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NUMERICAL ANALYSIS OF THE MESHLESS ELEMENT-FREE GALERKIN METHOD FOR HYPERBOLIC INITIAL-BOUNDARY VALUE PROBLEMS

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Abstract. The meshless element-free Galerkin method is developed for numerical analysis of hyperbolic initial-boundary value problems. In this method, only scattered nodes are required in the domain. Computational formulae of the method are analyzed in detail. Error estimates and convergence are also derived theoretically and verified numerically. Numerical examples validate the performance and efficiency of the method.

Keywords: meshless; element-free Galerkin method; hyperbolic partial differential equation; error estimate; convergence

MSC 2010: 65N12, 65M60, 65N30

1. Introduction

This paper concerns numerical analysis of the $(n+1)$-dimensional time-dependent hyperbolic partial differential equation

$$\frac{\partial^2 u(\mathbf{x},t)}{\partial t^2} + a_0(\mathbf{x})u(\mathbf{x},t) - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big[a_{ij}(\mathbf{x})\frac{\partial u(\mathbf{x},t)}{\partial x_j}\Big] = f(\mathbf{x},t), \quad \mathbf{x}\in\Omega,\ t>0, \tag{1.1}$$

with initial conditions

$$u(\mathbf{x},0) = \varphi_1(\mathbf{x}), \quad \mathbf{x}\in\Omega, \tag{1.2}$$

$$\frac{\partial u(\mathbf{x},t)}{\partial t}\Big|_{t=0} = \varphi_2(\mathbf{x}), \quad \mathbf{x}\in\Omega, \tag{1.3}$$

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and a boundary condition

$$\lambda(\mathbf{x})u(\mathbf{x},t)+\sum_{i,j=1}^{n}a_{ij}(\mathbf{x})n_i(\mathbf{x})\frac{\partial u(\mathbf{x},t)}{\partial x_j}=g(\mathbf{x},t),\quad \mathbf{x}\in\Gamma,\ t\geqslant 0, \tag{1.4}$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n=1,2,\ldots)$ with boundary Γ, $u(\mathbf{x},t)$ is an unknown function at position $\mathbf{x}=(x_1,x_2,\ldots,x_n)^{\mathrm{T}}$ and time t, $a_{ij}(\mathbf{x})$, $a_0(\mathbf{x})$, $\varphi_1(\mathbf{x})$, $\varphi_2(\mathbf{x})$, $\lambda(\mathbf{x})\geqslant 0$, $f(\mathbf{x},t)$ and $g(\mathbf{x},t)$ are given functions, and $\mathbf{n}=(n_1,n_2,\ldots,n_n)^{\mathrm{T}}$ is the unit normal exterior to Γ. In addition, the matrix $[a_{ij}(\mathbf{x})]_{i,j=1}^{n}$ satisfies $a_{ij}(\mathbf{x})=a_{ji}(\mathbf{x})$ and

$$\begin{aligned}C_{a1}\sum_{i=1}^{n}|\xi_i|^2\leqslant\sum_{i,j=1}^{n}a_{ij}(\mathbf{x})\xi_i\xi_j\\ \leqslant C_{a2}\sum_{i=1}^{n}|\xi_i|^2\quad\forall\,\mathbf{x}\in\Omega,\quad\forall\,(\xi_1,\xi_2,\ldots,\xi_n)^{\mathrm{T}}\in\mathbb{R}^n,\end{aligned} \tag{1.5}$$

where C_{a1} and C_{a2} are two positive constants.

The hyperbolic initial-boundary value problem given by (1.1)–(1.4) can be used to model many physical phenomena in mechanics, acoustics, optics, electromagnetism, and so on [10]. A number of analytical methods have been adopted to derive analytical solutions of this type of problem. However, due to the complexity of the hyperbolic problems and the domain, the research of analytical solutions is arduous in general. Thus, it is necessary to develop numerical methods for approximate solutions of hyperbolic problems.

The finite difference method (FDM) [3], [20], [22], the finite element method (FEM) [11] and the boundary element method (BEM) [7], [23] can be employed to obtain approximate solutions of hyperbolic problems. In these methods, the precision of approximate solutions relies acutely on the quality of meshes or elements. To overcome the meshing-related shortcomings, meshless (or meshfree) methods have been developed by using scattered nodes to discretize the solved domain [4], [19]. A variety of meshless methods have been proposed and many scientific and engineering problems have been solved successfully by these methods. Recently, some meshless methods, such as the meshless local weak-strong method [6], the boundary knot method [8] and radial basis functions methods [1], [9], [12] have been applied to hyperbolic problems.

The element-free Galerkin (EFG) method is an often used meshless method [2], [16], [18], [21]. In this method, the domain is discretized by scattered nodes, and the approximate solution is constructed by the moving least squares (MLS) approximation. Up to now, the EFG method has been applied to many problems in mathematical physics. In [5], the EFG method was used to solve a kind of two space dimensional linear hyperbolic equation. Detailed computational formulas have been given for a special two-dimensional domain. However, theoretical and numerical error analysis has not been established.

In this paper, the EFG method is developed for the numerical analysis of the generic n space dimensional hyperbolic problem given by (1.1)–(1.4). In this method, a time-stepping scheme is presented to approximate the time derivatives and then a full discretization scheme is deduced using the EFG method. Compared with [5], detailed computational formulas are given in a different way for arbitrary n-dimensional domain. Besides, using error results of the MLS approximation [13], [14], error and efficiency of the present EFG method is proved theoretically and verified numerically.

The rest of this paper is organized as follows. Section 2 develops a time-stepping scheme to approximate the time derivative. In Section 3, meshless numerical implementation of the EFG method for the hyperbolic problem is provided in detail. Then error analysis is given theoretically in Section 4. Finally, numerical results and conclusions are presented in Sections 5 and 6, respectively.

2. Approximation of time derivatives

Let $\tau > 0$ denote the time step size and let $u^{(k)}(\mathbf{x}) = u(\mathbf{x}, k\tau)$ for $k = 0, 1, 2, \ldots$ Then from Taylor's Theorem we have

$$u^{(k+1)}(\mathbf{x}) = u^{(k)}(\mathbf{x}) + \tau\frac{\partial u}{\partial t}(\mathbf{x}, k\tau) + \frac{\tau^2}{2}\frac{\partial^2 u}{\partial t^2}(\mathbf{x}, k\tau) + \frac{\tau^3}{6}\frac{\partial^3 u}{\partial t^3}(\mathbf{x}, k\tau) + \frac{\tau^4}{24}\frac{\partial^4 u}{\partial t^4}(\mathbf{x}, \xi_1),$$
$$u^{(k-1)}(\mathbf{x}) = u^{(k)}(\mathbf{x}) - \tau\frac{\partial u}{\partial t}(\mathbf{x}, k\tau) + \frac{\tau^2}{2}\frac{\partial^2 u}{\partial t^2}(\mathbf{x}, k\tau) - \frac{\tau^3}{6}\frac{\partial^3 u}{\partial t^3}(\mathbf{x}, k\tau) + \frac{\tau^4}{24}\frac{\partial^4 u}{\partial t^4}(\mathbf{x}, \xi_2),$$

where $\xi_1 \in (k\tau, (k+1)\tau)$ and $\xi_2 \in ((k-1)\tau, k\tau)$. Thus,

$$u^{(k)}(\mathbf{x}) = \frac{u^{(k+1)}(\mathbf{x}) + u^{(k-1)}(\mathbf{x})}{2} + \mathcal{O}(\tau^2), \tag{2.1}$$

$$\frac{\partial u(\mathbf{x}, t)}{\partial t}\Big|_{t=k\tau} = \frac{u^{(k+1)}(\mathbf{x}) - u^{(k-1)}(\mathbf{x})}{2\tau} + \mathcal{O}(\tau^2), \tag{2.2}$$

$$\frac{\partial^2 u(\mathbf{x}, t)}{\partial t^2}\Big|_{t=k\tau} = \frac{u^{(k+1)}(\mathbf{x}) - 2u^{(k)}(\mathbf{x}) + u^{(k-1)}(\mathbf{x})}{\tau^2} + \mathcal{O}(\tau^2). \tag{2.3}$$

Using (2.1) yields

$$u(\mathbf{x}, k\tau) = \frac{1}{3}(u^{(k+1)}(\mathbf{x}) + u^{(k)}(\mathbf{x}) + u^{(k-1)}(\mathbf{x})) + \mathcal{O}(\tau^2). \tag{2.4}$$

In (1.1), by setting $t = k\tau$ and invoking (2.2)–(2.4), we obtain

$$\frac{u^{(k+1)}(\mathbf{x}) - 2u^{(k)}(\mathbf{x}) + u^{(k-1)}(\mathbf{x})}{\tau^2} + a_0(\mathbf{x})u^{(k)}(\mathbf{x})$$
$$- \frac{1}{3}\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big[a_{ij}(\mathbf{x})\frac{\partial}{\partial x_j}(u^{(k+1)}(\mathbf{x}) + u^{(k)}(\mathbf{x}) + u^{(k-1)}(\mathbf{x}))\Big] = f^{(k)}(\mathbf{x}) + \frac{1}{3}R^{(k+1)},$$

i.e.,

$$\begin{aligned}\beta u^{(k+1)}(\mathbf{x}) &- \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big[a_{ij}(\mathbf{x})\frac{\partial}{\partial x_j}u^{(k+1)}(\mathbf{x})\Big] \\ &= [2\beta - 3a_0(\mathbf{x})]u^{(k)}(\mathbf{x}) - \beta u^{(k-1)}(\mathbf{x}) \\ &\quad + \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big[a_{ij}(\mathbf{x})\frac{\partial}{\partial x_j}(u^{(k)}(\mathbf{x}) + u^{(k-1)}(\mathbf{x}))\Big] + 3f^{(k)}(\mathbf{x}) + R^{(k+1)},\end{aligned} \tag{2.5}$$

where $f^{(k)}(\mathbf{x}) = f(\mathbf{x}, k\tau)$,

$$\beta = \frac{3}{\tau^2}, \tag{2.6}$$

and

$$|R^{(k+1)}| \leqslant C\tau^2 \tag{2.7}$$

for a positive constant C.

In (2.5), letting $k = 0$ yields

$$\begin{aligned}\beta u^{(1)}(\mathbf{x}) &- \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big[a_{ij}(\mathbf{x})\frac{\partial}{\partial x_j}u^{(1)}(\mathbf{x})\Big] \\ &= [2\beta - 3a_0(\mathbf{x})]u^{(0)}(\mathbf{x}) - \beta u^{(-1)}(\mathbf{x}) \\ &\quad + \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big[a_{ij}(\mathbf{x})\frac{\partial}{\partial x_j}(u^{(0)}(\mathbf{x}) + u^{(-1)}(\mathbf{x}))\Big] + 3f^{(0)}(\mathbf{x}) + R^{(1)}.\end{aligned} \tag{2.8}$$

According to (1.2), we have

$$u^{(0)}(\mathbf{x}) = \varphi_1(\mathbf{x}), \quad \mathbf{x} \in \Omega. \tag{2.9}$$

From (1.3) and (2.2) it follows that

$$\varphi_2(\mathbf{x}) = \frac{\partial u(\mathbf{x}, t)}{\partial t}\Big|_{t=0} = \frac{u^{(1)}(\mathbf{x}) - u^{(-1)}(\mathbf{x})}{2\tau} + \mathcal{O}(\tau^2),$$

thus

$$u^{(-1)}(\mathbf{x}) = u^{(1)}(\mathbf{x}) - 2\tau\varphi_2(\mathbf{x}) + \mathcal{O}(\tau^3), \quad \mathbf{x} \in \Omega. \tag{2.10}$$

Substituting (2.9) and (2.10) into (2.8) provides

$$\begin{aligned} 2\beta u^{(1)}(\mathbf{x}) - 2\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big[a_{ij}(\mathbf{x})\frac{\partial}{\partial x_j}u^{(1)}(\mathbf{x})\Big] &= [2\beta - 3a_0(\mathbf{x})]\varphi_1(\mathbf{x}) + 2\beta\tau\varphi_2(\mathbf{x}) \\ &\quad + \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big[a_{ij}(\mathbf{x})\frac{\partial}{\partial x_j}(\varphi_1(\mathbf{x}) - 2\tau\varphi_2(\mathbf{x}))\Big] + 3f^{(0)}(\mathbf{x}) + R^{(1)}. \end{aligned} \tag{2.11}$$

In view of (2.5) and (2.11), we recast the hyperbolic problem (1.1)–(1.4) in elliptic problems

$$\begin{cases} 2\beta u^{(1)}(\mathbf{x}) - 2\displaystyle\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big[a_{ij}(\mathbf{x})\frac{\partial}{\partial x_j}u^{(1)}(\mathbf{x})\Big] = b^{(0)}(\mathbf{x}) + R^{(1)}, & \mathbf{x} \in \Omega, \\ \lambda(\mathbf{x})u^{(1)}(\mathbf{x}) + \displaystyle\sum_{i,j=1}^{n} a_{ij}(\mathbf{x})n_i(\mathbf{x})\frac{\partial u^{(1)}(\mathbf{x})}{\partial x_j} = g^{(1)}(\mathbf{x}) \triangleq g(\mathbf{x},\tau), & \mathbf{x} \in \Gamma, \end{cases} \tag{2.12}$$

and

$$\begin{cases} \beta u^{(k+1)}(\mathbf{x}) - \displaystyle\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big[a_{ij}(\mathbf{x})\frac{\partial}{\partial x_j}u^{(k+1)}(\mathbf{x})\Big] \\ \qquad\qquad = b^{(k)}(\mathbf{x}) + R^{(k+1)}, & \mathbf{x} \in \Omega, \\ \lambda(\mathbf{x})u^{(k+1)}(\mathbf{x}) + \displaystyle\sum_{i,j=1}^{n} a_{ij}(\mathbf{x})n_i(\mathbf{x})\frac{\partial u^{(k+1)}(\mathbf{x})}{\partial x_j} \\ \qquad\qquad = g^{(k+1)}(\mathbf{x}) \triangleq g(\mathbf{x},(k+1)\tau), & \mathbf{x} \in \Gamma, \end{cases} \tag{2.13}$$

where $k = 1, 2, \ldots$,

$$\begin{aligned} b^{(0)}(\mathbf{x}) &= [2\beta - 3a_0(\mathbf{x})]\varphi_1(\mathbf{x}) + 2\beta\tau\varphi_2(\mathbf{x}) \\ &\quad + \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big[a_{ij}(\mathbf{x})\frac{\partial}{\partial x_j}(\varphi_1(\mathbf{x}) - 2\tau\varphi_2(\mathbf{x}))\Big] + 3f^{(0)}(\mathbf{x}), \\ b^{(k)}(\mathbf{x}) &= [2\beta - 3a_0(\mathbf{x})]u^{(k)}(\mathbf{x}) - \beta u^{(k-1)}(\mathbf{x}) + 3f^{(k)}(\mathbf{x}) \\ &\quad + \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big[a_{ij}(\mathbf{x})\frac{\partial}{\partial x_j}(u^{(k)}(\mathbf{x}) + u^{(k-1)}(\mathbf{x}))\Big]. \end{aligned}$$

3. Numerical implementation

Elliptic problems (2.12) and (2.13) can be solved directly using the EFG method. In problem (2.13), from the governing equation we have

$$\begin{aligned}\beta \int_\Omega u^{(k+1)} v \,\mathrm{d}\Omega - \sum_{i,j=1}^n \int_\Omega v \frac{\partial}{\partial x_i}\Big[a_{ij}\frac{\partial}{\partial x_j} u^{(k+1)}\Big] \mathrm{d}\Omega \\ = \sum_{i,j=1}^n \int_\Omega v \frac{\partial}{\partial x_i}\Big[a_{ij}\frac{\partial}{\partial x_j}(u^{(k)} + u^{(k-1)})\Big] \mathrm{d}\Omega \\ + \int_\Omega [(2\beta - 3a_0)u^{(k)} - \beta u^{(k-1)} + 3f^{(k)} + R^{(k+1)}] v \,\mathrm{d}\Omega,\end{aligned}$$

where $v \in H^1(\Omega)$. By applying the Gauss formula, we then obtain

$$\begin{aligned}\beta \int_\Omega u^{(k+1)} v \,\mathrm{d}\Omega + \sum_{i,j=1}^n \int_\Omega a_{ij}\frac{\partial v}{\partial x_i}\frac{\partial u^{(k+1)}}{\partial x_j} \mathrm{d}\Omega - \sum_{i,j=1}^n \int_\Gamma v a_{ij} n_i \frac{\partial u^{(k+1)}}{\partial x_j} \mathrm{d}\Gamma \\ = -\sum_{i,j=1}^n \int_\Omega a_{ij}\frac{\partial v}{\partial x_i}\frac{\partial}{\partial x_j}(u^{(k)} + u^{(k-1)}) \mathrm{d}\Omega + \sum_{i,j=1}^n \int_\Gamma v a_{ij} n_i \frac{\partial}{\partial x_j}(u^{(k)} + u^{(k-1)}) \mathrm{d}\Gamma \\ + \int_\Omega \{[2\beta - 3a_0]u^{(k)} - \beta u^{(k-1)} + 3f^{(k)} + R^{(k+1)}\} v \,\mathrm{d}\Omega.\end{aligned}$$

In light of the boundary conditions in problem (2.13), we finally conclude

$$\begin{aligned}\beta \int_\Omega u^{(k+1)} v \,\mathrm{d}\Omega + \sum_{i,j=1}^n \int_\Omega a_{ij}\frac{\partial v}{\partial x_i}\frac{\partial u^{(k+1)}}{\partial x_j} \mathrm{d}\Omega + \int_\Gamma v\lambda u^{(k+1)} \mathrm{d}\Gamma \\ = \int_\Gamma v(g^{(k+1)} + g^{(k)} + g^{(k-1)}) \mathrm{d}\Gamma - \int_\Gamma v\lambda(u^{(k)} + u^{(k-1)}) \mathrm{d}\Gamma \\ - \sum_{i,j=1}^n \int_\Omega a_{ij}\frac{\partial v}{\partial x_i}\frac{\partial}{\partial x_j}(u^{(k)} + u^{(k-1)}) \mathrm{d}\Omega \\ + \int_\Omega \{[2\beta - 3a_0]u^{(k)} - \beta u^{(k-1)} + 3f^{(k)} + R^{(k+1)}\} v \,\mathrm{d}\Omega.\end{aligned}$$

Thus, problem (2.13) is recast in the variational form: find $u^{(k+1)} \in H^1(\Omega)$ such that

$$a(u^{(k+1)}, v) = (b, v) + \int_\Omega R^{(k+1)} v \,\mathrm{d}\Omega \quad \forall\, v \in H^1(\Omega), \tag{3.1}$$

where

$$
\begin{aligned}
(b, v) = & \int_\Gamma v(g^{(k+1)} + g^{(k)} + g^{(k-1)})\,\mathrm{d}\Gamma - \int_\Gamma v\lambda(u^{(k)} + u^{(k-1)})\,\mathrm{d}\Gamma \\
& - \sum_{i,j=1}^{n} \int_\Omega a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial}{\partial x_j}(u^{(k)} + u^{(k-1)})\,\mathrm{d}\Omega \\
& + \int_\Omega \{[2\beta - 3a_0]u^{(k)} - \beta u^{(k-1)} + 3f^{(k)}\} v\,\mathrm{d}\Omega,
\end{aligned}
$$

and the continuous and coercive bilinear form $a(\cdot,\cdot)$ is defined as

$$
a(u^{(k+1)}, v) = \beta \int_\Omega u^{(k+1)} v\,\mathrm{d}\Omega + \sum_{i,j=1}^{n} \int_\Omega a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial u^{(k+1)}}{\partial x_j}\,\mathrm{d}\Omega + \int_\Gamma v\lambda u^{(k+1)}\,\mathrm{d}\Gamma. \tag{3.2}
$$

Let $\{\mathbf{x}_i\}_{i=1}^N$ be N scattered nodes in $\Omega \cup \Gamma$ and let

$$
h = \max_{1 \leqslant i \leqslant N} \min_{1 \leqslant j \leqslant N, j \neq i} |\mathbf{x}_i - \mathbf{x}_j|
$$

represent the nodal spacing. Then, according to the MLS approximation, we can express the approximate solution of $u^{(k+1)}(\mathbf{x})$ as

$$
u_h^{(k+1)}(\mathbf{x}) = \mathcal{M}u^{(k+1)}(\mathbf{x}) = \sum_{i=1}^{N} \Phi_i(\mathbf{x}) u_i^{(k+1)} = \mathbf{\Phi}(\mathbf{x})\mathbf{u}^{(k+1)}, \quad k = 0, 1, 2, \ldots \tag{3.3}
$$

where $\mathcal{M}$ is an approximation operator, $u_i^{(k+1)}$ is the nodal value of $u^{(k+1)}(\mathbf{x})$ at $\mathbf{x}_i$,

$$
\mathbf{u}^{(k+1)} = [u_1^{(k+1)}, u_2^{(k+1)}, \ldots, u_N^{(k+1)}]^{\mathrm{T}},
$$

and

$$
\mathbf{\Phi}(\mathbf{x}) = [\Phi_1(\mathbf{x}), \Phi_2(\mathbf{x}), \ldots, \Phi_N(\mathbf{x})],
$$

with the MLS shape function [14]

$$
\Phi_i(\mathbf{x}) = \begin{cases} \displaystyle\sum_{j=1}^{m} p_j(\mathbf{x})[\mathbf{A}^{-1}(\mathbf{x})\mathbf{B}(\mathbf{x})]_{jk}, & i = I_k \in \Lambda(\mathbf{x}), \\ 0, & i \notin \Lambda(\mathbf{x}), \end{cases} \qquad 1 \leqslant i \leqslant N. \tag{3.4}
$$

In (3.4),

$$
\begin{aligned}
\mathbf{A}(\mathbf{x}) &= \sum_{i \in \Lambda(\mathbf{x})} w_i(\mathbf{x}) \mathbf{p}(\mathbf{x}_i) \mathbf{p}^{\mathrm{T}}(\mathbf{x}_i), \\
\mathbf{B}(\mathbf{x}) &= [w_{I_1}(\mathbf{x})\mathbf{p}(\mathbf{x}_{I_1}), w_{I_2}(\mathbf{x})\mathbf{p}(\mathbf{x}_{I_2}), \ldots, w_{I_\nu}(\mathbf{x})\mathbf{p}(\mathbf{x}_{I_\nu})],
\end{aligned}
$$

where $w_i(\mathbf{x}) \geqslant 0$ is a weight function with compact support and $\mathbf{p}(\mathbf{x}) = [p_1(\mathbf{x}), p_2(\mathbf{x}), \ldots, p_m(\mathbf{x})]^{\mathrm{T}}$ is the basis vector. Besides, the set $\Lambda(\mathbf{x}) = \{I_1, I_2, \ldots, I_\nu\}$ is defined in such a way that $i \in \Lambda(\mathbf{x})$ if and only if $w_i(\mathbf{x}) > 0$. As proved in [15], [17], the basis function $p_j(\mathbf{x})$ should be chosen as the shifted and scaled polynomial function to enhance the stability and performance of the MLS approximation.

The shifted and scaled basis vector can be written as $\mathbf{p}(\mathbf{x}) = \{(\mathbf{x}-\mathbf{x}^e)^{\boldsymbol{\lambda}}/h^{|\boldsymbol{\lambda}|}\}_{|\boldsymbol{\lambda}|\geqslant 0}$, where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)^{\mathrm{T}}$ is a multi-index notation with $|\boldsymbol{\lambda}| = \sum\limits_{i=1}^{n} \lambda_i$, and $\mathbf{x}^e = (x_1^e, x_2^e, \ldots, x_n^e)^{\mathrm{T}}$ is fixed and relies on the evaluation point considered. In actual application, we can choose $\mathbf{x}^e$ as the evaluation point [17]. Then the n-dimensional linear basis is given by

$$\mathbf{p}(\mathbf{x}) = \Big[1, \frac{x_1 - x_1^e}{h}, \frac{x_2 - x_2^e}{h}, \ldots, \frac{x_n - x_n^e}{h}\Big]^{\mathrm{T}}, \quad \mathbf{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathbb{R}^n,$$

and the two-dimensional quadratic basis is given by

$$\mathbf{p}(\mathbf{x}) = \Big[1, \frac{x_1 - x_1^e}{h}, \frac{x_2 - x_2^e}{h}, \frac{(x_1 - x_1^e)^2}{h^2}, \frac{(x_1 - x_1^e)(x_2 - x_2^e)}{h^2}, \frac{(x_2 - x_2^e)^2}{h^2}\Big]^{\mathrm{T}}, \\ \mathbf{x} = (x_1, x_2)^{\mathrm{T}} \in \mathbb{R}^2.$$

Let

$$V_h(\Omega) = \operatorname{span}\{\Phi_i, 1 \leqslant i \leqslant N\}. \tag{3.5}$$

Then, the EFG approximation of the variational problem (3.1) is to calculate $u_h^{(k+1)} \in V_h(\Omega)$ such that

$$a(u_h^{(k+1)}, v) = (b, v) \quad \forall\, v \in V_h(\Omega). \tag{3.6}$$

Consequently, the elliptic problem (2.13) is discretized into the linear system

$$(\mathbf{G} + \mathbf{K} + \mathbf{H})\mathbf{u}^{(k+1)} = \mathbf{g}^{(k)} + (\bar{\mathbf{G}} - \mathbf{K} - \mathbf{H})\mathbf{u}^{(k)} - (\mathbf{G} + \mathbf{K} + \mathbf{H})\mathbf{u}^{(k-1)} + \mathbf{f}^{(k)}, \tag{3.7}$$

where $k = 1, 2, \ldots$,

$$[\mathbf{K}]_{ij} = \sum_{l,k=1}^{n} \int_\Omega a_{lk} \frac{\partial \Phi_i}{\partial x_l} \frac{\partial \Phi_j}{\partial x_k} \,\mathrm{d}\Omega, \tag{3.8}$$

$$[\mathbf{H}]_{ij} = \int_\Gamma \lambda \Phi_i \Phi_j \,\mathrm{d}\Gamma, \tag{3.9}$$

$$[\mathbf{G}]_{ij} = \beta \int_\Omega \Phi_i \Phi_j \,\mathrm{d}\Omega, \tag{3.10}$$

$$[\bar{\mathbf{G}}]_{ij} = \int_\Omega (2\beta - 3a_0) \Phi_i \Phi_j \,\mathrm{d}\Omega,$$

$$[\mathbf{g}^{(k)}]_i = \int_\Gamma \Phi_i(g^{(k+1)} + g^{(k)} + g^{(k-1)})\,\mathrm{d}\Gamma,$$
$$[\mathbf{f}^{(k)}]_i = \int_\Omega 3f^{(k)}\Phi_i\,\mathrm{d}\Omega.$$

Similarly, the elliptic problem (2.12) is discretized into the linear system

$$(\mathbf{G}+\mathbf{K}+\mathbf{H})\mathbf{u}^{(1)} = \mathbf{g}^{(0)} + \mathbf{f}^{(0)}, \tag{3.11}$$

where

$$[\mathbf{g}^{(0)}]_i = \int_\Gamma \Phi_i g^{(1)}\,\mathrm{d}\Gamma,$$
$$[\mathbf{f}^{(0)}]_i = \frac{1}{2}\int_\Omega \Big\{(2\beta - 3a_0)\varphi_1(\mathbf{x}) + 2\beta\tau\varphi_2 + \sum_{l,k=1}^{n} \frac{\partial}{\partial x_l}\Big[a_{lk}\frac{\partial}{\partial x_k}(\varphi_1 - 2\tau\varphi_2)\Big] + 3f^{(0)}\Big\}\Phi_i\,\mathrm{d}\Omega.$$

According to (3.4), the MLS shape function Φ_i has compact support $\operatorname{supp}\Phi_i$ [14]. Then we can deduce from (3.8)–(3.10) that the entries $[\mathbf{K}]_{ij}$, $[\mathbf{H}]_{ij}$ and $[\mathbf{G}]_{ij}$ need to be computed only when $\operatorname{supp}\Phi_i \cap \operatorname{supp}\Phi_j \neq \emptyset$. Otherwise, these entries are zero. Hence, the matrix $\mathbf{G}+\mathbf{K}+\mathbf{H}$ in (3.7) and (3.11) is sparse. In addition, in light of the bilinear form $a(\cdot,\cdot)$ defined in (3.2), the matrix is symmetric and positive definite. As a consequence, the present EFG method produces a symmetric positive definite and sparse system matrix $\mathbf{G}+\mathbf{K}+\mathbf{H}$.

4. Error analysis

Lemma 4.1 ([14]). *Let $\mathcal{M}v(\mathbf{x}) = \sum_{i=1}^{N}\Phi_i(\mathbf{x})v_i$ be the MLS approximation of $v(\mathbf{x}) \in H^{r+1}(\Omega)$. Then*

$$\|v - \mathcal{M}v\|_{H^l(\Omega)} \leqslant Ch^{\tilde{p}-l}\|v\|_{H^{\tilde{p}}(\Omega)}, \quad 0 \leqslant l \leqslant \min\{\tilde{p},\gamma\}, \quad \tilde{p} = \min\{r,\widehat{m}\}+1,$$

where C is a constant independent of h, $\gamma \geqslant 1$ is a positive number such that the weight function satisfies $w_i(\mathbf{x}) \in C^\gamma(\Omega)$, and $\widehat{m}$ denotes the highest degree of the basis vector $\mathbf{p}(\mathbf{x})$.

Theorem 4.1. *Let $u^{(k+1)} \in H^{r+1}(\Omega)$ and $u_h^{(k+1)} \in V_h(\Omega)$ be the solutions of* (3.1) *and* (3.6), *respectively. Then we can estimate the error of the EFG method for the hyperbolic problem* (1.1)–(1.4) *as*

$$\|u^{(k+1)} - u_h^{(k+1)}\|_{H^1(\Omega)} \leqslant C(\tau^2 + h^{\min\{r,\widehat{m}\}}), \tag{4.1}$$

where C is a constant independent of τ and h.

P r o o f. Subtracting (3.6) from (3.1), we have

$$a(u^{(k+1)} - u_h^{(k+1)}, v) = \int_\Omega R^{(k+1)} v \,\mathrm{d}\Omega \quad \forall\, v \in V_h(\Omega).$$

Then using the fact that $\mathcal{M}u^{(k+1)} - u_h^{(k+1)} \in V_h(\Omega)$ yields

$$\begin{aligned}(4.2)\quad a(u^{(k+1)} - u_h^{(k+1)}, \mathcal{M}u^{(k+1)} - u_h^{(k+1)}) &= \int_\Omega R^{(k+1)}(\mathcal{M}u^{(k+1)} - u_h^{(k+1)}) \,\mathrm{d}\Omega \\ &\leqslant \|R^{(k+1)}\|_{L^2(\Omega)} \|\mathcal{M}u^{(k+1)} - u_h^{(k+1)}\|_{L^2(\Omega)} \\ &\leqslant C_1 \|R^{(k+1)}\|_{L^2(\Omega)} \{\|\mathcal{M}u^{(k+1)} - u^{(k+1)}\|_{L^2(\Omega)} + \|u^{(k+1)} - u_h^{(k+1)}\|_{L^2(\Omega)}\}.\end{aligned}$$

For the bilinear form $a(\cdot,\cdot)$ defined in (3.2), let $\|v\|^2_{H^a(\Omega)} \triangleq a(v,v)$. Then

$$(4.3)\qquad a(u^{(k+1)} - u_h^{(k+1)}, u^{(k+1)} - u_h^{(k+1)}) = \|u^{(k+1)} - u_h^{(k+1)}\|^2_{H^a(\Omega)}.$$

Besides, since $a(\cdot,\cdot)$ is continuous, we also have

$$\begin{aligned}(4.4)\qquad & a(u^{(k+1)} - u_h^{(k+1)}, u^{(k+1)} - \mathcal{M}u^{(k+1)}) \\ &\leqslant C_2 \|u^{(k+1)} - u_h^{(k+1)}\|_{H^a(\Omega)} \|u^{(k+1)} - \mathcal{M}u^{(k+1)}\|_{H^a(\Omega)}.\end{aligned}$$

Clearly,

$$\begin{aligned}(4.5)\qquad & a(u^{(k+1)} - u_h^{(k+1)}, u^{(k+1)} - u_h^{(k+1)}) \\ &= a(u^{(k+1)} - u_h^{(k+1)}, u^{(k+1)} - \mathcal{M}u^{(k+1)}) \\ &\quad + a(u^{(k+1)} - u_h^{(k+1)}, \mathcal{M}u^{(k+1)} - u_h^{(k+1)}).\end{aligned}$$

By substituting (4.2)–(4.4) into (4.5), we get

$$\begin{aligned}&\|u^{(k+1)} - u_h^{(k+1)}\|^2_{H^a(\Omega)} \\ &\quad \leqslant C_1 \|R^{(k+1)}\|_{L^2(\Omega)} \{\|\mathcal{M}u^{(k+1)} - u^{(k+1)}\|_{L^2(\Omega)} + \|u^{(k+1)} - u_h^{(k+1)}\|_{L^2(\Omega)}\} \\ &\qquad + C_2 \|u^{(k+1)} - u_h^{(k+1)}\|_{H^a(\Omega)} \|u^{(k+1)} - \mathcal{M}u^{(k+1)}\|_{H^a(\Omega)}.\end{aligned}$$

And due to (2.7) and Lemma 3.1 we conclude

$$\begin{aligned}\|u^{(k+1)} - u_h^{(k+1)}\|^2_{H^a(\Omega)} &\leqslant C_3 \tau^2 \{h^{\min\{r,\widehat{m}\}+1} + \|u^{(k+1)} - u_h^{(k+1)}\|_{H^a(\Omega)}\} \\ &\quad + C_4 h^{\min\{r,\widehat{m}\}} \|u^{(k+1)} - u_h^{(k+1)}\|_{H^a(\Omega)}.\end{aligned}$$

Therefore,

$$\|u^{(k+1)} - u_h^{(k+1)}\|_{H^a(\Omega)} \leqslant C_5(\tau^2 + h^{\min\{r,\widehat{m}\}}).$$

Finally, together with $\|u^{(k+1)} - u_h^{(k+1)}\|_{H^1(\Omega)} \leqslant C_6 \|u^{(k+1)} - u_h^{(k+1)}\|_{H^a(\Omega)}$, we obtain (4.1). $\square$

Obviously, Theorem 4.1 implies that the EFG solution $u^{(k+1)}$ converges to the analytical solution u as $\tau \to 0$ and $h \to 0$.

5. Numerical results

Two numerical examples are given in this section to verify the efficiency of the present EFG method. The numerical results confirm the theoretical ones.

5.1. The $(1+1)$-dimensional hyperbolic problem. Consider the following hyperbolic partial differential equation in $(1+1)$ dimensions:

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t), \quad x \in \Omega = (0,1),\ t > 0,$$

with initial conditions

$$u(x,0) = \varphi_1(x), \quad \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = \varphi_2(x), \quad x \in \Omega,$$

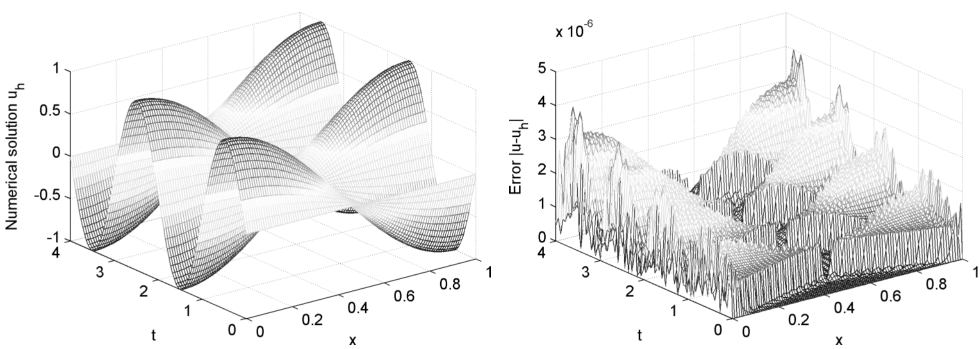
and a boundary condition

$$u(x,t) + \frac{\partial u(x,t)}{\partial x} = g(x,t), \quad x \in \Gamma,\ t \geqslant 0,$$

where $f(x,t)$, $\varphi_1(x)$, $\varphi_2(x)$, and $g(x,t)$ are taken such that we have the analytical solution [12]

$$u(x,t) = \cos(\pi x)\sin(\pi t).$$

Figure 1 presents the approximate solution u_h at $t = 4$ and the associated error $|u - u_h|$. The results are obtained by the EFG method using $\tau = 0.001$ and $h = 0.01$.



(a) (b)

Figure 1. Diagrams of (a) the approximate solution and (b) the associated error up to $t = 4$ for the $(1+1)$-dimensional hyperbolic problem.

Clearly, the EFG method produces very precise results. Besides, the structure of the system matrix $\mathbf{G}+\mathbf{K}+\mathbf{H}$ in (3.7) and (3.11) is shown in Figure 2. Clearly, a large number of its entries are zero, and thus the matrix is very sparse.

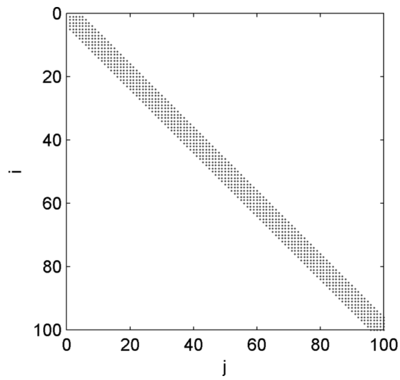
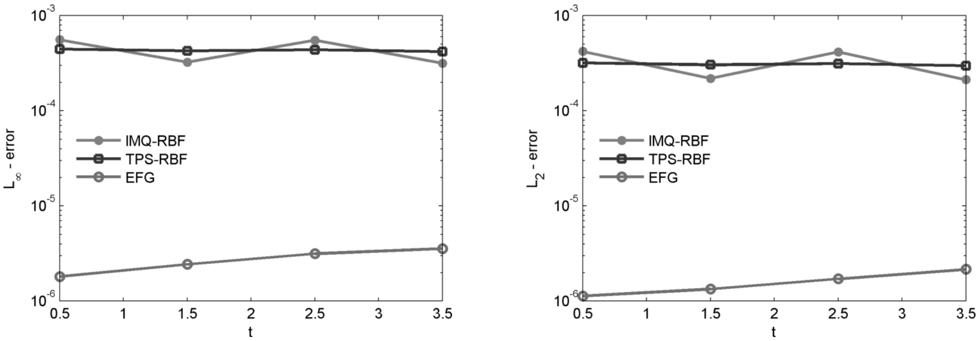


Figure 2. The sparse structure of the system matrix in (3.7) and (3.11) (the dots represent the nonzero entries $[\mathbf{G}+\mathbf{K}+\mathbf{H}]_{ij}$).

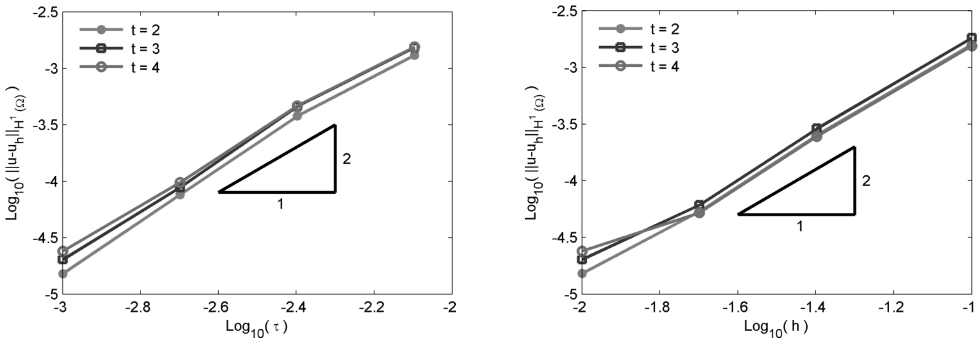
Figure 3 presents the L_∞-error and the L_2-error at times $t = 0.5, 1.5, 2.5$, and 3.5. These errors are obtained by the EFG method using $\tau = 0.001$ and $h = 0.01$. The errors of the inverse multiquadric radial basis function (IMQ-RBF) method and the thin plate splines radial basis function (TPS-RBF) method [12] are also given in this figure. Note that the errors of the two RBF methods are taken from Table 7 of [12]. Undoubtedly, the errors obtained by the present EFG method are much less than those obtained by the two RBF methods. In addition, the two RBF methods produce dense and asymmetric system matrices, while the present EFG method produces sparse and symmetric positive definite system matrices. Therefore, the EFG method is more efficient than the two RBF methods.



(a) (b)

Figure 3. (a) L_∞-error and (b) L_2-error of the EFG method and two RBF methods for the $(1+1)$-dimensional hyperbolic problem.

Figure 4(a) and Figure 4(b) give the error of the EFG method in the $H^1(\Omega)$ norm against the time step τ and the nodal spacing h, respectively. The errors in Figure 4(a) are obtained at times $t = 2$, 3 and 4 by using $h = 0.01$, while the errors in Figure 4(b) are obtained by using $\tau = 0.001$. We can find that the errors decrease monotonously as τ and h decrease, which implies that the approximate solution produced by the present EFG method converges to the analytical solution. Besides, we have experimental convergence forms of about $\mathcal{O}(\tau^2)$ and $\mathcal{O}(h^2)$ for this example. The numerical results confirm the theoretical ones.



(a) (b)

Figure 4. Error $\|u - u_h\|_{H^1(\Omega)}$ against (a) the time step τ and (b) the nodal spacing h for the $(1+1)$-dimensional hyperbolic problem.

5.2. The $(2+1)$-dimensional hyperbolic problem. Consider the following hyperbolic partial differential equation in $(2+1)$ dimensions:

$$\frac{\partial^2 u(\mathbf{x},t)}{\partial t^2} + 2u(\mathbf{x},t) - \frac{1}{2}\frac{\partial^2 u(\mathbf{x},t)}{\partial x_1^2} - \frac{1}{2}\frac{\partial^2 u(\mathbf{x},t)}{\partial x_2^2} = f(\mathbf{x},t), \quad \mathbf{x} = (x_1, x_2)^{\mathrm{T}} \in \Omega,\ t > 0,$$

with initial conditions

$$u(\mathbf{x},0) = \varphi_1(\mathbf{x}), \quad \left.\frac{\partial u(\mathbf{x},t)}{\partial t}\right|_{t=0} = \varphi_2(\mathbf{x}), \quad \mathbf{x} \in \Omega,$$

and a boundary condition

$$u(\mathbf{x},t) + \frac{1}{2}\sum_{j=1}^{2} n_j(\mathbf{x})\frac{\partial u(\mathbf{x},t)}{\partial x_j} = g(\mathbf{x},t), \quad \mathbf{x} \in \Gamma,\ t \geqslant 0,$$

where $\Omega = (-5,5)^2$, and $f(\mathbf{x},t)$, $\varphi_1(\mathbf{x})$, $\varphi_2(\mathbf{x})$, and $g(\mathbf{x},t)$ are taken such that we have the analytical solution

$$u(\mathbf{x},t) = \tan^{-1}\exp\Big(\sqrt{x_1^2 + x_2^2} - t\Big).$$

Figure 5 gives the approximate solution at $t = 4$, 7, and 10 and the associated error. The results are obtained by the EFG method using $\tau = 0.1$ and $h = 0.25$. Again, the EFG method produces very precise results.

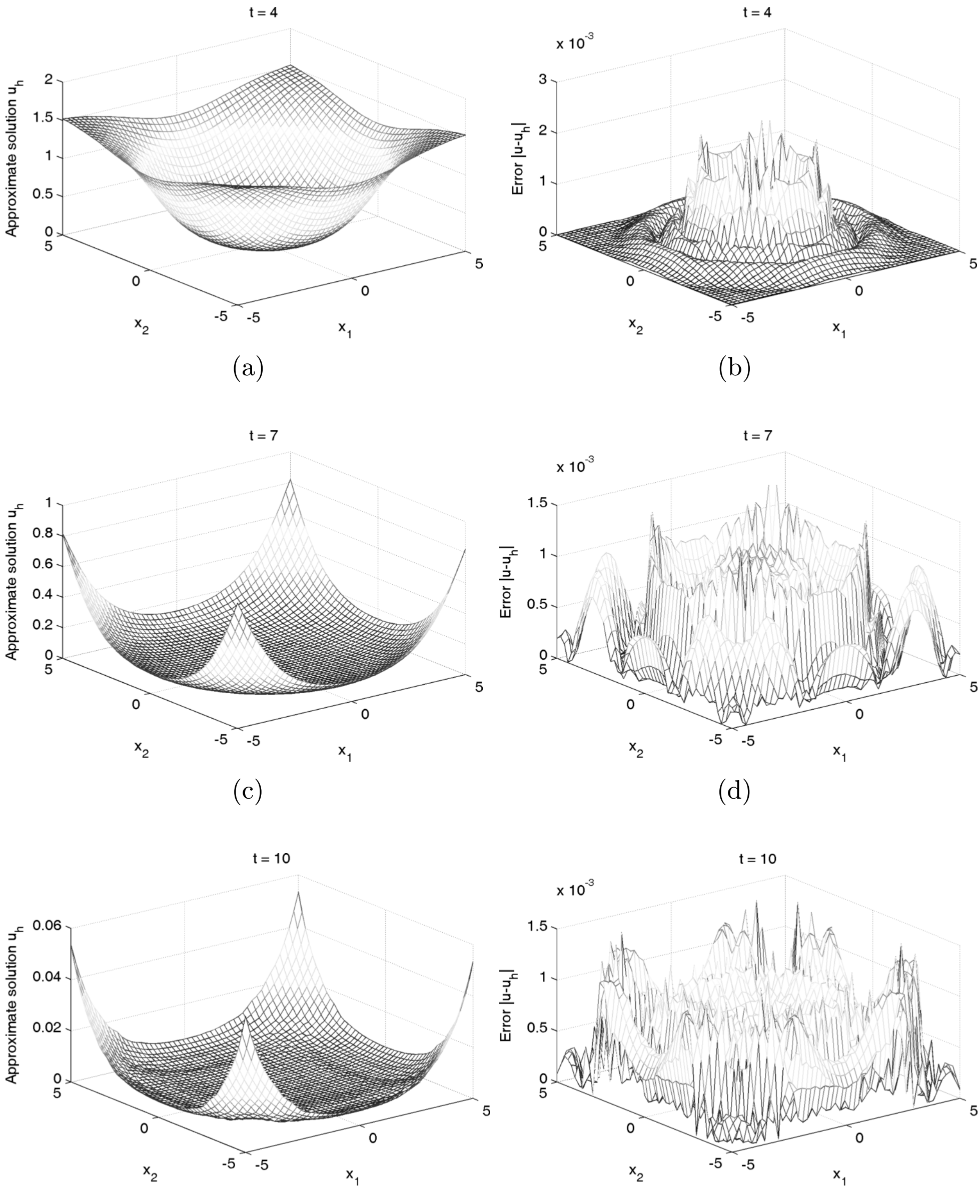
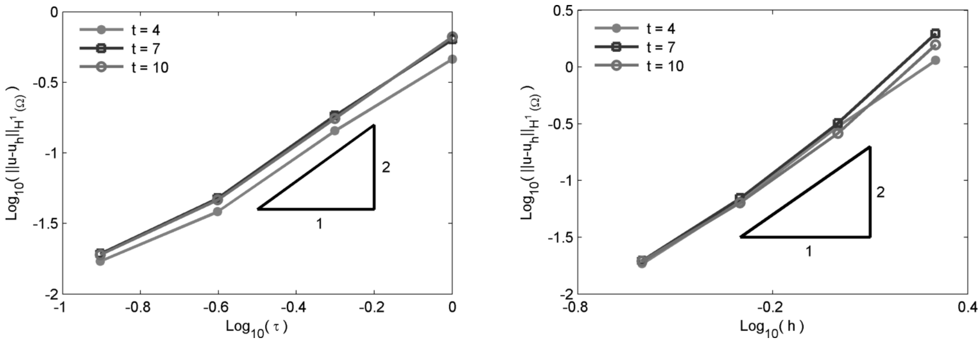


Figure 5. Diagrams of the approximate solution (left panel) and the associated error (right panel) at $t = 4, 7$, and 10 for the $(2 + 1)$-dimensional hyperbolic problem.

Figure 6(a) gives the error $\|u-u_h\|_{H^1(\Omega)}$ at times $t=4,7$ and 10 against the time step τ, while Figure 6(b) gives the error against the nodal spacing h. The errors in Figure 6(a) are obtained using $h=0.25$, and the errors in Figure 6(b) are obtained using $\tau=0.1$. The two figures indicate that the present EFG method converges to the analytical solution and has high convergence rate in all cases. The numerical results also confirm the theoretical ones.



(a) (b)

Figure 6. Error $\|u-u_h\|_{H^1(\Omega)}$ against (a) the time step τ and (b) the nodal spacing h for the $(2+1)$-dimensional hyperbolic problem.

6. Conclusions

In this paper, a numerical strategy for hyperbolic partial differential equations has been developed using the meshless element-free Galerkin (EFG) method. Theoretical error of this numerical strategy has been analyzed in detail. The theoretical error bound of the approximate solution depends on both the time step and the nodal spacing. Numerical results demonstrate the ability of the present EFG method, confirm the theoretical analysis and show that the present method has lower errors than other existing numerical methods.

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