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ON THE DERIVED LENGTH OF UNITS IN GROUP ALGEBRA

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Abstract. Let G be a finite group G, K a field of characteristic $p \geqslant 17$ and let U be the group of units in KG. We show that if the derived length of U does not exceed 4, then G must be abelian.

Keywords: group algebra; group of units; derived subgroup

MSC 2010: 16S34, 16U60

1. Introduction

Let KG be the group algebra of a group G over a field K of positive characteristic p and let $U(KG) = U$ denote its multiplicative group of units. An interesting problem is to relate the structural properties of G with those of $U(KG)$. The conditions under which U is solvable was started in 1970s independently by Bateman in [3], and Motose and Tominaga in [16] for finite groups. Then further results were given by Motose and Ninomiya in [15] and Bovdi and Khripta in [6], [7]. Finally Passman in [17] gave necessary and sufficient conditions to have U solvable when G is finite. Results on the derived length of the group of units of a modular group algebra are described well by Bovdi in [4]. The final classification for arbitrary groups was also given by Bovdi in [5]. From the result of Bateman in [3], one has that U is solvable if and only if the commutator subgroup G' of G is a finite p-group, if $|K| > 3$. As a corollary, we get that the p-elements of G form a normal subgroup such that G/P is abelian. The natural question is to ask about the derived length of U once it is assumed to be solvable. It seems quite difficult to give a general formula for the derived length of U. Only a few results have been proved. Shalev in [20], Kurdics in [13], Sahai and Chandra in [18], [19], [9] and [10] have investigated group algebras with units having derived length at most two and three, respectively, over fields of finite characteristic. Baginski in [1] showed that if G is a finite non-abelian p-group

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such that G' is cyclic, then the derived length of U is $\lceil \log_2(|G'|+1)\rceil$, where $\lceil r\rceil$ denotes the minimal integer not smaller than r for a real number r. This result was extended by Balogh and Li to an arbitrary group G with a cyclic derived subgroup of p-power order $p > 2$ in [2]. From these results it easily followed that if G is a torsion nilpotent non-abelian group, then the derived length of U is at least $\lceil \log_2(p+1)\rceil$. Finally Catino and Spinelli characterized group algebras over any torsion nilpotent group for which this lower bound is attained in [8]. The same for infinite groups was given recently by Lee-Sehgal-Spinelli in [14].

In this paper we extend the result from torsion nilpotent groups to groups G with the unit group $U = U(KG)$ in KG having derived length at most four. In an earlier paper [11], we discussed a special case when $(U^{(3)}, U') = \{1\}$ and G is of odd order using a combinatorial argument after considering several cases and subcases. In the present paper we resolve it completely for any G with $U = U(KG)$ of derived length four without any condition on the order of G. The present argument is far simpler and in particular, it avoids the combinatorial argument altogether. Our main result can be stated as follows:

Theorem 1.1. *Let K be a field of characteristic $p \geqslant 17$ and let G be a finite group. Then G is abelian if and only if U satisfies $U^{(4)} = \{1\}$.*

We now mention the notation to be followed. For subsets X, Y of a group G, we denote by (X,Y) the subgroup of G generated by all commutators $(x,y) = x^{-1}y^{-1}xy$ with $x \in X$ and $y \in Y$, i.e.,

$$(X,Y) = \langle x^{-1}y^{-1}xy\colon\ x \in X,\ y \in Y\rangle.$$

More generally, a commutator of weight $n \geqslant 2$ is defined inductively by the rule $(x_1, x_2, \dots, x_n) = ((x_1, x_2, \dots, x_{n-1}), x_n)$. The derived subgroups of G are denoted as $G^{(0)} = G$, $G^{(1)} = G' = (G,G)$, and $G^{(i)} = (G^{(i-1)}, G^{(i-1)})$ for all $i > 0$. If G is solvable, $G^{(n)} = 1$ for some integer n and the smallest such integer is called the derived length of G.

Lie algebraic properties of KG play an important role in our investigation. For $x, y \in KG$, we denote their Lie commutator by $[x,y]$, i.e., $[x,y] = xy - yx$. Also, $O_p(G)$ stands for the maximal normal p-subgroup of G, and $\Delta(G)$ denotes the augmentation ideal of the group algebra KG. For any two elements $x, h \in G$, x^h denotes the conjugation of x by h, that is, $h^{-1}xh$. By a p'-element or a p'-automorphism of a group G we mean an element or an automorphism of G whose order is not divisible by p. All groups considered are finite.

2. Key steps in the proof of Theorem 1.1

We first consider the simpler case when G is a finite p-group. Then we briefly outline the key steps that we are going to adopt when G is not a finite p-group.

2.1. G is a finite p-group. The following result can be found in [8].

Result 2.1. If KG is a non-commutative group algebra of a torsion nilpotent group G over a field K of positive characteristic p such that U is solvable, then the derived length of U is at least $\lceil \log_2(p+1) \rceil$.

Let KG be a group algebra such that $\operatorname{Char} K = p \geqslant 17$ and G is a finite p-group such that U satisfies $U^{(4)} = \{1\}$. A finite p-group is nilpotent and torsion. If G is non-abelian then by Result 2.1, the derived length of U for $p \geqslant 17$ is $\lceil \log_2(p+1) \rceil \geqslant \lceil \log_2(17+1) \rceil \approx \lceil 4.16 \rceil = 5$. Thus U can satisfy $U^{(4)} = \{1\}$ only, if G is abelian.

2.2. G is not a finite p-group. Consider a group G which is not a p-group such that the group U of units in KG satisfies $U^{(4)} = \{1\}$ and $\operatorname{char}(K) = p \geqslant 17$. We will proceed as follows.

(i) First we will show that G can be written as a semidirect product of P and H where P is a p-group and H is an abelian p'-group.

(ii) Our aim then is to express G as a direct product of P and H. If this is not possible, then there will exist an element h in H that will induce a non-identity p'-automorphism on P.

(iii) When P is elementary abelian, we will show that the non-identity p'-automorphism on P induced by h can be used to construct a non trivial element in $U^{(4)}$.

(iv) We will then reduce the argument for general P to the case when P is elementary abelian by exploiting the Frattini subgroup as follows.

When P is any p-group, let us assume $(P, h) \neq \{1\}$ for some $h \in H$. As the Frattini subgroup $\Phi(P)$ is a characteristic subgroup of P, we have $h\Phi(P) = \Phi(P)$ and hence h induces an automorphism on $P/\Phi(P)$. Now $P/\Phi(P)$ is elementary abelian (by [12], Theorem 5.1.3). If we assume the result for elementary abelian group then h will induce the identity automorphism on the elementary abelian group $P/\Phi(P)$. We can then invoke a well known result by Burnside ([12], Theorem 5.1.4). It states that if ψ is a p'-automorphism of a p-group P inducing the identity on $P/\Phi(P)$, then ψ must be the identity automorphism on P itself.

Hence Burnside's result allows us to conclude that h induces the identity automorphism on P as well. Thus we get $G = P \times H$.

(v) The fact that P will also be abelian now easily follows from Result 2.1 above regarding p-groups. Thus the sufficient part of the main result is proved.

(vi) The necessary part, that is, when G is abelian is a trivial case.

3. Certain useful results

In this section we write down certain well-known results as well some observations to be used in the later part. For any two elements x, y in KG, it is easy to observe that

$$xy - 1 = (x-1)(y-1) + (x-1) + (y-1). \tag{3.1}$$

Let J be any ideal of KG and let $x, y \in KG$ be such that $x - 1 \in J^i$ and $y - 1 \in J^j$ for some $i, j > 0$. Then it can be easily established using (3.1) that

$$(x, y) \equiv 1 + [x, y] \pmod{J^{i+j+1}}. \tag{3.2}$$

We are going to use the following necessary condition for U to be solvable ([5]).

Theorem 3.1. *Let K be a field of finite characteristic $p > 3$, and $O_p(G)$ a maximal normal p-subgroup of the finite group G. If the group $U(KG)$ is solvable then $G/O_p(G)$ is abelian.*

The next result can be found in [21], Theorem 3.5.

Theorem 3.2. *Let G be a group of order $p^a b$ and $(p, b) = 1$ and let K be a field of characteristic p. Assume that G has a normal Sylow p-subgroup P. Then the Jacobson radical $J = J(KG)$ of KG is $J = \Delta(P)KG$.*

The following can be found in [12], Theorem 5.3.6.

Theorem 3.3. *If A is a p'-group of automorphisms of the p-group Q, then $(Q, A, A) = (Q, A)$. In particular, if $(Q, A, A) = \{1\}$, then $A = \{1\}$.*

Note that $(Q, A) = \{q^{-1}\sigma_a(q)\colon\ q \in Q \text{ and } a \in A\}$, where σ_a denotes the automorphism of Q corresponding to $a \in A$.

4. Proof of theorem 1.1

In this section, we will provide proofs for the steps outlined in Section 2.2. In the first subsection, we will show that if $U^{(4)} = \{1\}$ then G is a semi-direct product of a p-group P and an abelian p'-group H. If that semi-direct product is not a direct product, then we will construct a nontrivial element in $U^{(4)} = (U^{(3)}, U^{(3)})$ in the second subsection. We may assume P to be elementary abelian as we indicated in Section 2.2.

4.1. G is a semidirect product of a p-group and an abelian p'-group. Let KG be the group algebra of a finite group G over a field K of characteristic $p \geqslant 5$, such that the unit group $U = U(KG)$ satisfies the condition $U^{(4)} = \{1\}$. Then U is solvable and according to Theorem 3.1, $G/O_p(G)$ is abelian.

Lemma 4.1. *If $G/O_p(G)$ is abelian, then G has a normal Sylow p-subgroup.*

Proof. Let P be a Sylow p-subgroup of G such that $O_p(G) \leqslant P$. Let $x \in P$ and $g \in G$. Since $G/O_p(G)$ is abelian, we have $G' \leqslant O_p(G)$. Therefore, $x^{-1}g^{-1}xg \in O_p(G)$, which implies that $g^{-1}xg \in P$ for every $x \in P$ and $g \in G$. Hence, $P \trianglerighteq G$. $\square$

So $O_p(G) = P$, a Sylow p-subgroup. Now, $|P|$ and $[G : P]$ are relatively prime, hence by the Schur-Zassenhaus theorem ([12], Theorem 6.2.1), we have $G = P \rtimes H$, where H is a p'-subgroup of G. Also, by the above conditions, H is abelian.

Further, the Jacobson radical $J = J(KG)$ is given by $\Delta(P)KG$ by Theorem 3.2.

4.2. G is a direct product of a p-group and an abelian p'-group. In this subsection we prove the following lemma, which is the most crucial ingredient in establishing Theorem 1.1.

Lemma 4.2. *Let* $\operatorname{Char} K = p \geqslant 17$. *Let G be a finite group. Suppose that $U = U(KG)$ satisfies $U^{(4)} = \{1\}$. Then $G = P \times H$, where P is a p-group and H is an abelian p'-group, where p' is odd.*

We now outline the strategy for the proof of the above lemma. First observe that we may assume P to be elementary abelian by (iv) in Section 2.2. We know from the last paragraph of Section 4.1 that $G = P \rtimes H$, where P is a p-group and H is an abelian p'-group. Also $P \trianglerighteq G$. We need to show that $(P, H) = \{1\}$. We will show that if $(P, H) \neq \{1\}$, then we can construct a nontrivial element in $U^{(4)}$. Clearly it will be enough to show nontriviality modulo a suitable power of the Jacobson radical J.

Construction of a nontrivial element in $U^{(4)}$ when $(P, H) \neq \{1\}$.

Note that by Theorem 3.3, $(P, H) = (P, H, H)$ and $(P, H) \leqslant P$ as P is normal in G. If $(P, H) = (P, H, H) \neq \{1\}$, then there exists $x \in (P, H) \subset G'$ and $h \in H$ such that $(x, h) \neq 1$. In the rest of this section, we will show that $(x, h) \neq 1$ results in a nontrivial element u_4 in $U^{(4)}$. Let x_i denote $\big(x, \underbrace{h, h, \ldots, h}_{i\,\text{times}}\big)$ for $i = 1, 2, \ldots$.

As $x - 1 \in \Delta(P)$ is contained in the Jacobson radical J, $u = 1 + h(x-1)$ is a unit in KG. We now proceed to form commutators of suitable elements in U and obtain elements in $U', U^{(2)}, U^{(3)}$ and finally in $U^{(4)}$. We first consider $u_1 = (u, h) \in U'$ and $v_1 = (u, x) \in U'$ and construct elements in $U^{(2)}$ using u_1, v_1, x and h. We also keep track of their behaviour modulo a suitable power of the Jacobson radical J. We begin by observing u_1 and v_1 modulo J^2:

$$\begin{aligned} u_1 = (u, h) &= 1 + u^{-1}h^{-1}[u, h] \\ &\equiv 1 + (1 - h(x-1))((x-1)h - h(x-1)) \pmod{J^2} \\ &\equiv 1 + (x-1)h - h(x-1) \pmod{J^2} \\ &= 1 + hx((x, h) - 1) \pmod{J^2} \\ &\equiv 1 + h(x_1 - 1) \pmod{J^2}. \end{aligned} \tag{4.1}$$

As $x - 1 \in J$, we use identity (3.2) to obtain the following:

$$\begin{aligned} v_1 = (u, x) &\equiv 1 + [u, x] \pmod{J^3} \\ &= 1 + (x + h(x-1)x - x - xh(x-1)) \pmod{J^3} \\ &= 1 + (hx - xh)(x-1) \pmod{J^3} \\ &= 1 + hx(1 - (x, h))(x-1) \pmod{J^3} \\ &\equiv 1 - h(x-1)(x_1 - 1) \pmod{J^3}. \end{aligned} \tag{4.2}$$

Next we consider $u_2 = (u_1, x)$ and $v_2 = (v_1, x)$. As $x \in (P, H) \subset G' \subset U'$, u_2 and v_2 are in $U^{(2)}$. Their residual properties modulo J can be observed as follows.

$$\begin{aligned} u_2 = (u_1, x) &\equiv 1 + [u_1, x] \pmod{J^3} \\ &= 1 + \{x + h(x_1 - 1)x - x - xh(x_1 - 1) \\ &= 1 + (hx - xh)(x_1 - 1) \pmod{J^3} \\ &= 1 + hx(1 - (x, h))(x_1 - 1) \pmod{J^3} \\ &\equiv 1 - h(x_1 - 1)^2 \pmod{J^3}, \end{aligned} \tag{4.3}$$

$$\begin{aligned}
(4.4) \qquad v_2 &= (v_1, x) \equiv 1 + [v_1, x] \quad (\mathrm{mod}\ J^4)\\
&= 1 - \{h(x-1)(x_1-1)x - xh(x-1)(x_1-1)\\
&= 1 + (xh - hx)(x-1)(x_1-1) \quad (\mathrm{mod}\ J^4)\\
&\equiv 1 + h(x_1-1)^2(x-1) \quad (\mathrm{mod}\ J^4).
\end{aligned}$$

Finally we obtain an element $u_3 = (u_2, v_2)$ in $U^{(3)}$. We have

$$\begin{aligned}
(4.5) \qquad u_3 &= (u_2, v_2) \equiv 1 + [u_2, v_2] \quad (\mathrm{mod}\ J^6)\\
&= 1 + \{-h(x_1-1)^2 h(x_1-1)^2(x-1)\\
&\quad + h(x_1-1)^2(x-1)h(x_1-1)^2\} \quad (\mathrm{mod}\ J^6)\\
&= 1 + h(x_1-1)^2\{(x-1)h\\
&\quad - h(x-1)\}(x_1-1)^2 \quad (\mathrm{mod}\ J^6)\\
&= 1 + h(x_1-1)^2 hx(x_1-1)^3 \quad (\mathrm{mod}\ J^6)\\
&\equiv 1 + h^2(x_1^h-1)^2(x_1-1)^3 \quad (\mathrm{mod}\ J^6).
\end{aligned}$$

With $u_2 \in U^{(2)} \subseteq U'$ and $x \in U'$, we obtain $w_2 = (u_2, x) \in (U', U') = U^{(2)}$ and $v_3 = (u_2, w_2) \in (U^{(2)}, U^{(2)}) = U^{(3)}$. Noting that $x_1 \equiv 1 \pmod{J}$, we examine w_2 and v_3 modulo powers of J:

$$\begin{aligned}
(4.6) \qquad w_2 &= (u_2, x)\\
&\equiv 1 + [1 - h(x_1-1)^2, x] \quad (\mathrm{mod}\ J^4)\\
&= 1 - hx(x_1-1)^2 + xh(x_1-1)^2 \quad (\mathrm{mod}\ J^4)\\
&\equiv 1 + h(x_1-1)^3 \quad (\mathrm{mod}\ J^4),
\end{aligned}$$

$$\begin{aligned}
(4.7) \qquad v_3 &= (u_2, w_2)\\
&\equiv 1 + [1 - h(x_1-1)^2, 1 + h(x_1-1)^3] \quad (\mathrm{mod}\ J^6)\\
&= 1 + h(x_1-1)^3 h(x_1-1)^2 - h(x_1-1)^2 h(x_1-1)^3 \quad (\mathrm{mod}\ J^6)\\
&= 1 + h(x_1-1)^2\{(x_1-1)h - h(x_1-1)\}(x_1-1)^2 \quad (\mathrm{mod}\ J^6)\\
&\equiv 1 + h(x_1-1)^2 h((x_1,h)-1)(x_1-1)^2 \quad (\mathrm{mod}\ J^6)\\
&= 1 + h^2(x_1^h-1)^2(x_2-1)(x_1-1)^2 \quad (\mathrm{mod}\ J^6).
\end{aligned}$$

From $u_3 \in U^{(3)} \subseteq U^{(2)}$ and $u_2 \in U^{(2)}$, we obtain $w_3 = (u_3, u_2) \in (U^{(2)}, U^{(2)}) = U^{(3)}$ and view it modulo J^8.

$$\begin{aligned}
(4.8)\ w_3 &= (u_3, u_2)\\
&\equiv 1 + [1 + h^2(x_1^h-1)^2(x_1-1)^3, 1 - h(x_1-1)^2]\\
&= 1 + h(x_1-1)^2 h^2(x_1^h-1)^2(x_1-1)^3 - h^2(x_1^h-1)^2(x_1-1)^3 h(x_1-1)^2\\
&= \{h^2(x_1^h-1)^2 h(x_1^h-1)^2(x_1-1) - h^2(x_1^h-1)^2(x_1-1)^3 h\}(x_1-1)^2
\end{aligned}$$

$$
\begin{aligned}
&= 1 + \{h^3(x_1^{h^2} - 1)^2(x_1^h - 1)^2(x_1 - 1) - h^3(x_1^{h^2} - 1)^2(x_1^h - 1)^3\}(x_1 - 1)^2 \\
&= 1 + h^3(x_1^{h^2} - 1)^2(x_1^h - 1)^2\{x_1 - x_1^h\}(x_1 - 1)^2 \\
&= 1 + h^3(x_1^{h^2} - 1)^2(x_1^h - 1)^2 x_1(1 - (x_1, h))(x_1 - 1)^2 \\
&\equiv 1 - h^3(x_1^{h^2} - 1)^2(x_1^h - 1)^2(x_2 - 1)(x_1 - 1)^2.
\end{aligned}
$$

Finally we construct $u_4 = (v_3, w_3)$ in $U^{(4)}$ and examine it modulo J^{13}. Note that $x_2 \equiv 1 \pmod{J}$.

$$
\begin{aligned}
(4.9)\, u_4 &= (v_3, w_3) \equiv 1 + [v_3 - 1, w_3 - 1] \\
&\equiv 1 + [h^2(x_1^h - 1)^2(x_1 - 1)^2(x_2 - 1), \\
&\qquad - h^3(x_1^{h^2} - 1)^2(x_1^h - 1)^2(x_1 - 1)^2(x_2 - 1)] \\
&= 1 - h^2(x_1^h - 1)^2(x_1 - 1)^2(x_2 - 1)h^3(x_1^{h^2} - 1)^2(x_1^h - 1)^2(x_1 - 1)^2(x_2 - 1) \\
&\quad + h^3(x_1^{h^2} - 1)^2(x_1^h - 1)^2(x_1 - 1)^2(x_2 - 1)h^2(x_1^h - 1)^2(x_1 - 1)^2(x_2 - 1) \\
&= 1 - \{h^2(x_1^h - 1)^2(x_1 - 1)^2(x_2 - 1)h^3(x_1^{h^2} - 1)^2 \\
&\quad - h^3(x_1^{h^2} - 1)^2(x_1^h - 1)^2(x_1 - 1)^2(x_2 - 1)h^2\}(x_1^h - 1)^2(x_1 - 1)^2(x_2 - 1) \\
&= 1 - \{h^5(x_1^{h^4} - 1)^2(x_1^{h^3} - 1)^2(x_2^{h^3} - 1)(x_1^{h^2} - 1)^2 \\
&\quad - h^5(x_1^{h^4} - 1)^2(x_1^{h^3} - 1)^2(x_1^{h^2} - 1)^2(x_2^{h^2} - 1)\}(x_1^h - 1)^2(x_1 - 1)^2(x_2 - 1) \\
&= 1 - \{h^5(x_1^{h^4} - 1)^2(x_1^{h^3} - 1)^2(x_1^{h^2} - 1)^2\}(x_2^{h^3} - x_2^{h^2}) \\
&\qquad \times (x_1^h - 1)^2(x_1 - 1)^2(x_2 - 1) \\
&= 1 - \{h^5(x_1^{h^4} - 1)^2(x_1^{h^3} - 1)^2(x_1^{h^2} - 1)^2\}h^{-2}x_2((x_2, h) - 1)h^2 \\
&\qquad \times (x_1^h - 1)^2(x_1 - 1)^2(x_2 - 1) \\
&\equiv 1 - h^5(x_1^{h^4} - 1)^2(x_1^{h^3} - 1)^2(x_1^{h^2} - 1)^2(x_3^{h^2} - 1)(x_1^h - 1)^2(x_1 - 1)^2(x_2 - 1).
\end{aligned}
$$

As $x_1^{h^j}$ is a conjugate of x_1 and $(x_1 - 1) \in J - J^2$, it follows that $(x_1^{h^j} - 1) \in J - J^2$. Further, x_2 is an element of order p in P as $x_2 = (x_1, h) = (x, h, h) = 1$ would imply $x_1 = (x, h) = 1$ by an obvious application of Theorem 3.3 with $Q = \langle x \rangle$ and $A = \langle h \rangle$. Therefore, it follows that $(x_2 - 1) \in J - J^2$. By the same argument, $(x_3 - 1) \in J - J^2$. Therefore the product on the right hand side in (4.9) belongs to $J^{12} - J^{13}$. Consequently by (4.9), u_4 is nontrivial modulo J^{13}, which contradicts our assumption that $U^{(4)} = \{1\}$. Therefore, we cannot have $(x, h) \neq 1$ which we have assumed for the construction of u_4. It follows that H must act trivially on P. Therefore, G must be a direct product of P and H and Lemma 4.2 follows.

Theorem 1.1 now follows, as G is a direct product of a p-group P and an abelian group H, and P must be abelian too as explained in Result 2.1.

5. When the derived length of U is smaller than $\lceil \log_2(2p) \rceil$

Let us consider the following result by Balogh and Li (see [2], Lemma 2.3).

Result 5.1. Let G be a group with a derived subgroup $G' = \langle u | u^{p^n} = 1 \rangle$, where p is an odd prime, and let $\mathrm{char}(K) = p$. Assume that the order of G/C, where $C = C_G(G')$=centraliser of G' in G, is divisible by an odd prime $q \neq p$. Then the derived length of U is greater than or equal to $\lceil \log_2(2p^n) \rceil$.

Very simple consequences of the above result tell us about the commutativity of the group G of odd order pm $[(p, m) = 1]$ when the derived length of the unit group $U(KG)$ is small compared to the characteristic p of the field K. We can write the following remark on its basis.

Remark 5.2. Let K be a field of characteristic p and let G be any group of odd order pm where m is co-prime to p. If the derived length of $U(KG)$ is strictly less than $\lceil \log_2(2p) \rceil$, then G must be abelian.

But the same statement can be proved also using our approach as described at the beginning of Section 4.1. Let p be the characteristic of the field K and d the derived length of the units $U(KG)$. As discussed at the beginning of Section 4.1, we can conclude that G has a normal p-Sylow subgroup P and $G = P \rtimes H$, where H is an abelian p'-subgroup of G.

Note that $C_G(G')$ is a normal subgroup of G since it contains G'.

P r o o f of Remark 5.2. Suppose G is of odd order pm where $(m, p) = 1$. Then the p-Sylow subgroup P is cyclic of order p. We assume that the derived length d of $U(KG)$ satisfies $d < \lceil \log_2(2p) \rceil$. We want to show that $G = P \times H$, i.e., $(P, H) = \{1\}$. If possible, let $(P, H) \neq \{1\}$. As P is normal in G, (P, H) is a subgroup of P and hence $(P, H) = P$. Therefore, we have $G' = (P, H) = P$.

We now apply the Result 5.1 to G and $G' = P$. Let C be the centraliser of G', i.e., $C = \{x \in G \colon xy = yx \text{ for all } y \in G'\}$. Then $h \notin C$ and hC is a nontrivial element of G/C. Let l be the order of h, so l is coprime to p. Then hC is a nontrivial element of order dividing l in G/C and a suitable power of hC gives an element of prime order q in G/C. As $(l, p) = 1$ and the order of G is odd, q must be an odd prime other than p. By Result 5.1, we can conclude that the derived length of $U(KG)$ must be at least $\lceil \log_2(2p) \rceil$. But this contradicts our assumption that the derived length of $U(KG)$ is smaller than $\lceil \log_2(2p) \rceil$. Hence, we cannot have $(P, H) \neq \{1\}$ and the proof of the remark is complete. $\square$

Again, very simple consequences of Result 5.1 give the following remark. But yet again it can be proved using our techniques. We denote the Frattini subgroup of a group G by $\Phi(G)$, which is the intersection of all maximal subgroups of G. It is well-known that $\Phi(G)$ is a characteristic subgroup of G. We can easily extend Remark 5.2 to any group G of odd order $p^n m$ with $(p,m)=1$ provided the quotient of its p-Sylow subgroup by the Frattini subgroup is cyclic.

Remark 5.3. Let K be a field of characteristic p and G any group of odd order $p^n m$ where m is co-prime to p. Let P be a p-Sylow subgroup and $\Phi(P)$ the Frattini subgroup of P. If the quotient $P/\Phi(P)$ is cyclic and the derived length of $U(KG)$ is strictly less than $\lceil \log_2(2p) \rceil$, then G must be abelian.

Proof of Remark 5.3. Assume that G is of odd order $p^n m$, and the derived length d of $U(KG)$ satisfies $d < \lceil \log_2(2p) \rceil$. As before, the p-Sylow subgroup P of G is normal and $G = P \rtimes H$ where H is an abelian p'-subgroup of G. By part (iv) of Section 2.2, it is enough to show that the induced conjugacy action of H on the group $P/\Phi(P)$ is trivial. Now $P/\Phi(P)$ is an elementary abelian group which moreover is assumed to be cyclic. Therefore $P/\Phi(P)$ is cyclic of order p and by Theorem 5.2, we know that the conjugacy action of H on $P/\Phi(P)$ has to be trivial. Hence the conjugacy action of H on P itself is trivial and G must be abelian. □

We conclude by observing that a group G of odd order with a cyclic p-Sylow subgroup must be abelian if the derived length of $U(KG)$ is smaller than $\lceil \log_2(2p) \rceil$ where p is the characteristic of the field K.

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