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THE FACTORIZATION METHOD FOR CRACKS IN INHOMOGENEOUS MEDIA

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Abstract. We consider the inverse scattering problem of determining the shape and location of a crack surrounded by a known inhomogeneous media. Both the Dirichlet boundary condition and a mixed type boundary conditions are considered. In order to avoid using the background Green function in the inversion process, a reciprocity relationship between the Green function and the solution of an auxiliary scattering problem is proved. Then we focus on extending the factorization method to our inverse shape reconstruction problems by using far field measurements at fixed wave number. We remark that this is done in a non intuitive space for the mixed type boundary condition as we indicate in the sequel.

Keywords: inverse scattering; factorization method; crack; inhomogeneous media

MSC 2010: 45Q05

1. Introduction

This study is concerned with the inverse problem of reconstruction of the shape and position of cracks embedded in a known inhomogeneous background medium from the knowledge of the far field due to incident plane waves at a fixed frequency. The particular application we have in mind is the nondestructive testing of flaws in materials in specific (typically thin) areas or the detection of thin air pockets inside structures. Earlier studies on the inverse obstacle scattering associated with outside

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inhomogeneity can be found in [14] and [18], where the uniqueness of the inverse problem of recovering the obstacle in an inhomogeneous background from the far field data is proved and the factorization method is developed for determining it, respectively. Very recently, the authors in [7] established the factorization method for a non-absorbing anisotropic background media containing penetrable defects. Some other qualitative methods such as the linear sampling method and reciprocity gap functional have been adopted to recover objects buried in inhomogeneous background media (possibly piecewise constant or anisotropic) [8], [6]. Inspired by the work [7], we develop in this paper a factorization method to reconstruct the crack inside an inhomogeneous media.

Precisely speaking, let Γ be an open arc as the cross section of an infinite cylinder in $\mathbb{R}^2$ and assume that Γ can be extended to an arbitrary smooth, simply connected, closed curve $\partial\Omega$ enclosing a bounded domain Ω such that the normal vector ν on Γ coincides with the outward normal vector on $\partial\Omega$, which we again denote by ν. We denote by $n(x)$ the refraction index of an inhomogeneous media which is represented by a domain D, the support of $(n(x)-1)$ such that $\Omega \subset D$. We note that the real valued function $n(x) > 0$ belongs to $C(D)$ and satisfies $n(x) = 1$ outside the region D. Assume that the incident electromagnetic field is a time harmonic plane wave polarized in the TM mode, hence we arrive at the following boundary value problem for the Helmholtz equation with positive wave number k:

$$\Delta u + k^2 n u = 0 \quad \text{in } \mathbb{R}^2 \setminus \bar{\Gamma}, \tag{1.1}$$

where the total field $u = u^s + u^i$ is decomposed into the given incident plane wave $u^i = \mathrm{e}^{\mathrm{i}kx\cdot d}$ with unitary $d \in S^1$ (the unit circle in $\mathbb{R}^2$) and the unknown scattered field u^s which is required to satisfy the Sommerfeld radiation condition

$$\lim_{r\to\infty} \sqrt{r}\Big(\frac{\partial u^s}{\partial r} - \mathrm{i}ku^s\Big) = 0 \tag{1.2}$$

uniformly in $\hat{x} = x/|x|$ with $r = |x|$. It is known that $u^s(x)$ has the asymptotic representation

$$u^s(x) = \frac{\mathrm{e}^{\mathrm{i}k|x|}}{\sqrt{|x|}}\Big\{u^\infty(\hat{x}) + O\Big(\frac{1}{|x|}\Big)\Big\} \quad \text{as } |x| \to \infty \tag{1.3}$$

uniformly for all directions $\hat{x}$, where u^∞ is the far field pattern of the scattered field u^s. We impose the Dirichlet boundary condition for a thin perfect conductor

$$u_\pm = 0 \quad \text{on } \Gamma, \tag{1.4}$$

and the mixed type boundary conditions with impedance function λ is also considered:

$$
\begin{cases} \dfrac{\partial u_-}{\partial \nu} = 0 & \text{on } \Gamma, \\ \left[\dfrac{\partial u}{\partial \nu}\right] + \lambda u_+ = 0 & \text{on } \Gamma. \end{cases} \tag{1.5}
$$

We assume that λ is a constant and satisfies $\operatorname{Re}\lambda > 0$ and $\operatorname{Im}\lambda \geqslant 0$. Notice that $u_\pm(x) = \lim\limits_{h\to 0^+} u(x \pm h\nu)$, $\partial u_\pm/\partial\nu = \lim\limits_{h\to 0^+} \nu \cdot \nabla u(x \pm h\nu)$ and $[\partial u/\partial\nu] = \partial u_+/\partial\nu - \partial u_-/\partial\nu$ for $x \in \Gamma$.

When the background media is homogeneous or a multilayer with piecewise constant index of refraction, one can use the integral equation method or the variational approach to solve the direct scattering problem, see for example [21], [4], [3], [11]. Although it seems that the well posedness of the forward problem is an intuitive conclusion, for convenience of the reader we will employ the variational method to investigate the direct scattering problems (1.1), (1.2), (1.4) and (1.1), (1.2), (1.5) in the subsequent section.

The crack inverse scattering problem was initiated in 1995 by Kress who used Newton's method to reconstruct the shape of the crack. Since then, some others categorized the qualitative methods such as the linear sampling method [4], [11], the factorization method [3], [15], the reciprocity gap functional method [21] and have developed them to recover the crack in homogeneous or piecewise homogeneous background media. In this paper, we propose to use the factorization method to reconstruct a crack surrounded by inhomogeneous media as we mentioned above.

There are two challenges we will encounter during the process of analysis of the inverse problem. The main difficulty is due to the inhomogeneous background medium. Since the far field data arise simultaneously by the crack and the background medium, thus the inversion formula for the crack should exclude the influence of the background and a modified factorization method should be built. On the other hand, in order to avoid using the background Green's function as the test function, as we will see later, a new reciprocity relationship between Green's function and the solution to an auxiliary scattering problem corresponding to the background medium should be established. Those solving ideas are reflected in the works [7], [2], where the factorization method is used to recover the obstacle in inhomogeneous and piecewise homogeneous media, respectively. Here we investigate our inverse problems with help of their method and generalize the factorization method to our cases. Another block is generated by the mixed boundary conditions attached to the crack. As we know, the key ingredient in the factorization method is to connect the detection object to the far field measurements. This can be accomplished by an appropriate decomposition of the far field operator involving a data-to-pattern operator and another auxiliary operator such that the range identity theorem [13] can be applied. We will see that the factorization method can be directly extended to the case of the Dirichlet boundary condition but not so straightforward to the mixed boundary conditions. Generally, we can use a linear sampling method to reconstruct cracks with mixed boundary conditions [4], [1], while the factorization method requires more restrictions [3], [10]. We note that the same hypothesis on the function space has been made in [3], but our proof is different from theirs.

The paper is organized as follows. Using a variational method, we show the well posedness of the direct scattering problems in Section 2. Section 3 is dedicated to some preparatory work for the reconstruction of the crack and mainly focuses on the reciprocity relationship between the background Green's function and the solution to a scattering problem of the background medium. In Section 4, we give a rigorous proof of a modified version of the factorization method for our inverse scattering problems in a nonintuitive function space. This goal is achieved by verifying the properties of some associated operators.

2. The direct scattering problems

In this section, we consider the direct scattering problems (1.1), (1.2), (1.4) and (1.1), (1.2), (1.5). In order to formulate our scattering problems more precisely we need to properly define the trace space on Γ. If $L^2(\Gamma)$, $H^{1/2}(\partial\Omega)$, and $H^{-1/2}(\partial\Omega)$ denote the usual Sobolev spaces on the closed regular curve $\partial\Omega$, we define the spaces

$$\begin{aligned}
H^{1/2}(\Gamma) &= \{u|_\Gamma\colon\ u \in H^{1/2}(\partial\Omega)\},\\
\widetilde{H}^{1/2}(\Gamma) &= \{u \in H^{1/2}(\partial\Omega)\colon\ \operatorname{supp} u \subseteq \bar{\Gamma}\},\\
H^{-1/2}(\Gamma) &= (\widetilde{H}^{1/2}(\Gamma))',\ \text{the dual space of } \widetilde{H}^{1/2}(\Gamma),\\
\widetilde{H}^{-1/2}(\Gamma) &= (H^{1/2}(\Gamma))',\ \text{the dual space of } H^{1/2}(\Gamma),
\end{aligned}$$

and we have the inclusions [17]

$$\widetilde{H}^{1/2}(\Gamma) \subset H^{1/2}(\Gamma) \subset L^2(\Gamma) \subset \widetilde{H}^{-1/2}(\Gamma) \subset H^{-1/2}(\Gamma).$$

We consider the more general problems for the scattered field u^s.

Dirichlet Crack Problem (DP): given $f \in H^{1/2}(\Gamma)$ and $p \in L^2(D \setminus \bar{\Gamma})$ find $V^d \in H^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus \bar{\Gamma})$ such that

$$(2.1)\qquad \begin{cases} \Delta V^d + k^2 n V^d = p & \text{in } \mathbb{R}^2 \setminus \bar{\Gamma},\\ V^d_{\pm} = f & \text{on } \Gamma,\\ \lim\limits_{r\to\infty} \sqrt{r}\Big(\dfrac{\partial V^d}{\partial r} - \mathrm{i}kV^d\Big) = 0, & r = |x|. \end{cases}$$

Mixed Crack Problem (MP): given $g \in H^{-1/2}(\Gamma)$, $h \in \widetilde{H}^{-1/2}(\Gamma)$ and $p \in L^2(D \setminus \bar{\Gamma})$ find $V^m \in H^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus \bar{\Gamma})$ such that

$$
\begin{cases}
\Delta V^m + k^2 n V^m = p & \text{in } \mathbb{R}^2 \setminus \bar{\Gamma}, \\
\dfrac{\partial V^m_-}{\partial \nu} = g & \text{on } \Gamma, \\
\Big[\dfrac{\partial V^m}{\partial \nu}\Big] + \lambda V^m_+ = h & \text{on } \Gamma, \\
\lim\limits_{r\to\infty} \sqrt{r}\Big(\dfrac{\partial V^m}{\partial r} - \mathrm{i}kV^m\Big) = 0, & r = |x|.
\end{cases}
\tag{2.2}
$$

Note that we extend p by zero to the whole space $\mathbb{R}^2$ and still denote it by p. We choose to adopt a variational approach in the study of the direct scattering problems. Such a method has been introduced in [3] to solve the direct scattering problem for cracks with impedance boundary conditions; after some modifications it can be applied to our problems.

Denote by B_R a sufficiently large ball with radius R containing D and by S_R its boundary. We introduce $T_R\colon H^{1/2}(S_R) \to H^{-1/2}(S_R)$, the Dirichlet to Neumann operator, defined by

$$
T_R(\varphi) = \frac{\partial \omega}{\partial \nu} \quad \text{on } S_R,
\tag{2.3}
$$

with $\omega \in H^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus \overline{B}_R)$ being the unique solution satisfying the Sommerfeld radiation condition and verifying

$$
\begin{cases}
\Delta\omega + k^2\omega = 0 & \text{in } \mathbb{R}^2 \setminus \overline{B}_R, \\
\omega = \varphi & \text{on } S_R.
\end{cases}
$$

Let $\langle , \rangle_{S_R}$ denote the duality product between $H^{1/2}(S_R)$ and $H^{-1/2}(S_R)$ that coincides with $L^2(S_R)$ scalar product for regular functions. We have the following important properties of the Dirichlet to Neumann map [5].

Lemma 1. *The Dirichlet to Neumann map T_R is a bounded linear operator from $H^{1/2}(S_R)$ to $H^{-1/2}(S_R)$. Furthermore, there exists a bounded operator T_0: $H^{1/2}(S_R) \to H^{-1/2}(S_R)$ satisfying*

$$
-\langle T_0\varphi, \varphi\rangle_{S_R} \geqslant C\|\varphi\|^2_{H^{\frac{1}{2}}(S_R)}
\tag{2.4}
$$

for some constant $C > 0$ such that $T_R - T_0\colon H^{1/2}(S_R) \to H^{-1/2}(S_R)$ is compact.

Let V_0 be the solution to the boundary value problem

$$\begin{cases} \Delta V_0 + k^2 V_0 = 0 & \text{in } \mathbb{R}^2 \setminus \bar{\Gamma}, \\ V_{0\pm} = f & \text{on } \Gamma, \\ \lim\limits_{r\to\infty} \sqrt{r}\Big(\dfrac{\partial V_0}{\partial r} - \mathrm{i}kV_0\Big) = 0, & r = |x|. \end{cases}$$

It has been proved in [4] that this problem is well posedness and the following inequality relation holds:

$$\|V_0\|_{H^1_{\mathrm{loc}}(\mathbb{R}^2\setminus\bar{\Gamma})} \leqslant C\|f\|_{H^{1/2}(\Gamma)}.$$

Let U_0 be the solution to the boundary value problem

$$\begin{cases} \Delta U_0 + k^2 U_0 = 0 & \text{in } \mathbb{R}^2 \setminus \bar{\Gamma}, \\ \frac{\partial U_{0\pm}}{\partial \nu} = g & \text{on } \Gamma, \\ \lim\limits_{r\to\infty} \sqrt{r}\Big(\dfrac{\partial U_0}{\partial r} - \mathrm{i}kU_0\Big) = 0, & r = |x|. \end{cases}$$

It is easy to prove that there exists a unique solution (ref. [20]) satisfying

$$\|U_0\|_{H^1_{\mathrm{loc}}(\mathbb{R}^2\setminus\bar{\Gamma})} \leqslant C\|g\|_{H^{-1/2}(\Gamma)}.$$

We now formulate the problems (2.1) and (2.2) as two variational problems. To this end we define the Sobolev spaces

$$\begin{aligned} X &:= \{\varphi \in H^1(B_R\setminus\bar{\Gamma})\colon\ \varphi_\pm = 0 \quad \text{on } \Gamma\}, \\ Y &:= \Big\{\varphi \in H^1(B_R\setminus\bar{\Gamma})\colon\ \frac{\partial \varphi_-}{\partial \nu} = 0 \quad \text{on } \Gamma\Big\}. \end{aligned}$$

Then $U^d := V^d - V_0 \in X$ and $U^m := V^m - U_0 \in Y$, where V^d and V^m are solutions to problems (2.1) and (2.2), respectively. Furthermore U^d satisfies

$$\begin{cases} \Delta U^d + k^2 n U^d = p - k^2(n-1)V_0 & \text{in } B_R\setminus\bar{\Gamma}, \\ U^d_\pm = 0 & \text{on } \Gamma, \\ \dfrac{\partial U^d}{\partial \nu} = T_R V^d - \dfrac{\partial V_0}{\partial \nu} & \text{on } S_R \end{cases} \tag{2.5}$$

and U^m satisfies

$$\begin{cases} \Delta U^m + k^2 n U^m = p - k^2(n-1)U_0 & \text{in } B_R\setminus\bar{\Gamma}, \\ \dfrac{\partial U^m_-}{\partial \nu} = 0 & \text{on } \Gamma, \\ \Big[\dfrac{\partial U^m}{\partial \nu}\Big] + \lambda U^m_+ = h - \lambda U_{0+} & \text{on } \Gamma, \\ \dfrac{\partial U^m}{\partial \nu} = T_R V^m - \dfrac{\partial U_0}{\partial \nu} & \text{on } S_R. \end{cases} \tag{2.6}$$

Therefore, U^d is a solution of problem (2.5) if and only if $U^d \in X$ and satisfies for all $\varphi \in X$

$$
\begin{aligned}
(2.7)\qquad & \int_{B_R\setminus\bar\Gamma} (\nabla U^d \cdot \nabla\bar\varphi - k^2 n U^d \bar\varphi)\,\mathrm{d}x - \int_{S_R} T_R U^d \bar\varphi\,\mathrm{d}s \\
&\quad = \int_{D\setminus\bar\Gamma} \{k^2(n-1)V_0 - p\}\bar\varphi\,\mathrm{d}x + \int_{S_R}\Big(T_R V_0 - \frac{\partial V_0}{\partial\nu}\Big)\bar\varphi\,\mathrm{d}s,
\end{aligned}
$$

and U^m is a solution of problem (2.6) if and only if $U^m \in Y$ and satisfies for all $\varphi \in Y$

$$
\begin{aligned}
(2.8)\qquad & \int_{B_R\setminus\bar\Gamma} (\nabla U^m \cdot \nabla\bar\varphi - k^2 n U^m \bar\varphi)\,\mathrm{d}x - \int_{S_R} T_R U^m \bar\varphi\,\mathrm{d}s - \lambda\int_\Gamma U^m_+\bar\varphi_+\,\mathrm{d}s \\
&\quad = \int_{D\setminus\bar\Gamma} \{k^2(n-1)U_0 - p\}\bar\varphi\,\mathrm{d}x + \int_{S_R}\Big(T_R U_0 - \frac{\partial U_0}{\partial\nu}\Big)\bar\varphi\,\mathrm{d}s \\
&\quad\quad - \int_\Gamma h\bar\varphi\,\mathrm{d}s + \lambda\int_\Gamma U_{0+}\bar\varphi_+\,\mathrm{d}s,
\end{aligned}
$$

where the third integral of the above identity over Γ is interpreted as the duality pairing between $\widetilde{H}^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$, while the fourth is an inner product of $L^2(\Gamma)$.

Theorem 2. *Assume that* $f \in H^{1/2}(\Gamma)$, $g \in H^{-1/2}(\Gamma)$, $h \in \widetilde{H}^{-1/2}(\Gamma)$ *and* $p \in L^2(D\setminus\bar\Gamma)$. *Then the variational formulas* (2.7) *and* (2.8) *have a unique solution* $U^d \in X$ *and* $U^m \in Y$, *respectively, which depend on the boundary data and the source.*

Proof. Since this is a classical exercise, we will give here only the proof for variational formula (2.8), and the assertion for (2.7) can be obtained similarly. To this end, we denote by $A(U^m,\varphi)$ an operator associated with the right-hand side of (2.8) and by $l(\varphi)$ the one associated with the left-hand side.

Noting that $\big[\partial U_0/\partial\nu\big] = 0$ on Γ, Green's first identity for U_0 and φ in the domain $B_R\setminus\bar\Gamma$ implies that

$$
-\int_{S_R}\frac{\partial U_0}{\partial\nu}\bar\varphi\,\mathrm{d}s = -\int_\Gamma \frac{\partial U_0}{\partial\nu}[\bar\varphi]\,\mathrm{d}s - \int_{B_R\setminus\bar\Gamma}\nabla U_0\cdot\nabla\bar\varphi\,\mathrm{d}x + \int_{B_R\setminus\bar\Gamma} k^2 U_0\cdot\bar\varphi\,\mathrm{d}x.
$$

Hence, by the Cauchy-Schwarz inequality, the trace theorem, the properties of the Dirichlet to Neumann map T_R, and the assumptions on n and λ, we have that

$$
\begin{aligned}
|l(\varphi)| &\leqslant C_1(\|U_0\|_{H^1(B_R\setminus\bar\Gamma)} + \|p\|_{L^2(D\setminus\bar\Gamma)})\|\varphi\|_{H^1(B_R\setminus\bar\Gamma)} + C_2\|U_0\|_{H^{1/2}(S_R)}\|\varphi\|_{H^{1/2}(S_R)} \\
&\quad + C_3\|g\|_{H^{-1/2}(\Gamma)}\|[\varphi]\|_{\widetilde{H}^{1/2}(\Gamma)} + C_4\|h\|_{\widetilde{H}^{-1/2}(\Gamma)}\|\varphi\|_{H^{1/2}(\Gamma)} \\
&\quad + C_5\|U_0\|_{H^{1/2}(\Gamma)}\|\varphi\|_{H^{1/2}(\Gamma)} \\
&\leqslant C(\|p\|_{L^2(D\setminus\bar\Gamma)} + \|g\|_{H^{-1/2}(\Gamma)} + \|h\|_{\widetilde{H}^{-1/2}(\Gamma)})\|\varphi\|_{H^1(B_R\setminus\bar\Gamma)},
\end{aligned}
$$

which shows that l is a bounded conjugate linear functional.

The remaining part is to prove that $A(U^m,\varphi)$ is invertible by the Fredholm theorem. We decompose A into a coercive part

$$A_0(U^m,\varphi)=\int_{B_R\setminus\bar\Gamma}(\nabla U^m\cdot\nabla\bar\varphi+U^m\bar\varphi)\,\mathrm{d}x-\int_{S_R}T_0U^s\bar\varphi\,\mathrm{d}s,$$

and a compact one

$$A_1(U^m,\varphi)=-\int_{B_R\setminus\bar\Gamma}(k^2n+1)U^m\bar\varphi\,\mathrm{d}x-\int_{S_R}(T_R-T_0)U^m\bar\varphi\,\mathrm{d}s-\lambda\int_\Gamma U^m_+\bar\varphi_+\,\mathrm{d}s.$$

The coercivity of $A_0(\cdot,\cdot)$ directly follows from the property of T_0, while the compactness of $A_1(\cdot,\cdot)$ follows from the properties of T_R-T_0, the trace theorem, the Rellich compact imbedding theorem and the assumption on λ. Thus the operator A is a Fredholm operator of index 0 and then we just need to show its injectivity. That is the uniqueness of a weak solution to the problem (2.2) with homogeneous equation and boundary conditions.

Indeed, letting the data p, g, and h in the problem (2.2) equal zero and using Green's first identity for V^m, we get

$$\begin{aligned}\int_{B_R\setminus\bar\Gamma}&(\nabla V^m\cdot\nabla\overline{V}^m-k^2n|V^m|^2)\,\mathrm{d}x\\&=\int_{S_R}V^m\frac{\partial\overline{V}^m}{\partial\nu}\,\mathrm{d}s-\int_\Gamma V^m_+\frac{\partial\overline{V}^m_+}{\partial\nu}\,\mathrm{d}s+\int_\Gamma V^m_-\frac{\partial\overline{V}^m_-}{\partial\nu}\,\mathrm{d}s.\end{aligned}$$

By using the boundary conditions in problem (2.2), we have

$$\int_{S_R}V^m\frac{\partial\overline{V}^m}{\partial\nu}\,\mathrm{d}s=\int_{B_R\setminus\bar\Gamma}(|\nabla V^m|^2-k^2n|V^m|^2)\,\mathrm{d}x-\bar\lambda\int_\Gamma|V^m_+|^2\,\mathrm{d}s.$$

Hence, we obtain that by the assumption $\operatorname{Im}\lambda\geqslant0$

$$\operatorname{Im}\int_{\partial B_R}V^m\frac{\partial\overline{V}^m}{\partial\nu}\,\mathrm{d}s=-\bar\lambda\int_\Gamma|V^m_+|^2\,\mathrm{d}s\geqslant0.$$

So, from Rellich's lemma and the unique continuation principle we obtain that $V^m=0$ in $\mathbb{R}^2\setminus\bar\Gamma$. The proof is then completed. □

Remark 1. Theorem 2 implies that the *Dirichlet Crack Problem* and *Mixed Crack Problem* both have a unique solution, moreover, there exists a positive constant C depending on B_R, n and λ such that

$$\|U^d\|_{H^1(B_R\setminus\bar\Gamma)}\leqslant C(\|f\|_{H^{1/2}(\Gamma)}+\|p\|_{L^2(D\setminus\bar\Gamma)}),\tag{2.9}$$

$$\|U^m\|_{H^1(B_R\setminus\bar\Gamma)}\leqslant C(\|g\|_{H^{-1/2}(\Gamma)}+\|h\|_{\widetilde{H}^{-1/2}(\Gamma)}+\|p\|_{L^2(D\setminus\bar\Gamma)}).\tag{2.10}$$

3. The preliminaries for inverse problems

In this part, we will give two theorems which will play an important role in our inverse problems. First, we establish a mixed reciprocity relation for the scattering by an inhomogeneous media. Despite the reciprocity principle being a common conclusion in the scattering theory, there is no relevant one about inhomogeneous media which is required in this paper. Some related results can be found in [18], [7], [19]. Secondly, a connection between an outgoing wave and incoming wave, associated with the background inhomogeneous media, will also be proved. Although this relation has been obtained in [9], we follow closely the ideas in [2] and provide a simpler and direct version.

Let $K(\cdot,\cdot)$ be the Green function of the background media, i.e., $K(\cdot,z) \in H^1_{\rm loc}(\mathbb{R}^2 \setminus \{z\})$ solves

$$\begin{cases} \Delta K(\cdot,z) + k^2 n K(\cdot,z) = \delta(\cdot - z) & \text{in } \mathbb{R}^2 \setminus \{z\}, \\ \lim\limits_{r\to\infty} \sqrt{r}\Big(\dfrac{\partial K(\cdot,z)}{\partial r} - \mathrm{i}kK(\cdot,z)\Big) = 0, & r = |x|. \end{cases}$$

We denote by $K^\infty(\hat{x},z)$ the far field pattern of the Green function $K(x,z)$ and consider the scattering problem by the background media due to the incident plane wave $u^i = \mathrm{e}^{-\mathrm{i}kz\cdot\hat{x}}$

$$\begin{cases} \Delta u_0 + k^2 n u_0 = 0 & \text{in } \mathbb{R}^2, \\ \lim\limits_{r\to\infty} \sqrt{r}\Big(\dfrac{\partial u_0^s}{\partial r} - \mathrm{i}ku_0^s\Big) = 0, & r = |x|, \end{cases} \tag{3.1}$$

where $u_0(z,-\hat{x})$ at $z \in \mathbb{R}^2$ is the total field which is the sum of the scattered field $u_0^s(z,-\hat{x})$ and the incident wave $u^i(z,-\hat{x})$ with incident direction $-\hat{x} \in S^1$ (the unit disc in $\mathbb{R}^2$). Then we have the following mixed reciprocity relation.

Theorem 3 (Mixed reciprocity relation). *For $-\hat{x} \in S^1$ and $\gamma = \mathrm{e}^{\mathrm{i}\pi/4}/\sqrt{8k\pi}$ we have*

$$K^\infty(\hat{x},z) = \gamma u_0(z,-\hat{x}), \quad z \in \mathbb{R}^2 \setminus \partial D. \tag{3.2}$$

P r o o f. We first consider $z \in \mathbb{R}^2 \setminus \overline{D}$. Assume that $\Phi(\cdot,\cdot)$ is the fundamental solution to the Helmholtz equation $\Delta u + k^2 u = 0$, which is defined by

$$\Phi(x,z) = \frac{\mathrm{i}}{4} H_0^{(1)}(k|x-z|)$$

with $H_0^{(1)}$ being the Hankel function of the first kind of order zero. Since the difference $K(\cdot, z) - \Phi(\cdot, z)$ is a smooth radiation solution to the Helmholtz equation in $\mathbb{R}^2 \setminus \overline{D}$, Green's representation formula shows us that

$$\begin{aligned} &K(x,z) - \Phi(x,z) \\ &\quad = \int_{\partial D} \Big([K(y,z) - \Phi(y,z)] \frac{\partial \Phi(x,y)}{\partial \nu(y)} - \frac{\partial [K(y,z) - \Phi(y,z)]}{\partial \nu(y)} \Phi(x,y) \Big) \, \mathrm{d}s(y) \end{aligned}$$

for $x \in \mathbb{R}^2 \setminus \overline{D}$. Applying Green's second theorem to Φ in the domain D, we have

$$0 = \int_{\partial D} \Big(\Phi(y,z) \frac{\partial \Phi(x,y)}{\partial \nu(y)} - \frac{\partial \Phi(y,z)}{\partial \nu(y)} \Phi(x,y) \Big) \, \mathrm{d}s(y)$$

for $x \in \mathbb{R}^2 \setminus \overline{D}$. The sum of these two equations yields

$$\begin{aligned} &K(x,z) - \Phi(x,z) \\ &\quad = \int_{\partial D} \Big(K(y,z) \frac{\partial \Phi(x,y)}{\partial \nu(y)} - \frac{\partial K(y,z)}{\partial \nu(y)} \Phi(x,y) \Big) \, \mathrm{d}s(y), \quad x \in \mathbb{R}^2 \setminus \overline{D}. \end{aligned} \tag{3.3}$$

On the other hand, from the Green theorem and the Sommerfeld radiation condition we obtain that

$$0 = \int_{\partial D} \Big([K(y,z) - \Phi(y,z)] \frac{\partial u_0^s(y, -\hat{x})}{\partial \nu(y)} - \frac{\partial [K(y,z) - \Phi(y,z)]}{\partial \nu(y)} u_0^s(y, -\hat{x}) \Big) \, \mathrm{d}s(y)$$

in the domain $\mathbb{R}^2 \setminus \overline{D}$. Green's representation formula for u_0^s outside D implies that

$$u_0^s(z, -\hat{x}) = \int_{\partial D} \Big(u_0^s(y, -\hat{x}) \frac{\partial \Phi(y,z)}{\partial \nu(y)} - \frac{\partial u_0^s(y, -\hat{x})}{\partial \nu(y)} \Phi(y,z) \Big) \, \mathrm{d}s(y).$$

Adding the above two equations yields

$$u_0^s(z, -\hat{x}) = \int_{\partial D} \Big(u_0^s(y, -\hat{x}) \frac{\partial K(y,z)}{\partial \nu(y)} - \frac{\partial u_0^s(y, -\hat{x})}{\partial \nu(y)} K(y,z) \Big) \, \mathrm{d}s(y).$$

Using Green's second theorem for u_0 and K in D, we see that

$$\begin{aligned} &\int_{\partial D} \Big(u_0(y, -\hat{x}) \frac{\partial K(y,z)}{\partial \nu(y)} - \frac{\partial u_0(y, -\hat{x})}{\partial \nu(y)} K(y,z) \Big) \, \mathrm{d}s(y) \\ &\qquad = \int_D (\Delta u_0(y, -\hat{x}) K(y,z) - u_0(y, -\hat{x}) \Delta K(y,z)) \, \mathrm{d}y = 0. \end{aligned}$$

Subtracting these two equations we arrive at

$$u_0^s(z,-\hat{x}) = \int_{\partial D} \Big(\frac{\partial u^i(y,-\hat{x})}{\partial \nu(y)} K(y,z) - u^i(y,-\hat{x}) \frac{\partial K(y,z)}{\partial \nu(y)}\Big)\, \mathrm{d}s(y), \quad z \in \mathbb{R}^2 \setminus \overline{D}. \tag{3.4}$$

Then we obtain the mixed reciprocity relation from the equations (3.3) and (3.4).

Next, we consider the case for $z \in D$. Since $K(y,z)$ is a smooth radiation solution to the Helmholtz equation in $\mathbb{R}^2 \setminus \overline{D}$, we have

$$K^\infty(\hat{x},z) = \gamma \int_{\partial D} \Big(K(y,z) \frac{\partial \mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y}}{\partial \nu(y)} - \frac{\partial K(y,z)}{\partial \nu(y)} \mathrm{e}^{-\mathrm{i}k\hat{x}\cdot y}\Big)\, \mathrm{d}s(y).$$

Applying Green's second theorem for $u_0(y,-\hat{x})$ and $K(y,z)$ in D, we observe that

$$\begin{aligned}\int_{\partial D} \Big(u_0(y,-\hat{x}) \frac{\partial K(y,z)}{\partial \nu(y)} - \frac{\partial u_0(y,-\hat{x})}{\partial \nu(y)} K(y,z)\Big)\, \mathrm{d}s(y) \\ = \int_D (\Delta u_0(y,-\hat{x}) K(y,z) - u_0(y,-\hat{x}) \Delta K(y,z))\, \mathrm{d}y \\ = -\int_D \delta(y-z) u_0(y,-\hat{x})\, \mathrm{d}y = -u_0(z,-\hat{x}).\end{aligned}$$

Noting that $u_0^s(y,-\hat{x})$ and $K(y,z)$ are the radiating solution to the Helmholtz equation in $\mathbb{R}^2 \setminus \overline{D}$, Green's theorem shows that

$$0 = \int_{\partial D} \Big(u_0^s(y,-\hat{x}) \frac{\partial K(y,z)}{\partial \nu(y)} - \frac{\partial u_0^s(y,-\hat{x})}{\partial \nu(y)} K(y,z)\Big)\, \mathrm{d}s(y).$$

The subtraction of these two equations yields that

$$\gamma u_0(z,-\hat{x}) = \gamma \int_{\partial D} \Big(\frac{\partial u^i(y,-\hat{x})}{\partial \nu(y)} K(y,z) - u^i(y,-\hat{x}) \frac{\partial K(y,z)}{\partial \nu(y)}\Big)\, \mathrm{d}s(y), \tag{3.5}$$

which is just $K^\infty(\hat{x},z)$. So, we complete the proof of Theorem 3. $\square$

The far field patterns $u_0^\infty(\hat{x},d)$, $\hat{x},d \in S^1$, corresponding to the incident plane waves $u^i(x,d)$ define the far field operator $F_0\colon\ L^2(S^1) \to L^2(S^1)$ by

$$(F_0 g)(\hat{x}) = \int_{S^1} u_0^\infty(\hat{x},d) g(d)\, \mathrm{d}s(d). \tag{3.6}$$

Let us define the scattering operator $\mathcal{S}_0\colon\ L^2(S^1) \to L^2(S^1)$ by

$$\mathcal{S}_0 = I + 2\mathrm{i}k|\gamma|^2 F_0,$$

where I denotes the identity operator and $\gamma = \mathrm{e}^{\mathrm{i}\pi/4}/\sqrt{8k\pi}$. Then by Theorem 4.4 in [13], we see that $\mathcal{S}_0$ is unitary, that is $\mathcal{S}_0^* \mathcal{S}_0 = \mathcal{S}_0 \mathcal{S}_0^* = I$. Furthermore, we have the following statement.

Theorem 4. *For all $z \in D$, $\hat{x} \in S^1$, we have*

$$u_0(z, -\hat{x}) = (\mathcal{S}_0 \overline{u_0(z, \cdot)})(\hat{x}). \tag{3.7}$$

P r o o f. Let B_R be a sufficiently large ball with radius R containing D and y for some point $y \in \mathbb{R}^2 \setminus \overline{D}$, and denote by S_R the boundary of B_R. In such a case, noting that $K(y, z) - \overline{K(y, z)}$ is smooth solutions of the Helmholtz equation for $y \in \mathbb{R}^2 \setminus \overline{D}$, by Green's representation formula, we have

$$\begin{aligned} K(y, z) &- \overline{K(y, z)} \\ &= \int_{S_R} \Big(\frac{\partial [K(x, z) - \overline{K(x, z)}]}{\partial \nu(x)} K(x, y) - [K(x, z) - \overline{K(x, z)}] \frac{\partial K(x, y)}{\partial \nu(x)} \Big) \, \mathrm{d}s(x) \\ &\quad - \int_{\partial D} \Big(\frac{\partial [K(x, z) - \overline{K(x, z)}]}{\partial \nu(x)} K(x, y) - [K(x, z) - \overline{K(x, z)}] \frac{\partial K(x, y)}{\partial \nu(x)} \Big) \, \mathrm{d}s(x). \end{aligned}$$

Applying Green's second theorem to $K(\cdot, \cdot)$ outside the domain B_R and noting the radiation condition, we have

$$0 = \int_{S_R} \Big(K(x, z) \frac{\partial K(x, y)}{\partial \nu(x)} - \frac{\partial K(x, z)}{\partial \nu(x)} K(x, y) \Big) \, \mathrm{d}s(x).$$

Since $K(x, z) - \overline{K(x, z)}$ and $K(x, y)$ are smooth solutions in D, by Green's theorem we see that

$$\begin{aligned} \int_{\partial D} &\Big(\frac{\partial [K(x, z) - \overline{K(x, z)}]}{\partial \nu(x)} K(x, y) - [K(x, z) - \overline{K(x, z)}] \frac{\partial K(x, y)}{\partial \nu(x)} \Big) \, \mathrm{d}s(x) \\ &= \int_D (\Delta [K(x, z) - \overline{K(x, z)}] K(x, y) - [K(x, z) - \overline{K(x, z)}] \Delta K(x, y)) \, \mathrm{d}x = 0. \end{aligned}$$

The sum of this three equations yields

$$\begin{aligned} K(y, z) - \overline{K(y, z)} &= - \int_{S_R} \Big(K(x, y) \frac{\partial \overline{K(x, z)}}{\partial \nu(x)} + \frac{\partial K(x, y)}{\partial \nu(x)} \overline{K(x, z)} \Big) \, \mathrm{d}s(y) \\ &= 2\mathrm{i}k |\gamma|^2 \int_{S^1} K^\infty(\hat{x}, y) \overline{K^\infty(\hat{x}, z)} \, \mathrm{d}s(\hat{x}), \quad y \in \mathbb{R}^2 \setminus \overline{D}, \end{aligned} \tag{3.8}$$

where the second identity is obtained from the Sommerfeld radiation condition.

On the other hand, we have from (3.5) for $z \in D$

$$\overline{u_0(z, \hat{x})} = \int_{\partial D} \Big(\frac{\partial u^i(y, -\hat{x})}{\partial \nu(y)} \overline{K(y, z)} - u^i(y, -\hat{x}) \frac{\partial \overline{K(y, z)}}{\partial \nu(y)} \Big) \, \mathrm{d}s(y).$$

By using Theorem 3 and the equation (3.8), the above identity and (3.5) show that

$$\begin{aligned}
&u_0(z,-\hat{x})-\overline{u_0(z,\hat{x})}\\
&=\int_{\partial D}\Big(\frac{\partial u^i(y,-\hat{x})}{\partial\nu(y)}[K(y,z)-\overline{K(y,z)}]-u^i(y,-\hat{x})\frac{\partial[K(y,z)-\overline{K(y,z)}]}{\partial\nu(y)}\Big)\,\mathrm{d}s(y)\\
&=2\mathrm{i}k|\gamma|^2\int_{S^1}\overline{u_0(z,-\alpha)}\int_{\partial D}\Big(\frac{\partial u^i(y,-\hat{x})}{\partial\nu(y)}K^\infty(\alpha,y)-u^i(y,-\hat{x})\frac{\partial K^\infty(\alpha,y)}{\partial\nu(y)}\,\mathrm{d}s(y)\Big)\,\mathrm{d}s(\alpha)\\
&=2\mathrm{i}k|\gamma|^2\int_{S^1}\overline{u_0(z,-\alpha)}u_0^\infty(\alpha,-\hat{x})\,\mathrm{d}s(\alpha),
\end{aligned}$$

where the third identity is obtained by the far field pattern of $u_0^s(\cdot,-\hat{x})$ in accordance with (3.4). Furthermore, due to the relation $u_0^\infty(\alpha,-\hat{x})=u_0^\infty(\hat{x},-\alpha)$ (see for example [13]) and the definition of F_0, we see that

$$\begin{aligned}
(3.9)\qquad u_0(z,-\hat{x})-\overline{u_0(z,\hat{x})}&=2\mathrm{i}k|\gamma|^2\int_{S^1}\overline{u_0(z,-\alpha)}u_0^\infty(\hat{x},-\alpha)\,\mathrm{d}s(\alpha)\\
&=2\mathrm{i}k|\gamma|^2\int_{S^1}\overline{u_0(z,\alpha)}u_0^\infty(\hat{x},\alpha)\,\mathrm{d}s(\alpha)=2\mathrm{i}k|\gamma|^2(F_0\overline{u_0(z,\cdot)})(\hat{x}).
\end{aligned}$$

The proof is thus completed by the definition of $\mathcal{S}_0$. □

Remark 2. We note that Theorem 4 is not true for $z\in\mathbb{R}^2\setminus\overline{D}$, but this conclusion is sufficient for us, since we just need to consider testing points $z\in D$ (known a priori) for our inverse crack problems.

4. The factorization method

The inverse problems we are considering are, in the situation of knowing the background medium in advance, to determine the 'Dirichlet crack' and 'Mixed crack', named the *Inverse Dirichlet Crack Problem* (IDP) and *Inverse Mixed Crack Problem* (IMP), respectively, from the knowledge of the far field pattern $u^\infty(\hat{x},d)$ of the corresponding scattered field $u^s(x,d)$ for all $\hat{x},d\in S^1$. We recall that u^s is the scattered solution of problem (1.1), (1.2), (1.4) or problem (1.1), (1.2), (1.5) from the incident plane waves $u^i=\mathrm{e}^{\mathrm{i}kx\cdot d}, d\in S^1$. This section aims at establishing the theoretical foundation of the factorization method for the above two inverse problems.

For the sake of reader's convenience, here we state the well-known range identity theorem [16], which is the theoretical basis of the factorization method. For a generic bounded linear operator A between two Banach spaces, we define its real and imaginary parts by $\mathrm{Re}(A)=\frac{1}{2}(A+A^*)$ and $\mathrm{Im}(A)=\frac{1}{2\mathrm{i}}(A-A^*)$, respectively, where A^* is the adjoint of A.

Lemma 5. *Let $X \subset U \subset X^*$ be a Gelfand triple with a Hilbert space U and a reflexive Banach space X such that the embeddings are dense. Moreover, let Y be another Hilbert space and $F\colon Y \to Y$, $H\colon Y \to X$, and $T\colon X \to X^*$ linear bounded operators such that*

$$F = H^*TH. \tag{4.1}$$

We make the following assumptions:
(a) *H^* is compact with dense range.*
(b) *$\mathrm{Re}(T) = T_0 + T_1$ with some compact operator T_1 and some self-adjoint and coercive operator $T_0\colon X \to X^*$, i.e., there exists $c > 0$ such that*

$$\langle \varphi, T_0\varphi \rangle \geqslant c\|\varphi\|^2 \quad \forall\, \varphi \in X,$$

where $\langle \cdot, \cdot \rangle$ is the duality pair between X and X^*.
(c) *$\mathrm{Im}(T)$ is non-negative on X or $\mathrm{Im}(T)$ is non-positive on X, i.e.,*

$$\langle \mathrm{Im}(T)\varphi, \varphi \rangle \geqslant 0 \quad \text{or} \quad \langle \mathrm{Im}(T)\varphi, \varphi \rangle \leqslant 0 \quad \forall\ \varphi \in X.$$

(d) *T is injective.*
Then the operator $F_\sharp := |\mathrm{Re}(F)| + |\mathrm{Im}(F)|$ is positive and the ranges of $H^*\colon X^* \to Y$ and $F_\sharp^{1/2}\colon Y \to Y$ coincide.

4.1. Inverse Dirichlet Crack Problem. Recall that u is the solution to problem (1.1), (1.2), (1.4) and u_0 is the solution of problem (3.1). It is easy to verify that the field $u_\Gamma^s := u - u_0$ solves the following boundary value problem with $\eta = -u_0$:

$$\begin{cases} \Delta u_\Gamma^s + k^2 n u_\Gamma^s = 0 & \text{in } \mathbb{R}^2 \setminus \bar{\Gamma}, \\ u_{\Gamma\pm}^s = \eta & \text{on } \Gamma, \\ \lim\limits_{r\to\infty} \sqrt{r}\Big(\dfrac{\partial u_\Gamma^s}{\partial r} - \mathrm{i}k u_\Gamma^s\Big) = 0, & r = |x|. \end{cases} \tag{4.2}$$

Define the data-to-pattern operator $G\colon H^{1/2}(\Gamma) \to L^2(S^1)$ by

$$(G\eta)(\hat{x}) = u_\Gamma^\infty(\hat{x}), \tag{4.3}$$

where $u_\Gamma^\infty(\hat{x})$ is the far field pattern of the scattered field u_Γ^s of problem (4.2). The auxiliary operator $H\colon L^2(S^1) \to H^{1/2}(\Gamma)$ is given as

$$(Hg)(x) = \int_{S^1} u_0(x, d) g(d)\, \mathrm{d}s(d), \quad x \in \Gamma, \tag{4.4}$$

where u_0 is the solution of (3.1) corresponding to the incident wave $u^i(\cdot, d)$. The far field operator $F\colon L^2(S^1) \to L^2(S^1)$ is defined as

$$(Fg)(\hat{x}) = \int_{S^1} u^\infty(\hat{x}, d) g(d)\, \mathrm{d}s(d), \quad g \in L^2(S^1), \tag{4.5}$$

where u^∞ is the far field pattern of the scattered wave u^s of problem (1.1), (1.2), (1.4).

Noticing the definition of F_0 (see (3.6)), we deduce that

$$[(F - F_0)g](\hat{x}) = \int_{S^1} [u^\infty(\hat{x}, d) - u_0^\infty(\hat{x}, d)] g(d)\, \mathrm{d}s(d)$$

is just the far field pattern of the radiating function

$$\begin{aligned}\int_{S^1} &\{[u(x, d) - u^i(x, d)] - [u_0(x, d) - u^i(x, d)]\} g(d)\, \mathrm{d}s(d)\\ &= \int_{S^1} [u(x, d) - u_0(x, d)] g(d)\, \mathrm{d}s(d) = \int_{S^1} u_\Gamma^s(x, d) g(d)\, \mathrm{d}s(d).\end{aligned}$$

Due to the boundary condition (4.2), i.e., $u_\Gamma^s = -u_0$, the operator $F - F_0$ can be factorized as follows by the definition of G and H:

$$F - F_0 = -GH. \tag{4.6}$$

Next, we continue to transform the relation (4.7) so that it possesses the essential form of (4.1) on which the factorization method is based.

Note that the adjoint operator H^* of H is $H^*\colon \widetilde{H}^{-1/2}(\Gamma) \to L^2(S^1)$ with

$$(H^*\mu)(\hat{x}) = \int_\Gamma \overline{u_0(z, \hat{x})} \mu(z)\, \mathrm{d}s(z), \quad \mu \in \widetilde{H}^{-1/2}(\Gamma). \tag{4.7}$$

Due to the property of the scattered operator $\mathcal{S}_0$ (Theorem 4) and the mixed reciprocity relation (3.2) in Theorem 3, we have

$$\begin{aligned}(\mathcal{S}_0 H^*\mu)(\hat{x}) &= \left(\mathcal{S}_0 \int_\Gamma \overline{u_0(z, \cdot)} \mu(z)\, \mathrm{d}s(z)\right)(\hat{x})\\ &= \int_\Gamma u_0(z, -\hat{x}) \mu(z)\, \mathrm{d}s(z)\\ &= \int_\Gamma K^\infty(\hat{x}, z) \mu(z)\, \mathrm{d}s(z).\end{aligned}$$

Define the single-layer potential function

$$v(x) = \int_\Gamma K(x, z) \mu(z)\, \mathrm{d}s(z), \quad x \in \mathbb{R}^2 \setminus \bar{\Gamma}. \tag{4.8}$$

Let $S_\Gamma\colon \widetilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ be the single-layer operator given by

$$(4.9) \qquad (S_\Gamma \mu)(x) = \int_\Gamma K(x,z)\mu(z)\,\mathrm{d}s(z), \quad x \in \Gamma.$$

Then v satisfies the problem (4.2) with boundary data $S_\Gamma\mu$, its far field pattern being exactly $\mathcal{S}_0 H^*\mu$. By the definition of the operator G, we have $GS_\Gamma = \mathcal{S}_0 H^*$ and thus $H = S_\Gamma^* G^* \mathcal{S}_0$. So the equation (4.6) can be changed into

$$(4.10) \qquad (F_0 - F)\mathcal{S}_0^* = GS_\Gamma^* G^*.$$

The properties of the operator S_Γ are stated as follows.

Theorem 6. (a) *Let S_i be the single-layer operator corresponding to the wave number $k = i$. Then S_i is self-adjoint and coercive, i.e.,*

$$\langle \mu, S_i\mu\rangle \geqslant c\|\mu\|^2_{\widetilde{H}^{-1/2}(\Gamma)} \quad \forall\, \mu \in \widetilde{H}^{-1/2}(\Gamma),$$

where $\langle\cdot,\cdot\rangle$ is the duality pair between $\widetilde{H}^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$.

(b) *The difference $S_\Gamma - S_i$ is compact from $\widetilde{H}^{-1/2}(\Gamma)$ into $H^{1/2}(\Gamma)$.*

(c) *S_Γ is an isomorphism from $\widetilde{H}^{-1/2}(\Gamma)$ onto $H^{1/2}(\Gamma)$.*

(d) $\operatorname{Im}\langle S_\Gamma\mu, \mu\rangle > 0$ *for all $\mu \in \widetilde{H}^{-1/2}(\Gamma)$ with $\mu \neq 0$.*

Proof. (a) Set $k = i$ in the definition of Green's function and denote by $K_i(\cdot,\cdot)$ the Green function in such a case and by S_i the associated single-layer operator. Let the single-layer potential v be defined by (4.8) with $K(\cdot,\cdot)$ replaced by $K_i(\cdot,\cdot)$. Note that the crack Γ can be extended to an arbitrary smooth, simply connected, closed curve $\partial\Omega$ enclosing a bounded domain Ω which is completely contained in D.

Applying Green's first identity to v and $\overline{v}$ in Ω and $B_R \setminus \bar\Omega$ (a disk of radius R containing D with boundary S_R) and using the jump relation of the single-layer potential, we have that

$$\begin{aligned}
\langle \mu, S_i\mu\rangle &= \Big\langle \frac{\partial \overline{v}_-}{\partial\nu} - \frac{\partial \overline{v}_+}{\partial\nu}, v\Big\rangle_{\partial\Omega} = \int_{\Omega\cup\{B_R\setminus\bar\Omega\}} (|\nabla v|^2 + n|v|^2)\,\mathrm{d}x - \int_{S_R} v\frac{\partial\overline{v}}{\partial\nu}\,\mathrm{d}s \\
&= \int_{\Omega\cup\{B_R\setminus\bar\Omega\}} (|\nabla v|^2 + n|v|^2)\,\mathrm{d}x + \int_{S_R} |v|^2\,\mathrm{d}s + o(1) \\
&= \int_{\mathbb{R}^2\setminus\partial\Omega} (|\nabla v|^2 + n|v|^2)\,\mathrm{d}x \\
&\geqslant \widetilde{c}\|v\|^2_{H^1(\mathbb{R}^2\setminus\partial\Omega)} \geqslant c\Big\|\frac{\partial v_-}{\partial\nu} - \frac{\partial v_+}{\partial\nu}\Big\|_{H^{-1/2}(\partial\Omega)} = c\|\mu\|_{H^{-1/2}(\Gamma)}.
\end{aligned}$$

The first identity is true since $\mu \in \widetilde{H}^{-1/2}(\Gamma)$ can be extended to the boundary $\partial\Omega$ by zero and the trace of the potential function v on $\partial\Omega$ belongs to $H^{1/2}(\Gamma)$. The third identity is deduced by the radiation condition, and the forth is due to the fact that v decays exponentially.

(b) The potential function ω defined by

$$\omega(x) = \int_\Gamma (K(x,z) - K_i(x,z))\mu(z)\,\mathrm{d}s(z), \quad \mu \in \widetilde{H}^{-1/2}(\Gamma),\ x \in \mathbb{R}^2 \setminus \bar{\Gamma},$$

is a smooth function of $C^\infty(\mathbb{R}^2 \setminus \bar{\Gamma})$, furthermore, ω and $\frac{\partial\omega}{\partial\nu}$ have no jump across the crack Γ due to the continuous integral kernel, which results in ω belonging to $C(\mathbb{R}^2)$. Therefore, the potential operator

$$((S_\Gamma - S_i)\mu)(x) = \int_\Gamma (K(x,z) - K_i(x,z))\mu(z)\,\mathrm{d}s(z), \quad \mu \in \widetilde{H}^{-1/2}(\Gamma),\ x \in \Gamma,$$

is a bounded operator from $\widetilde{H}^{-1/2}(\Gamma)$ into $C(\Gamma)$ and certainly bounded from $\widetilde{H}^{-1/2}(\Gamma)$ into $L^2(\Gamma)$. By the compact embedding theorem we obtain that $S_\Gamma - S_i\colon \widetilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is compact.

(c) We conclude that $S_i^{-1}\colon H^{1/2}(\Gamma) \to \widetilde{H}^{-1/2}(\Gamma)$ exists and is bounded from the first part. By the Fredholm theorem and using the result in part 2, S_Γ is an isomorphism if and only if S_Γ is injective. Now let $S_\Gamma\mu = 0$ for $\mu \in \widetilde{H}^{-1/2}(\Gamma)$, then the potential v defined by (4.8) satisfies problem (4.2) with boundary data $\eta = 0$. By the well posedness of this problem, we deduce that $v = 0$ in $\mathbb{R}^2 \setminus \bar{\Gamma}$. The jump relation shows that

$$\mu = \frac{\partial v_-}{\partial\nu} - \frac{\partial v_+}{\partial\nu} = 0,$$

which proves that S_Γ is injective.

(d) The same argument as in part 1 yields

$$\begin{aligned}\langle S_\Gamma\mu, \mu\rangle = \left\langle v, \frac{\partial\overline{v}_-}{\partial\nu} - \frac{\partial\overline{v}_+}{\partial\nu}\right\rangle_{\partial\Omega} &= \int_{\Omega\cup\{B_R\setminus\bar{\Omega}\}} (|\nabla v|^2 - n|v|^2)\,\mathrm{d}x - \int_{S_R} v\frac{\partial\overline{v}}{\partial\nu}\,\mathrm{d}s \\ &= \int_{\Omega\cup\{B_R\setminus\bar{\Omega}\}} (|\nabla v|^2 - n|v|^2)\,\mathrm{d}x + \mathrm{i}k\int_{S_R} |v|^2\,\mathrm{d}s + o(1).\end{aligned}$$

Taking the imaginary part we see that

$$\operatorname{Im}\langle S_\Gamma\mu, \mu\rangle = k\lim_{R\to\infty}\int_{S_R} |v|^2\,\mathrm{d}s \geqslant 0.$$

If $\operatorname{Im}\langle S_\Gamma\mu, \mu\rangle = 0$, then Rellich's lemma implies that v vanishes in $\mathbb{R}^2 \setminus \bar{\Gamma}$ and thus $S_\Gamma\mu = 0$ on Γ by the trace theorem. Hence, part 3 shows that $\mu = 0$. □

We now define a testing function $\varphi_L(\hat{x})$ by

$$\varphi_L(\hat{x}) := \int_L K^\infty(\hat{x}, z)\beta(z)\,\mathrm{d}s(z) = \int_L u_0(z, -\hat{x})\beta(z)\,\mathrm{d}s(z) \tag{4.11}$$

with density $\beta \in \widetilde{H}^{-1/2}(L)$, for any smooth non intersecting open arc L. We will characterize the crack Γ by the behavior of an approximate solution $g_L \in L^2(S^1)$ of the far field equation given by

$$((F_0 - F)\mathcal{S}_0^*)(g_L)(\hat{x}) = \varphi_L(\hat{x}). \tag{4.12}$$

To this end, we next turn to the operator G, and present the following conclusions.

Theorem 7. *For any smooth non intersecting arc L and function $\beta \in \widetilde{H}^{-1/2}(L)$ with $\operatorname{supp}\beta = L$, $\varphi_L \in L^2(S^1)$ is in the range of G if and only if $L \subset \Gamma$. Moreover, the data-to-pattern operator $G\colon\ H^{1/2}(\Gamma) \to L^2(S^1)$ is compact and has dense range in $L^2(S^1)$.*

P r o o f. Note that $\varphi_L(\hat{x})$ is just the far field pattern of the potential

$$\Phi_L(x) := \int_L K(x, z)\beta(z)\,\mathrm{d}s(z).$$

First assume that $L \subset \Gamma$. Since $\widetilde{H}^{-1/2}(L) \subset \widetilde{H}^{-1/2}(\Gamma)$, Φ_L solves problem (4.2) with boundary data $\eta = \Phi_L|_\Gamma$. It follows from the definition of G that $\varphi_L(\hat{x}) \in R(G)$.

Now we assume that $L \not\subset \Gamma$, and on the contrary, $\varphi_L(\hat{x}) \in R(G)$, i.e., there exists $\eta \in H^{1/2}(\Gamma)$ such that the far field patterns of $\Phi_L(x)$ and u^s_Γ (the solution of problem (4.2)) coincide. Then by Rellich's lemma and the unique continuation principle we have that $\Phi_L(x)$ and u^s_Γ coincide in $\mathbb{R}^2 \setminus \{\bar{\Gamma} \cup \bar{L}\}$. Since $L \neq \Gamma$, without loss of generality, there exists $x_0 \in L$ and $x_0 \notin \Gamma$ such that there is a small ball $B_\varepsilon(x_0) \cap \Gamma = \emptyset$ with center at x_0. Hence u^s_Γ has a continuous derivative in $B_\varepsilon(x_0)$ while $\Phi_L(x)$ does not at x_0, which leads to a contradiction. This proves the theorem. □

This theorem plays a key role in the connection between the testing function φ_L and the location of the crack. Observe the far field equation (4.12): once the range identity theorem is valid for the decomposition of (4.10), this far field equation is solvable if and only if the testing arc $L \subset \Gamma$. In addition, this solution g_L can be seen as an indicator function of the crack.

Theorem 8. *Assume that* $\mathcal{K}\colon\ L^2(S^1) \to H^{1/2}(\Gamma)$ *defined by*

$$(\mathcal{K}g)(x) := \int_{S^1} K^\infty(-d, x)g(d)\,\mathrm{d}s(d), \quad x \in \Gamma, \tag{4.13}$$

has a trivial kernel, i.e., $N(\mathcal{K}) = \{0\}$*. Then the data-to-pattern operator* G: $H^{1/2}(\Gamma) \to L^2(S^1)$ *is compact and has a dense range in* $L^2(S^1)$*.*

P r o o f. The compactness can be obtained by an argument analogous to Lemma 1.13 in [13]. Note that $GS_\Gamma = S_0 H^*$; since S_Γ is an isomorphism and S_0 is unitary, the injectivity of H indicates the denseness of G. Let $Hg = 0$. It is enough to prove $g = 0$.

By the mixed reciprocity relation (3.2), we have

$$(Hg)(x) = \int_{S^1} u_0(x, d)g(d)\,\mathrm{d}s(d) = \int_{S^1} K^\infty(-d, x)g(d)\,\mathrm{d}s(d), \quad x \in \Gamma.$$

The assumption implies that $g = 0$, which completes the proof of the theorem. $\square$

R e m a r k 3. The condition on $\mathcal{K}$ is essentially decided by the background inhomogeneous media. The injectivity can be ensured if the Dirichlet problem

$$\begin{cases} \Delta u + k^2 n u = 0 & \text{in } D, \\ u = 0 & \text{on } \partial D \end{cases}$$

possesses only zero solution.

Combining the previous results leads to the following main result for the solution of IDP.

Theorem 9. *Assume that the operator* $\mathcal{K}$ *defined by* (4.13) *satisfies the condition in Theorem* 8*. Let the far field equation be given by* (4.12)*. Then*

$$L \subset \Gamma \iff \varphi_L \in \mathrm{Range}((F_\sharp)^{1/2})$$

and consequently

$$L \subset \Gamma \iff \sum_{j=1}^{\infty} \frac{|\langle \varphi_L, \psi_j \rangle_{L^2(S^1)}|^2}{|\lambda_j|} < \infty, \tag{4.14}$$

where (λ_j, ψ_j) *is an eigensystem of the operator* $F_\sharp = |\mathrm{Re}((F_0 - F)\mathcal{S}_0^*)| + |\mathrm{Im}((F_0 - F)\mathcal{S}_0^*)|$*. In other words, the sign of the function*

$$W(L) = \left[\sum_{j=1}^{\infty} \frac{|\langle \varphi_L, \psi_j \rangle_{L^2(S^1)}|^2}{|\lambda_j|}\right]^{-1}$$

is just the characteristic function of Γ*.*

4.2. Inverse Mixed Crack Problem. In this part, we proceed with the IMP again by the factorization method. For the sake of simplicity, we continue to use some of the previous notation but with different meanings. Moreover, we omit the proof in some unnecessary places to avoid duplication. As one will see later, the factorization method we use here bases on a constraint on the solution space and a different decomposition strategy.

Let u be the solution to problem (1.1), (1.2), (1.5), and u_0 the solution of problem (3.1). It is easy to verify that the field $w := u - u_0$ solves the following boundary value problem with $\eta = -\partial u_0/\partial\nu$, $\theta = -\lambda u_0$:

$$\begin{cases} \Delta w + k^2 n w = 0 & \text{in } \mathbb{R}^2 \setminus \bar{\Gamma}, \\ \dfrac{\partial w_-}{\partial\nu} = \eta & \text{on } \Gamma, \\ \Big[\dfrac{\partial w}{\partial\nu}\Big] + \lambda w_+ = \theta & \text{on } \Gamma, \\ \lim\limits_{r\to\infty} \sqrt{r}\Big(\dfrac{\partial w}{\partial r} - \mathrm{i}kw\Big) = 0, & r = |x|. \end{cases} \tag{4.15}$$

Same as in the previous issue, we need to define the following operators. The operator $G\colon H^{-1/2}(\Gamma) \times L^2(\Gamma) \to L^2(S^1)$ is given by

$$(G(\eta,\theta))(\hat{x}) = w^\infty(\hat{x}), \tag{4.16}$$

where w^∞ is the far field pattern of the wave field w. The operator $H\colon L^2(S^1) \to H^{-1/2}(\Gamma) \times L^2(\Gamma)$ is defined by

$$(Hg)(x) = \Big(\frac{\partial}{\partial\nu}\int_{S^1} u_0(x,d)g(d)\,\mathrm{d}s(d), \lambda\int_{S^1} u_0(x,d)g(d)\,\mathrm{d}s(d)\Big), \quad x \in \Gamma. \tag{4.17}$$

The far field operator $F\colon L^2(S^1) \to L^2(S^1)$ is defined as

$$(Fg)(\hat{x}) = \int_{S^1} u^\infty(\hat{x},d)g(d)\,\mathrm{d}s(d), \quad g \in L^2(S^1), \tag{4.18}$$

where u^∞ is the far field pattern of the scattered wave u^s of problem (1.1), (1.2), (1.5). Notice the definition of F_0 (see (3.6)); we deduce that by superposition

$$F - F_0 = -GH. \tag{4.19}$$

Remark 4. We set the boundary data in the Sobolev space $H^{-1/2}(\Gamma) \times L^2(\Gamma)$ for the purpose of being able to apply the factorization method to solve IMP. Furthermore, the direct problem (4.15) is solvable in this case, since the variational formula (2.8) is still solvable under this condition.

The adjoint operator of H is $H^*\colon \widetilde{H}^{1/2}(\Gamma) \times L^2(\Gamma) \to L^2(S^1)$ with

$$(H^*(\mu,\tau)^\top)(\hat{x}) = \int_\Gamma \frac{\partial \overline{u_0(z,\hat{x})}}{\partial \nu(z)} \mu(z)\,\mathrm{d}s(z) + \bar{\lambda}\int_\Gamma \overline{u_0(z,\hat{x})}\tau(z)\,\mathrm{d}s(z), \tag{4.20}$$

for $\mu \in \widetilde{H}^{1/2}(\Gamma)$, $\tau \in L^2(\Gamma)$. The property of the scattered operator $\mathcal{S}_0$ (Theorem 4) and the mixed reciprocity relation (3.2) show that

$$(\mathcal{S}_0 H^*(\mu,\tau)^\top)(\hat{x}) = \int_\Gamma \frac{K^\infty(\hat{x},z)}{\partial \nu(z)}\mu(z)\,\mathrm{d}s(z) + \bar{\lambda}\int_\Gamma K^\infty(\hat{x},z)\tau(z))\,\mathrm{d}s(z).$$

It is just the far field pattern of the combined potential function

$$v(x) = \int_\Gamma \frac{K(x,z)}{\partial \nu(z)}\mu(z)\,\mathrm{d}s(z) + \bar{\lambda}\int_\Gamma K(x,z)\tau(z)\,\mathrm{d}s(z), \quad x \in \mathbb{R}^2 \setminus \bar{\Gamma}. \tag{4.21}$$

Let us recall the single-layer operator S_Γ,

$$(S_\Gamma \varphi)(x) := \int_\Gamma K(x,z)\varphi(z)\,\mathrm{d}s(z), \quad x \in \Gamma,$$

and define the double-layer operator N_Γ,

$$(N_\Gamma \varphi)(x) := \int_\Gamma \frac{\partial K(x,z)}{\partial \nu(z)}\varphi(z)\,\mathrm{d}s(z), \quad x \in \Gamma.$$

Their normal derivative operators N'_Γ and T_Γ are given by

$$\begin{aligned}
(N'_\Gamma \varphi)(x) &:= \int_\Gamma \frac{\partial K(x,z)}{\partial \nu(x)}\varphi(z)\,\mathrm{d}s(z), \quad x \in \Gamma,\\
(T_\Gamma \varphi)(x) &:= \frac{\partial}{\partial \nu(x)}\int_\Gamma \frac{\partial K(x,z)}{\partial \nu(z)}\varphi(z)\,\mathrm{d}s(z), \quad x \in \Gamma.
\end{aligned}$$

They have the following mapping properties [17] for $-\frac{1}{2} \leqslant s \leqslant \frac{1}{2}$:

$$\begin{aligned}
S_\Gamma\colon \widetilde{H}^{s-1/2}(\Gamma) &\to H^{s+1/2}(\Gamma), \quad & N_\Gamma\colon \widetilde{H}^{s+1/2}(\Gamma) &\to H^{s+1/2}(\Gamma),\\
N'_\Gamma\colon \widetilde{H}^{s-1/2}(\Gamma) &\to H^{s-1/2}(\Gamma), \quad & T_\Gamma\colon \widetilde{H}^{s+1/2}(\Gamma) &\to H^{s-1/2}(\Gamma).
\end{aligned}$$

Then v satisfies the problem (4.15) with the boundary data

$$\frac{\partial v_-}{\partial \nu}(x) = T_\Gamma \mu + \bar{\lambda} N'_\Gamma \tau + \frac{\bar{\lambda}}{2}\tau, \quad x \in \Gamma,$$

and

$$\Big(\Big[\frac{\partial v}{\partial \nu}\Big] + \lambda v_+\Big)(x) = \lambda N_\Gamma \mu + \frac{\lambda}{2}\mu + |\lambda|^2 S_\Gamma \tau - \bar{\lambda}\tau, \quad x \in \Gamma.$$

If we define $M\colon\ \widetilde{H}^{1/2}(\Gamma)\times L^2(\Gamma)\to H^{-1/2}(\Gamma)\times L^2(\Gamma)$ by

$$M=-\begin{pmatrix} T_\Gamma & \bar\lambda N'_\Gamma+\frac{\bar\lambda}{2}I \\ \lambda N_\Gamma+\frac{\lambda}{2}I & |\lambda|^2S_\Gamma-\bar\lambda I\end{pmatrix}, \tag{4.22}$$

then by the definition of G we get the relation

$$\mathcal{S}_0H^*(\mu,\tau)^\top=-GM(\mu,\tau)^\top.$$

Combining this with (4.19) leads to the decomposition

$$(F-F_0)\mathcal{S}_0^*=GM^*G^*. \tag{4.23}$$

Theorem 10. (a) *The operator M defined by (4.22) can be written as $M=M_0+M_1$ with a compact operator M_1 and a self-adjoint and coercive operator* $\operatorname{Re}M_0$.

(b) *The imaginary of M is non-positive.*

(c) *M is invertible.*

Proof. (a) The operator $-T_\Gamma$ admits a decomposition $-T_\Gamma=T_0+T_1$ such that $T_0\colon\ \widetilde{H}^{1/2}(\Gamma)\to H^{-1/2}(\Gamma)$ is compact and $\operatorname{Re}T_1$ self-adjoint and coercive [17]. We rewrite M as

$$M=\begin{pmatrix} T_0 & -\bar\lambda N'_\Gamma-\frac{\bar\lambda}{2}I \\ -\lambda N_\Gamma-\frac{\lambda}{2}I & -|\lambda|^2S_\Gamma\end{pmatrix}+\begin{pmatrix} T_1 & 0\\ 0 & \bar\lambda I\end{pmatrix}:=M_0+M_1. \tag{4.24}$$

Since $S_\Gamma,N_\Gamma,N'_\Gamma$ is compact from $L^2(\Gamma)$ into itself [12], these facts and the compact imbedding theorem show that $M_0\colon\ \widetilde{H}^{1/2}(\Gamma)\times L^2(\Gamma)\to H^{-1/2}(\Gamma)\times L^2(\Gamma)$ is a compact operator. Obviously,

$$\operatorname{Re}\langle(\mu,\tau),M_1(\mu,\tau)^\top\rangle\geqslant c(\|\mu\|_{\widetilde{H}^{-1/2}(\Gamma)}+\|\tau\|_{L^2(\Gamma)})$$

due to $\operatorname{Re}\lambda>0$, which indicates that $\operatorname{Re}M_1$ is self-adjoint and coercive.

(b) From the expression of v (4.21), the jump relations of the single-and double-layer potential yield that

$$\mu=v_+-v_-\quad\text{and}\quad\bar\lambda\tau=\frac{\partial v_-}{\partial\nu}-\frac{\partial v_+}{\partial\nu}\quad\text{on }\Gamma.$$

Then by the definition of M, we get

$$
\begin{aligned}
-\langle M(\mu,\tau)^{\top},(\mu,\tau)\rangle &= \int_\Gamma \frac{\partial v_-}{\partial\nu}\bar\mu\,\mathrm{d}s + \int_\Gamma\Big(\frac{\partial v_+}{\partial\nu}-\frac{\partial v_-}{\partial\nu}+\lambda v_+\Big)\bar\tau\,\mathrm{d}s\\
&= \int_\Gamma \frac{\partial v_-}{\partial\nu}(\overline{v_+-v_-})\,\mathrm{d}s + \frac{1}{\lambda}\int_\Gamma\Big(\frac{\partial v_+}{\partial\nu}-\frac{\partial v_-}{\partial\nu}+\lambda v_+\Big)\overline{\Big(\frac{\partial v_-}{\partial\nu}-\frac{\partial v_+}{\partial\nu}\Big)}\,\mathrm{d}s\\
&= \int_\Gamma \frac{\partial v_-}{\partial\nu}\overline{v}_+\,\mathrm{d}s + \int_\Gamma \frac{\partial \overline{v}_-}{\partial\nu}v_+\,\mathrm{d}s - \frac{1}{\lambda}\int_\Gamma\Big[\frac{\partial v}{\partial\nu}\Big]\Big[\frac{\partial \overline{v}}{\partial\nu}\Big]\,\mathrm{d}s\\
&\quad - \int_\Gamma \frac{\partial v_-}{\partial\nu}\overline{v}_-\,\mathrm{d}s - \int_\Gamma \frac{\partial \overline{v}_+}{\partial\nu}v_+\,\mathrm{d}s.
\end{aligned}
$$

Taking the imaginary part, using Green's theorem and noting the fact that $\operatorname{Im}\lambda\geqslant 0$, we have

$$
\begin{aligned}
&\operatorname{Im}\langle M(\mu,\tau)^{\top},(\mu,\tau)\rangle \qquad (4.25)\\
&= \operatorname{Im}\Big(\frac{1}{\lambda}\int_\Gamma\Big[\frac{\partial v}{\partial\nu}\Big]\Big[\frac{\partial \overline{v}}{\partial\nu}\Big]\,\mathrm{d}s + \int_\Gamma \frac{\partial v_-}{\partial\nu}\overline{v}_-\,\mathrm{d}s - \int_\Gamma \frac{\partial v_+}{\partial\nu}\overline{v}_+\,\mathrm{d}s\Big)\\
&= \operatorname{Im}\int_{B_R\setminus\partial\Omega}(|\nabla v|^2-k^2n|v|^2)\,\mathrm{d}x + \operatorname{Im}\frac{\bar\lambda}{|\lambda|^2}\int_\Gamma\Big|\Big[\frac{\partial v}{\partial\nu}\Big]\Big|^2\,\mathrm{d}s - \operatorname{Im}\int_{S_R}\frac{\partial\overline{v}}{\partial\nu}v\,\mathrm{d}s\\
&= \operatorname{Im}\frac{\bar\lambda}{|\lambda|^2}\int_\Gamma\Big|\Big[\frac{\partial v}{\partial\nu}\Big]\Big|^2\,\mathrm{d}s - \frac{1}{8\pi}\int_{S^1}|v^\infty|^2\,\mathrm{d}s,
\end{aligned}
$$

where the last equality is obtained due to the radiation condition and the asymptotic behavior of the scattered field. From this we observe that $\operatorname{Im}\langle M(\mu,\tau)^{\top},(\mu,\tau)\rangle\leqslant 0$.

(c) We only need to prove that $M\colon \widetilde{H}^{1/2}(\Gamma)\times L^2(\Gamma)\to H^{-1/2}(\Gamma)\times L^2(\Gamma)$ is injective in accordance with the Fredholm theorem. To this end, let $M(\mu,\tau)^{\top}=0$, then the potential function v given by (4.21) satisfies problem (4.15) with homogeneous boundary conditions. The well posedness of this direct problem shows us that $v=0$ in $\mathbb{R}^2\setminus\bar\Gamma$. The result is obtained by the jump relations of single- and double-layer potentials. □

Remark 5. We observe that S_Γ is a compact operator from $L^2(\Gamma)$ into $L^2(\Gamma)$; this is the key precondition for our factorization method, which has been proposed in the paper [3]. Indeed, in the standard spaces setting, $S_\Gamma\colon \widetilde{H}^{-1/2}(\Gamma)\to H^{1/2}(\Gamma)$ and $-T_\Gamma\colon \widetilde{H}^{1/2}(\Gamma)\to H^{-1/2}(\Gamma)$ are positive and bounded below up to a compact perturbation. However, their coefficients in the matrix operator M are positive and therefore do not match. Usually, the linear sampling method is applicable for this case [4].

By an analogous argument as in Theorem 7 and 8, we have the following conclusion for the operator G given by (4.16).

Theorem 11. *For any smooth non intersecting arc L and functions $\varrho \in \widetilde{H}^{-1/2}(L)$, $\sigma \in L^2(L)$, the operator $\varphi_L(\hat{x}) \in L^2(S^1)$ defined by*

$$\begin{aligned}\varphi_L(\hat{x}) &:= \int_L \frac{K^\infty(\hat{x},z)}{\partial\nu(z)}\varrho(z)\,\mathrm{d}s(z) + \lambda\int_L K^\infty(\hat{x},z)\sigma(z))\,\mathrm{d}s(z)\\ &= \int_L \frac{u_0(z,-\hat{x})}{\partial\nu(z)}\varrho(z)\,\mathrm{d}s(z) + \lambda\int_L u_0(z,-\hat{x})\sigma(z))\,\mathrm{d}s(z)\end{aligned}$$

is in the range of $R(G)$ if and only if $L \subset \Gamma$. Moreover, $G\colon H^{-1/2}(\Gamma)\times L^2(\Gamma) \to L^2(S^1)$ is compact and has dense range in $L^2(S^1)$.

Finally, we get a result similar to Theorem 8 from Theorems 10, 11, and Lemma 5 along with an application of the Picard theorem [13].

Theorem 12. *Consider the far field equation*

$$((F-F_0)\mathcal{S}_0^*)(g_L)(\hat{x}) = \varphi_L(\hat{x}).$$

For any smooth non intersecting arc L

$$L \subset \Gamma \Longleftrightarrow \varphi_L \in \mathrm{Range}((F_\sharp)^{1/2})$$

and consequently

$$L \subset \Gamma \Longleftrightarrow \sum_{j=1}^\infty \frac{|\langle\varphi_L,\psi_j\rangle_{L^2(S^1)}|^2}{|\lambda_j|} < \infty, \tag{4.26}$$

where (λ_j,ψ_j) is an eigensystem of the operator

$$F_\sharp = |\mathrm{Re}((F-F_0)\mathcal{S}_0^*)| + |\mathrm{Im}((F-F_0)\mathcal{S}_0^*)|.$$

In other words, the sign of the function

$$W(L) = \left[\sum_{j=1}^\infty \frac{|\langle\varphi_L,\psi_j\rangle_{L^2(S^1)}|^2}{|\lambda_j|}\right]^{-1}$$

is just the characteristic function of Γ.

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