A note on spaces with countable extent

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Abstract. Let P be a topological property. A space X is said to be *star* P if whenever $\mathcal{U}$ is an open cover of X, there exists a subspace $A \subseteq X$ with property P such that $X = St(A,\mathcal{U})$. In this note, we construct a Tychonoff pseudocompact SCE-space which is not star Lindelöf, which gives a negative answer to a question of Rojas-Sánchez and Tamariz-Mascarúa.

Keywords: star properties; star Lindelöf; space with star countable extent

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1. Introduction

By a space, we mean a topological space. In this section, we give definitions of terms which are used in this paper. Let X be a space and $\mathcal{U}$ a collection of subsets of X. For $A \subseteq X$, let $St(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$. As usual, we write $St(x,\mathcal{U})$ instead of $St(\{x\},\mathcal{U})$.

Definition 1.1 ([1], [2], [3], [7]). Let P be a topological property. A space X is said to be *star* P if whenever $\mathcal{U}$ is an open cover of X, there exists a subspace $A \subseteq X$ with property P such that $X = St(A,\mathcal{U})$. The set A will be called a *star kernel* of the cover $\mathcal{U}$.

The term star P was coined in [1], [2], [3], [7] but certain star properties, specifically those corresponding to "P=finite" and "P=countable" were first studied by van Douwen et al. in [4] and later by many other authors. A survey of star covering properties with a comprehensive bibliography can be found in [4], [6]. The author believes the terminology from [1], [2], [3], [7] and the terminology used in the paper to be simple and logical. But we must mention that authors of previous works have used many different notations to define properties of this sort. For example, in [6] and earlier [4], a star finite space is called starcompact and strongly 1-starcompact, a star countable space is called star Lindelöf and strongly 1-star Lindelöf.

In [9], a space with star countable extent was called SCE-space. Rojas-Sánchez and Tamariz-Mascarúa investigated the relationships between SCE-spaces and

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related spaces, and also studied topological properties of SCE-spaces. They asked the following question:

Question 1.2 ([9]). Is every SCE-space and pseudocompact Tychonoff space star-Lindelöf?

The purpose of this paper is to construct the example stated in the abstract, which gives a negative answer to the above question.

Throughout the paper, the extent $e(X)$ of a space X is the smallest infinite cardinal κ such that every discrete closed subset of X has cardinality at most κ. Let $\mathfrak{c}$ denote the cardinality of the continuum and ω the first infinite cardinal. For a pair of ordinals α, β with $\alpha < \beta$, we write $(\alpha, \beta) = \{\gamma : \alpha < \gamma < \beta\}$, $(\alpha, \beta] = \{\gamma : \alpha < \gamma \leq \beta\}$ and $[\alpha, \beta] = \{\gamma : \alpha \leq \gamma \leq \beta\}$. As usual, a cardinal is an initial ordinal and an ordinal is the set of smaller ordinals. Every cardinal is often viewed as a space with the usual order topology. Other terms and symbols that we do not define will be used as in [5].

2. An example of spaces with countable extent

In this section, we construct the example stated in the abstract.

For a Tychonoff space X, let βX denote the Čech-Stone compactification of X. For the following example, we use the Noble plank. Let τ be an infinite cardinal and X be a Tychonoff space. Consider the subspace $N_\tau X$ ([8], see also [5])

$$N_\tau X = (\beta X \times (\tau + 1)) \setminus ((\beta X \setminus X) \times \{\tau\})$$

of the product space $\beta X \times (\tau + 1)$. Note that $N_\tau X$ contains a closed subspace $\widetilde{X} = X \times \{\tau\}$ which is homeomorphic to X. We need the following lemma from [9].

Lemma 2.1. *Let $\mathcal{P}$ be a topological property. If a space X has a dense subspace $D \subseteq X$ with the property $\mathcal{P}$, then X is star $\mathcal{P}$.*

Example 2.2. *There exists a Tychonoff pseudocompact SCE-space which is not star Lindelöf.*

Proof: Let $D = \{d_\alpha : \alpha < \mathfrak{c}\}$ be a discrete space of cardinality $\mathfrak{c}$ and let $X = N_{\mathfrak{c}} D$. Then X is Tychonoff pseudocompact, since $\beta D \times \mathfrak{c}$ is a dense countably compact subspace of X. Since the extent of countably compact space is countable, then $e(\beta D \times \mathfrak{c}) \leq \omega$, thus X is SCE-space by Lemma 2.1.

Now we show that X is not star Lindelöf. For each $\alpha < \mathfrak{c}$, let $U_\alpha = \{d_\alpha\} \times (\alpha, \mathfrak{c}]$. Let

$$\mathcal{U} = \{U_\alpha : \alpha < \mathfrak{c}\} \cup \{\beta D \times [0, \mathfrak{c})\}.$$

Then $\mathcal{U}$ is an open cover of X. It suffices to show that there exists $x \in X$ such that $x \notin st(F, \mathcal{U})$ for any Lindelöf subset F of X. Let F be a Lindelöf subset of X and let $A = \{\alpha : \langle d_\alpha, \mathfrak{c}\rangle \in F\}$. The set A is countable, since F is a Lindelöf subset of X and $D \times \{\mathfrak{c}\}$ is a discrete and closed subset of X. Then there exists $\alpha_1 < \mathfrak{c}$ such that $\langle d_\alpha, \mathfrak{c}\rangle \notin F$ for each $\alpha > \alpha_1$. Let $F' = F \setminus \cup\{U_\alpha : \alpha \in A\}$. We have the following two cases:

(a) If $F' = \emptyset$, choose $\alpha < \mathfrak{c}$ such that $\alpha > \alpha_1$. Then $F \cap U_\alpha = \emptyset$, hence $\langle d_\alpha, \mathfrak{c}\rangle \notin st(F, \mathcal{U})$, since U_α is the only element of $\mathcal{U}$ containing $\langle d_\alpha, \mathfrak{c}\rangle$.

(b) If $F' \neq \emptyset$, since $F'$is closed in F, then F' is Lindelöf and $F' \subseteq \beta D \times \mathfrak{c}$, hence $\pi(F')$ is a Lindelöf subset of the countably compact space $\mathfrak{c}$, where $\pi : \beta D \times \mathfrak{c} \to \mathfrak{c}$ is the projection. Hence there exists $\alpha_2 < \mathfrak{c}$ such that $\pi(F') \cap (\alpha_2, \mathfrak{c}) = \emptyset$. Choose $\alpha < \mathfrak{c}$ such that $\alpha > \max\{\alpha_1, \alpha_2\}$. Then $\langle d_\alpha, \mathfrak{c}\rangle \notin st(F, \mathcal{U})$, since U_α is the only element of $\mathcal{U}$ containing $\langle d_\alpha, \mathfrak{c}\rangle$ and $U_\alpha \cap F = \emptyset$, which completes the proof. □

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References

[1] Aiken L.P., *Star-covering properties: generalized Ψ-spaces, countability conditions, reflection*, Topology Appl. **158** (2011), 1732–1737.

[2] Alas O.T., Junqueira L.R., Wilson R.G., *Countability and star covering properties*, Topology Appl. **158** (2011), 620–626.

[3] Alas O.T., Junqueira L.R., van Mill J., Tkachuk V.V., Wilson R.G., *On the extent of star countable spaces*, Cent. Eur. J. Math. **9** (2011), no 3, 603–615.

[4] van Douwen E.K., Reed G.M., Roscoe A.W., Tree I.J., *Star covering properties*, Topology Appl. **39** (1991), 71–103.

[5] Engelking R., *General Topology*, revised and completed edition, Heldermann, Berlin, 1989.

[6] Matveev M.V., *A survey on star-covering properties*, Topological Atlas preprint no. 330, 1998.

[7] van Mill J., Tkachuk V.V., Wilson R.G., *Classes defined by stars and neighborhood assignments*, Topology Appl. **154** (2007), 2127–2134.

[8] Noble N., *Countably compact and pseudocompact products*, Czechoslovak Math. J. **19(3)** (1969), 390–397.

[9] Rojas-Sánchez A.D., Tamariz-Mascarúa Á., *Spaces with star countable extent*, Comment. Math. Univ. Carolin. **57** (2016), no 3, 381–395.

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