min\{(c_1\bar{b}_1 - c_2 d_{21})(x_1^*)^2, (c_2\bar{b}_2 - c_1 d_{12})(x_2^*)^2\} \doteq 0.18544$, and from Theorems 3.7, 3.9 and 4.1, we know that SDE (6.1) is stochastically permanent, persistent in mean and has a stationary distribution.

In Figure 5, we select $\sigma_1 = 0.01$, $\sigma_2 = 0.6$. The conditions of Theorems 3.7, 3.9 and 4.1 are satisfied, and x_2 suffers relatively large white noise. From the left pictures in Figure 1 and Figure 5, we see that the fluctuations of the two curves are different, and the reason is that larger white noise of x_2 impacts x_1 in Figure 5. In other words, due to the presence of diffusion, the relatively big white noise intensity in the individual patches will be evenly distributed to the other patches. Therefore, system (6.1) is stochastically permanent and has a stationary distribution.

Case III. White noise of large intensity. In comparison with small white noise in Figures 1 and 2, we choose $\sigma_1 = 0.9$, $\sigma_2 = 1.0$ in Figure 6. Both x_1 and x_2 suffer large white noise. We find that $a_i < \frac{1}{2}\sigma_i^2$, $i = 1, 2$, $\delta_2 \doteq 0.53637 > \min\{(c_1\bar{b}_1 - c_2 d_{21})(x_1^*)^2, (c_2\bar{b}_2 - c_1 d_{12})(x_2^*)^2\} \doteq 0.18544$, so the conditions of The-

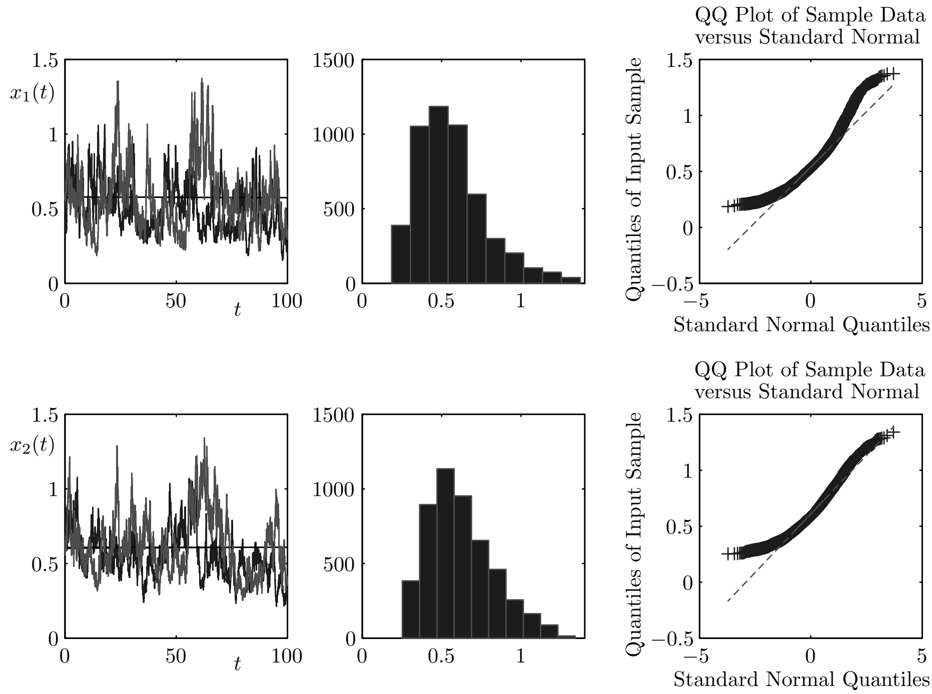

Figure 3. The subgraphs are defined in Figure 1. Here $\sigma_1 = 0.4$, $\sigma_2 = 0.3$. The stochastic system is stochastically permanent, persistent in mean and has a stationary distribution.

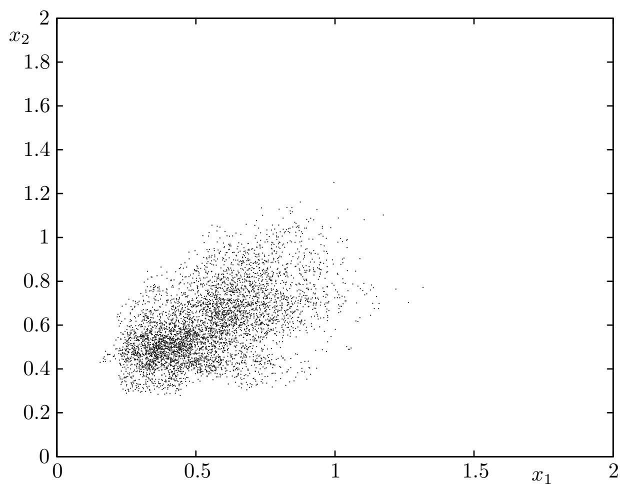

Figure 4. Population distribution of stochastic system (6.1) around the deterministic model's positive equilibrium $x^* \doteq (0.5734, 0.6090)$. Here $\sigma_1 = 0.4$, $\sigma_2 = 0.3$.

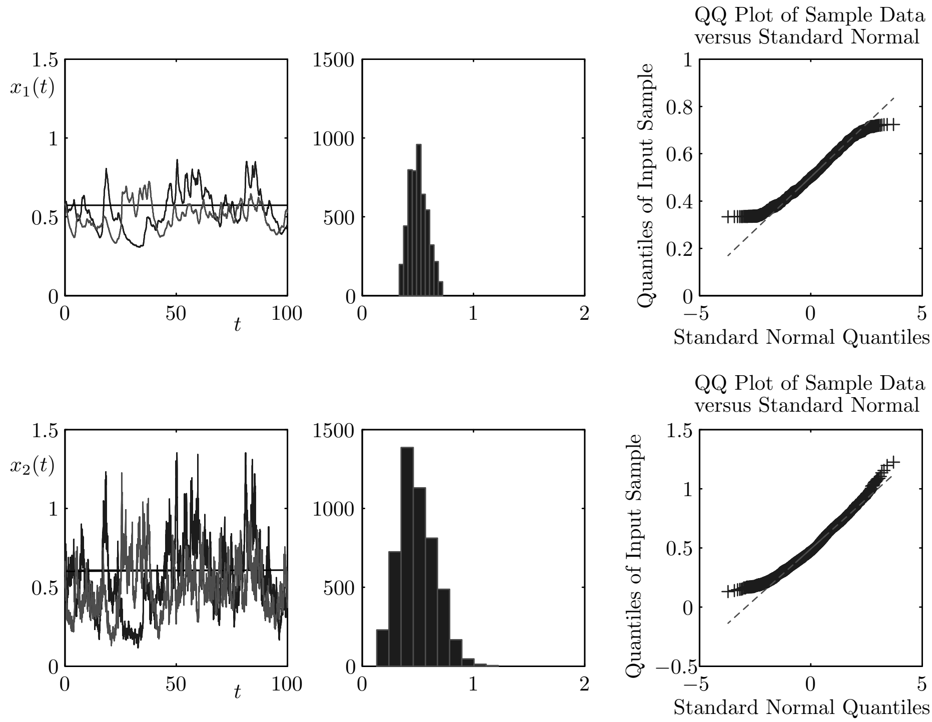

Figure 5. The subgraphs are defined in Figure 1. Here $\sigma_1 = 0.01$, $\sigma_2 = 0.6$. The SDE (6.1) is stochastically permanent, persistent in mean and has a stationary distribution.

orems 3.7, 3.9 and Theorem 4.1 are not satisfied and the extinction conditions in Theorem 5.1 are satisfied. That is, $\check{a} - \frac{1}{2}\hat{\sigma}^2 \doteq -0.04751 < 0$, as the case in Theorem 5.1 expected, and the species x_1 and x_2 will become extinct although the deterministic system is globally asymptotically stable.

7. Conclusion

In this paper, we study the stochastic logistic single-species model with nonlinear directed diffusion. We divide the white noise intensity into small, medium and large cases, and through numerical simulation, we are able to understand the important role played by the white noise and diffusion phenomena in biological populations. In addition, we can see from the left subgraphs in Figures 1, 3, 5, 6 that, due to the random disturbance, the curves starting from the same initial value are not overlapped. From these figures, we find that when the white noise is small, system (6.1) imitates its deterministic system and it is stochastically permanent and has a stationary distribution (see Figures 1–2). When the white noise is relatively large in

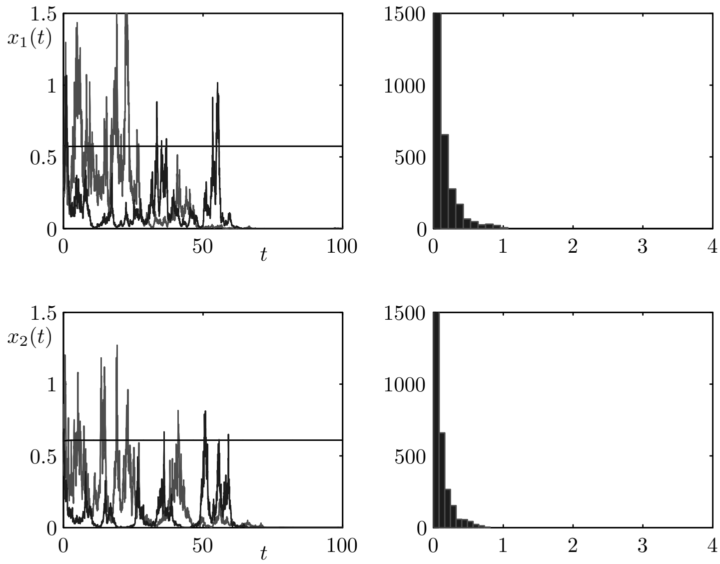

Figure 6. The subgraphs are defined in Figure 1. Here $\sigma_1 = 0.9$, $\sigma_2 = 1$. The populations of $x_1(t)$ and $x_2(t)$ will become extinct.

some groups, it will produce relatively large deviation (see Figure 5) but will not produce the species extinction due to the presence of diffusion. When the noise is sufficiently large in all the groups (see Figure 6), the species will become extinct even if diffusion exists. In the real world, the large white noise may be bad weather or serious epidemic, which can be considered decisive factors responsible for the extinction of populations. Therefore, our research and analysis on population has great practical significance.

8. Appendix

In this section, we list some results about the stationary distribution (see [12], page 101) which we used in the previous sections.

Let $X(t)$ be a regular time-homogeneous Markov process in $\mathbb{E}_l$ ($\mathbb{E}_l$ denotes the Euclidean l-space i.e., $\xi(\omega) = (\xi_1(\omega), \xi_2(\omega), \ldots, \xi_l(\omega))$ is a vector in $\mathbb{E}_l$) described by SDE

$$\mathrm{d}X(t) = b(X)\,\mathrm{d}t + \sum_{r=1}^{k} g_r(X)\,\mathrm{d}B_r(t).$$

The diffusion matrix is

$$\Lambda(x) = (\lambda_{ij}(x)), \quad \lambda_{ij}(x) = \sum_{r=1}^{k} g_r^i(x) g_r^j(x).$$

Remark 8.1. We say that the Markov process $X(t)$ is regular if for any $(s,x) \in E$ ($E = \mathbb{E}_l \times I$ and $I = I_\infty$), $\mathbf{P}^{s,x}\{\tau = \infty\} = 1$; here $\mathbf{P}$ is a probability measure and the random variable τ is the first exit time of the sample function from every bounded domain, or briefly the explosion time (see [12], page 75). The Markov process $X(t)$ is called time-homogeneous if the transition probability function $\mathbf{P}(s, X, t+s, A)$ is independent of s for $0 \leqslant s \leqslant t$ and $A \in \mathfrak{B}$ (recall that $\mathfrak{B}$ denotes the σ-algebra of Borel sets in $\mathbb{E}_l$) (see [12], page 68 for details of transition probability function).

Assume a Markov semigroup $(\mathbf{P}_t)_{t \geqslant 0}$ is strong Feller and irreducible, so then there exists at most one invariant measure for it. An invariant measure μ of $\mathbf{P}_t$ is said to be ergodic if

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbf{P}_t \varphi \, \mathrm{d}t = \bar{\varphi}, \quad \varphi \in L^2(H, \mu),$$

where $\bar{\varphi}$ is the mean of φ,

$$\bar{\varphi} = \int_H \varphi(x) \mu(\mathrm{d}x), \quad x \in H,$$

and H is a separable Hilbert space (see [4], page 14 and [2] for more details about the ergodic property).

Assumption B ([12]). There exists a bounded domain $U \subset \mathbb{E}_l$ with regular boundary Γ, having the following properties:

(B.1) In the domain U and some neighborhood thereof, the smallest eigenvalue of the diffusion matrix $\Lambda(x)$ is bounded away from zero.

(B.2) If $x \in \mathbb{E}_l \setminus U$, the mean time τ (that is $E_x\tau$) at which a path issuing from x reaches the set U is finite, and $\sup_{x\in K} E_x\tau < \infty$ for every compact subset $K \subset \mathbb{E}_l$.

Lemma 8.2. *If Assumption* B *holds, then the Markov process $X(t)$ has a stationary distribution μ. Let $f(\cdot)$ be a function integrable with respect to the measure μ. Then*

$$\mathbf{P}\left\{ \lim_{T\to\infty} \frac{1}{T} \int_0^T f(X(t)) \, \mathrm{d}t = \int_{\mathbb{E}_l} f(x)\mu(\mathrm{d}x) \right\} = 1 \tag{8.1}$$

for all $x \in \mathbb{E}_l$.

Remark 8.3. (i) Note (8.1) is called the strong law of large numbers. The proof of Lemma 8.2 can be found in [12]. More precisely, the existence of a stationary distribution with density can be found in Theorem 4.1, page 108. The weak convergence and the ergodicity is obtained in Theorem 4.2 on page 110, and Corollary 4.3, Corollary 4.4 on page 112.

(ii) To verity (B.1), it is sufficient to show that F is uniformly elliptical in U, where $Fu = b(x) \cdot u_x + \frac{1}{2}\mathrm{trace}(\Lambda(x)u_{xx})$, that is to say, there is a positive number M such that

$$\sum_{i,j=1}^{l} \lambda_{ij}(x)\xi_i\xi_j \geqslant M|\xi|^2, \quad x \in U,\ \xi \in \mathbb{E}_l$$

(for details we refer to [5], page 103, and Rayleigh's principle in [21], page 349). To verify (B.2), it suffices to prove that there exist a neighborhood U and a non-negative C^2-function V such that $\Lambda(x)$ is uniformly elliptical in U and for any $x \in \mathbb{E}_l \setminus U$, LV is negative (see [22], page 1163).

Lemma 8.4 ([12]). *Let $X(t)$ be a regular time-homogeneous Markov process in $\mathbb{E}_l$. If $X(t)$ is recurrent relative to some bounded domain U, then it is recurrent relative to any nonempty domain in $\mathbb{E}_l$.*

References

[1] *L. J. S. Allen*: Persistence and extinction in single-species reaction-diffusion models. Bull. Math. Biol. *45* (1983), 209–227.

[2] *S. Cerrai*: Second Order PDE's in Finite and Infinite Dimension. A Probabilistic Approach. Lecture Notes in Mathematics 1762, Springer, Berlin, 2001.

[3] *L. S. Chen, J. Chen*: Nonlinear Biological Dynamical System. Science Press, Beijing, 1993.

[4] *G. Da Prato*: Kolmogorov Equations for Stochastic PDEs. Advanced Courses in Mathematics-CRM Barcelona, Birkhäuser, Basel, 2004.

[5] *T. C. Gard*: Introduction to Stochastic Differential Equations. Pure and Applied Mathematics 114, Marcel Dekker, New York, 1988.

[6] *D. J. Higham*: An algorithmic introduction to numerical simulation of stochastic differential equations. SIAM Rev. *43* (2001), 525–546.

[7] *N. Iked, S. Watanabe*: Stochastic Differential Equations and Diffusion Processes. North-Holland Mathematical Library 24, North-Holland, Amsterdam; Kodansha, Tokyo, 1989.

[8] *C. Ji, D. Jiang, H. Liu, Q. Yang*: Existence, uniqueness and ergodicity of positive solution of mutualism system with stochastic perturbation. Math. Probl. Eng. *2010* (2010), Article ID 684926, 18 pages.

[9] *C. Ji, D. Jiang, N. Shi*: Analysis of a predator-prey model with modified Leslie-Gower and Holling-type II schemes with stochastic perturbation. J. Math. Anal. Appl. *359* (2009), 482–498.

[10] *D. Jiang, N. Shi*: A note on nonautonomous logistic equation with random perturbation. J. Math. Anal. Appl. *303* (2005), 164–172.

[11] *D. Jiang, N. Shi, X. Li*: Global stability and stochastic permanence of a non-autonomous logistic equation with random perturbation. J. Math. Anal. Appl. *340* (2008), 588–597.

[12] *R. Khas'minskiĭ*: Stochastic Stability of Differential Equations. Monographs and Textbooks on Mechanics of Solids and Fluids. Mechanics: Analysis, 7. Sijthoff Noordhoff, USA; Alphen aan den Rijn, The Netherlands, 1980.

[13] *X. Li, A. Gray, D. Jiang, X. Mao*: Sufficient and necessary conditions of stochastic permanence and extinction for stochastic logistic populations under regime switching. J. Math. Anal. Appl. *376* (2011), 11–28.

[14] *H. Liu, Q. Yang, D. Jiang*: The asymptotic behavior of stochastically perturbed DI SIR epidemic models with saturated incidences. Automatica *48* (2012), 820–825.

[15] *M. Liu, K. Wang*: Persistence and extinction in stochastic non-autonomous logistic systems. J. Math. Anal. Appl. *375* (2011), 443–457.

[16] *Z. Lu, Y. Takeuchi*: Global asymptotic behavior in single-species discrete diffusion systems. J. Math. Biol. *32* (1993), 67–77.

[17] *X. Mao*: Stochastic Differential Equations and Their Applications. Ellis Horwood Series in Mathematics and Its Applications, Horwood Publishing, Chichester, 1997.

[18] *X. Mao, C. Yuan*: Stochastic Differential Equations with Markovian Switching. Imperial College Press, London, 2006.

[19] *X. Mao, C. Yuan, J. Zou*: Stochastic differential delay equations of population dynamics. J. Math. Anal. Appl. *304* (2005), 296–320.

[20] *A. Okubo*: Diffusion and Ecological Problems: Mathematical Models. Biomathematics, vol. 10, Springer, Berlin, 1980.

[21] *G. Strang*: Linear Algebra and Its Applications. Academic Press (A Subsidiary of Harcourt Brace Jovanovich, Publishers), New York, 1980.

[22] *C. Zhu, G. Yin*: Asymptotic properties of hybrid diffusion systems. SIAM J. Control Optim. *46* (2007), 1155–1179.

Authors' addresses: Li Zu, College of Mathematics and Statistics, Hainan Normal University, Longkun South Road, Hainan 571158, Haikou, Hainan, China, e-mail: `zulihsd2014@163.com`; Daqing Jiang (corresponding author), School of Science, China University of Petroleum, 66 Changjiang West Road, Qingdao 266580, Huangdao, Shandong, China, and Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, King Abdulaziz University, Umm Al Muminin, Al-Sharafeyah, Jeddah 23218, Saudi Arabia, e-mail: `daqingjiang2010@hotmail.com`; Donal O'Regan, School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, University Road, Galway H91 TK33, Ireland, e-mail: `donal.oregan@nuigalway.ie`.