Xiaoqi Wei; Yan Gu

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# DEPTH AND STANLEY DEPTH OF THE FACET IDEALS OF SOME CLASSES OF SIMPLICIAL COMPLEXES

Xiaoqi Wei, Yan Gu, Suzhou

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*Abstract.* Let $\Delta_{n,d}$ (resp. $\Delta'_{n,d}$) be the simplicial complex and the facet ideal $I_{n,d} = (x_1 \dots x_d, x_{d-k+1} \dots x_{2d-k}, \dots, x_{n-d+1} \dots x_n)$ (resp. $J_{n,d} = (x_1 \dots x_d, x_{d-k+1} \dots x_{2d-k}, \dots, x_{n-2d+2k+1} \dots x_{n-d+2k}, x_{n-d+k+1} \dots x_n x_1 \dots x_k)$). When $d \geqslant 2k+1$, we give the exact formulas to compute the depth and Stanley depth of quotient rings $S/J_{n,d}$ and $S/I^t_{n,d}$ for all $t \geqslant 1$. When $d = 2k$, we compute the depth and Stanley depth of quotient rings $S/J_{n,d}$ and $S/I_{n,d}$, and give lower bounds for the depth and Stanley depth of quotient rings $S/I^t_{n,d}$ for all $t \geqslant 1$.

*Keywords*: monomial ideal; facet ideal; depth; Stanley depth

*MSC 2010*: 13C15, 13P10, 13F20, 13F55

## 1. Introduction

Let $K$ be a field and $S = K[x_1, \dots, x_n]$ the polynomial ring over $K$ in $n$ variables. Let $M$ be a finitely generated $\mathbb{Z}^n$-graded $S$-module. A Stanley decomposition $\mathcal{D}$ of $M$ is a finite direct sum of $K$-vector spaces

$$\mathcal{D}\colon\ M = \bigoplus_{i=1}^{r} u_i K[Z_i],$$

where $u_i \in M$ is homogeneous and $Z_i \subseteq \{x_1, \dots, x_n\}$, $i = 1, \dots, r$, and its Stanley depth, $\operatorname{sdepth}(\mathcal{D})$, is defined as $\min\{|Z_i|\colon\ i = 1, \dots, r\}$. The number

$$\operatorname{sdepth}(M) = \max\{\operatorname{sdepth}(\mathcal{D})\colon\ \mathcal{D} \text{ is a Stanley decomposition of } M\}$$

is called the Stanley depth of $M$.

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Stanley conjectured in [13] that $\operatorname{sdepth}(M) \geqslant \operatorname{depth}(M)$ for any $\mathbb{Z}^n$-graded $S$-module $M$. There are many researches on this conjecture, especially when $M$ has the form $S/I$ or $I$ with $I$ a squarefree monomial ideal of $S$, e.g., [1], [8], [11], [12]. In [6], Duval et al. constructed an explicit counterexample to disprove the Stanley conjecture for $S/I$, where $I$ is a monomial ideal of $S$. Thus the Stanley conjecture is open for monomial ideals $I \subset S$.

Let $\Delta \subset 2^{[n]}$ be a simplicial complex. Each element of $\Delta$ is called a face of $\Delta$, and a face $F$ is called a facet if $F$ is a maximal face with respect to inclusion. Let $\mathcal{F}(\Delta)$ denote the set of facets of $\Delta$. If $F \in \mathcal{F}(\Delta)$, we denote $x_F = \prod\limits_{j \in F} x_j$. Then the facet ideal of $\Delta$ is a squarefree monomial ideal $I(\Delta)$ of $S$, $I(\Delta) = (x_F \colon F \in \mathcal{F}(\Delta))$. The facet ideal was studied by Faridi in [7] from the depth perspective. In this paper, we consider depth and Stanley depth of $I(\Delta)$ of some classes of simplicial complexes.

A line graph of length $n$, denoted by $L_n$, is a graph with the vertex set $V = [n]$ and edge set $E = \{\{1,2\},\{2,3\},\dots,\{n-1,n\}\}$. The depth and Stanley depth of the edge ideal associated to $L_n$ (which is in fact the facet ideal of $L_n$) were computed by Morey in [9] and Ştefan in [14], respectively. A cyclic graph $C_n$ is a graph with the vertex set $V = [n]$ and edge set $E \cup \{n,1\}$. The depth and Stanley depth of the edge ideal associated to $C_n$ were computed by Cimpoeaş in [5].

Let $\Delta_{n,d}$ be the simplicial complex with the set of facets $\mathcal{F}(\Delta_{n,d}) = \{\{1,2,\dots,d\}, \{d-k+1, d-k+2,\dots,2d-k\},\dots,\{n-2d+k+1, n-2d+k+2,\dots,n-d+k\},\{n-d+1,n-d+2,\dots,n\}\}$, where $n \geqslant d > k \geqslant 1$. It is easy to see that $d-k \mid n-k$. We denote the facet ideal $I(\Delta_{n,d})$ of $\Delta_{n,d}$ by $I_{n,d}$, where $I_{n,d} = (x_1\dots x_d, x_{d-k+1}\dots x_{2d-k},\dots,x_{n-d+1}\dots x_n)$. When $d = 2$ and $k = 1$, then $I_{n,d} = I(L_n)$.

Let $\Delta'_{n,d}$ be the simplicial complex with the set of facets $\mathcal{F}(\Delta'_{n,d}) = \{\{1,2,\dots,d\}, \{d-k+1, d-k+2,\dots,2d-k\},\dots,\{n-2d+2k+1, n-2d+2k+2,\dots,n-d+2k\}, \{n-d+k+1,\dots,n,1,\dots,k\}\}$, where $d \geqslant 2k \geqslant 2$ and $n \geqslant 3d-3k$. It is easy to see that $d-k \mid n$. We denote the facet ideal $I(\Delta'_{n,d})$ of $\Delta'_{n,d}$ by $J_{n,d}$, where $J_{n,d} = (x_1\dots x_d, x_{d-k+1}\dots x_{2d-k},\dots,x_{n-2d+2k+1}\dots x_{n-d+2k}, x_{n-d+k+1}\dots x_n x_1\dots x_k)$. If $d = 2$ and $k = 1$, then $J_{n,d} = I(C_n)$.

The followings are our main results, which generalize some results of [5], [9], [14].

**Theorem 1.1.** *Let $d \geqslant 2k+1$. Then*

(1) $\operatorname{sdepth}(S/I_{n,d}^t) = \operatorname{depth}(S/I_{n,d}^t) = n - \frac{n-k}{d-k}$ *for all* $t \geqslant 1$,

(2) $\operatorname{sdepth}(S/J_{n,d}) = \operatorname{depth}(S/J_{n,d}) = n - \frac{n}{d-k}$.

**Theorem 1.2.** *Let $d = 2k$. Then*

(1) $\operatorname{sdepth}(S/I_{n,d}) = \operatorname{depth}(S/I_{n,d}) = \frac{(d-2)n}{d} + \lceil \frac{2n}{3d} \rceil$,

(2) $\operatorname{depth}(S/I^t_{n,d}) \geqslant \max\left\{1, \frac{(d-2)n}{d} + \lceil \frac{2n-dt+d}{3d} \rceil\right\}$ *for all* $t \geqslant 1$, $\operatorname{sdepth}(S/I^t_{n,d}) \geqslant \max\left\{1, \frac{(d-2)n}{d} + \lceil \frac{2n-dt+d}{3d} \rceil\right\}$ *for all* $t \geqslant 1$,

(3) $\operatorname{depth}(S/J_{n,d}) = \frac{(d-2)n}{d} + \lceil \frac{2n-d}{3d} \rceil$,
$\operatorname{sdepth}(S/J_{n,d}) = \frac{(d-2)n}{d} + \lceil \frac{2n-d}{3d} \rceil$ *for* $\frac{n}{k} \equiv 0 \pmod 3$ *and* $\frac{n}{k} \equiv 2 \pmod 3$,
$\frac{(d-2)n}{d} + \lceil \frac{2n-d}{3d} \rceil \leqslant \operatorname{sdepth}(S/J_{n,d}) \leqslant \frac{(d-2)n}{d} + \lceil \frac{2n}{3d} \rceil$ *for* $\frac{n}{k} \equiv 1 \pmod 3$.

## 2. Depth and Stanley depth of the facet ideals

First, we recall a well-known result, referred to as the Depth lemma, that will be heavily used in the proofs in this article. Two different versions of the lemma will be used in this article, so both are stated here for ease of reference.

**Lemma 2.1** (Depth lemma). *Let $S$ be a local ring or a Noetherian graded ring with $S_0$ local. If*

$$0 \to A \to B \to C \to 0$$

*is a short exact sequence of finitely generated $S$-modules, where the maps are all homogeneous, then (*[15], *Lemma* 1.3.9*):*

a) *If* $\operatorname{depth}(B) < \operatorname{depth}(C)$, *then* $\operatorname{depth}(A) = \operatorname{depth}(B)$.

b) *If* $\operatorname{depth}(B) = \operatorname{depth}(C)$, *then* $\operatorname{depth}(A) \geqslant \operatorname{depth}(B)$.

c) *If* $\operatorname{depth}(B) > \operatorname{depth}(C)$, *then* $\operatorname{depth}(A) = \operatorname{depth}(C) + 1$.

*Also (see* [3], *Proposition* 1.2.9*):*

d) $\operatorname{depth}(A) \geqslant \min\{\operatorname{depth}(B), \operatorname{depth}(C) + 1\}$.

e) $\operatorname{depth}(B) \geqslant \min\{\operatorname{depth}(A), \operatorname{depth}(C)\}$.

f) $\operatorname{depth}(C) \geqslant \min\{\operatorname{depth}(A) - 1, \operatorname{depth}(B)\}$.

In [12], Rauf proved the analog of Lemma 2.1 (e) for sdepth:

**Lemma 2.2.** *Let $0 \to U \to M \to N \to 0$ be a short exact sequence of finitely generated $\mathbb{Z}^n$-graded $S$-modules. Then*

$$\operatorname{sdepth}(M) \geqslant \min\{\operatorname{sdepth}(U), \operatorname{sdepth}(N)\}.$$

Next, we will discuss our main results in two cases.

**2.1. The case $d \geqslant 2k + 1$.** Let $I \subset S$ be a monomial ideal. The big height of $I$, denoted by $\operatorname{bight}(I)$, is the maximum height of the minimal prime ideals of $I$. The arithmetical rank of $I$, denoted by $\operatorname{ara}(I)$, is the minimum number $r$ of elements of $S$ such that the ideal $(a_1, a_2, \ldots, a_r)$ has the same radical as $I$. It is well-known that

$$\operatorname{ht}(I) \leqslant \operatorname{bight}(I) \leqslant \operatorname{pd}(S/I) \leqslant \operatorname{ara}(I) \leqslant |G(I)|,$$

where $\mathrm{pd}(S/I)$ denotes the projective dimension of $S/I$ and $G(I)$ denotes the set of minimal monomial generators of $I$. We see that $|G(I_{n,d})| = \frac{n-k}{d-k}$ and $|G(J_{n,d})| = \frac{n}{d-k}$.

A prime ideal $P$ is associated to $I$ if $P = (I : c)$ for some monomial $c \in S$. The set of prime ideals associated to $I$ will be denoted by $\mathrm{Ass}(S/I)$. The associated prime ideals of a monomial ideal are monomial prime ideals. The set $\mathrm{Min}(S/I)$ consists of all prime ideals that are minimal over $I$ with respect to inclusion. It is known that $\mathrm{Min}(S/I) \subset \mathrm{Ass}(S/I)$. When $I$ is squarefree, $\mathrm{Ass}(S/I) = \mathrm{Min}(S/I)$.

Our proofs of the main results make heavy use of the following lemma.

**Lemma 2.3.** *$P = (x_{k+1}, x_{k+1+(d-k)}, x_{k+1+2(d-k)}, \ldots, x_{n-d+k+1}) \in \mathrm{Min}(S/I_{n,d})$, and $P' = (x_{k+1}, x_{k+1+(d-k)}, x_{k+1+2(d-k)}, \ldots, x_{n-d+2k+1}) \in \mathrm{Min}(S/J_{n,d})$.*

Proof. Let we have $a_i = x_{1+(i-1)(d-k)}x_{2+(i-1)(d-k)} \ldots x_{d+(i-1)(d-k)}$ and $b_j = x_{k+1+(j-1)(d-k)}$, where $i, j = 1, 2, \ldots, \frac{n-k}{d-k}$. Then $I_{n,d} = (a_1, a_2, \ldots, a_{(n-k)/(d-k)})$ and $P = (b_1, b_2, \ldots, b_{(n-k)/(d-k)})$. It is easy to see that $b_j$ divides $a_i$ if and only if $i = j$, so $I_{n,d} \subset P$. We assume that $P$ is not minimal over $I_{n,d}$. Let $P_0 \subsetneq P$ be a minimal prime ideal of $I_{n,d}$. Since $I_{n,d}$ is squarefree, $P_0 \subsetneq P$ is a monomial prime ideal, and there exists $a_i$ such that none of $G(P_0)$ divides $a_i$. Hence $I_{n,d} \nsubseteq P_0$, a contradiction. Similarly, $P'$ is a minimal prime ideal of $S/J_{n,d}$. □

**Proposition 2.4.** $\mathrm{bight}(I_{n,d}) = \mathrm{pd}(S/I_{n,d}) = \mathrm{ara}(I_{n,d}) = |G(I_{n,d})| = \frac{n-k}{d-k}$.

Proof. Let we have $P = (x_{k+1}, x_{k+1+(d-k)}, x_{k+1+2(d-k)}, \ldots, x_{n-d+k+1}) \in \mathrm{Min}(S/I_{n,d})$ and $\mathrm{ht}(P) = \frac{n-k}{d-k}$ by Lemma 2.3. Then $\frac{n-k}{d-k} \leqslant \mathrm{bight}(I_{n,d}) \leqslant \mathrm{pd}(S/I_{n,d}) \leqslant \mathrm{ara}(I_{n,d}) \leqslant |G(I_{n,d})| = \frac{n-k}{d-k}$. Now the result is clear. □

Now, we give the exact formulas for $\mathrm{sdepth}(S/I_{n,d})$ and $\mathrm{depth}(S/I_{n,d})$.

**Theorem 2.5.** $\mathrm{sdepth}(S/I_{n,d}) = \mathrm{depth}(S/I_{n,d}) = n - \frac{n-k}{d-k}$.

Proof. Since $|G(I_{n,d})| = \frac{n-k}{d-k}$, by [4], Proposition 1.2, we have $\mathrm{sdepth}(S/I_{n,d}) \geqslant n - \frac{n-k}{d-k}$. On the other hand, there exists a prime ideal $P \in \mathrm{Ass}(S/I_{n,d})$ such that $\mathrm{ht}(P) = \frac{n-k}{d-k}$ by Lemma 2.3. Then $\mathrm{sdepth}(S/I_{n,d}) \leqslant n - \frac{n-k}{d-k}$ by [8], Proposition 1.3. By the Auslander-Buchsbaum formula and Proposition 2.4, we get $\mathrm{depth}(S/I_{n,d}) = n - \mathrm{pd}(S/I_{n,d}) = n - \frac{n-k}{d-k}$. □

The following corollary states that the Stanley inequality holds for $I_{n,d}$.

**Corollary 2.6.** $\mathrm{sdepth}(I_{n,d}) \geqslant n - \lfloor \frac{n-k}{2(d-k)} \rfloor \geqslant \mathrm{depth}(I_{n,d})$.

Proof. Since $|G(I_{n,d})| = \frac{n-k}{d-k}$, it follows that $\mathrm{sdepth}(I_{n,d}) \geqslant \max\{1, n - \lfloor \frac{1}{2}|G(I_{n,d})| \rfloor\} = n - \lfloor \frac{n-k}{2(d-k)} \rfloor \geqslant n - \frac{n-k}{d-k} + 1 = \mathrm{depth}(I_{n,d})$ by [10], Theorem 2.3, and Theorem 2.5. □

Next, we present a main result.

**Theorem 2.7.** *For all* $t \geqslant 1$, $\operatorname{sdepth}(S/I^t_{n,d}) = \operatorname{depth}(S/I^t_{n,d}) = n - \frac{n-k}{d-k}$.

P r o o f. We use induction on $n$ and $t$. If $n = d$, then $I^t_{n,d} = (x_1^t \dots x_d^t)$ is principal. It follows that $\operatorname{sdepth}(S/I^t_{n,d}) = \operatorname{depth}(S/I^t_{n,d}) = d - 1 = d - \frac{d-k}{d-k}$ for all $t \geqslant 1$. Assume that $n \geqslant 2d - k$ in the following.

If $t = 1$, the result holds for all $n$ by Theorem 2.5. Now let $t \geqslant 2$. We denote $u := x_{n-d+1} \dots x_{n-d+k}$, $v := x_{n-d+k+1} \dots x_n$ and consider the short exact sequence

$$0 \to \frac{S}{(I^t_{n,d} : uv)} \to \frac{S}{(I^t_{n,d} : v)} \to \frac{S}{((I^t_{n,d} : v), u)} \to 0.$$

Let $G(I_{n,d}) = \{a_1, a_2, \dots, a_{(n-k)/(d-k)}\}$, the same as in the proof of Lemma 2.3, and $w \in G(I^t_{n,d})$. If $a_{(n-k)/(d-k)} \mid w$, then $\frac{w}{a_{(n-k)/(d-k)}} \in G(I^t_{n,d} : uv) \cap I^{t-1}_{n,d}$. If $a_{(n-k)/(d-k)} \nmid w$ and $a_{(n-d)/(d-k)} \mid w$, then $\frac{w}{u} \in G(I^t_{n,d} : uv)$ and $\frac{w}{a_{(n-d)/(d-k)}} \mid \frac{w}{u}$, where $\frac{w}{a_{(n-d)/(d-k)}} \in I^{t-1}_{n,d}$. If $a_{(n-k)/(d-k)} \nmid w$ and $a_{(n-d)/(d-k)} \nmid w$, then $\frac{w}{1} \in G(I^t_{n,d} : uv)$ and $w$ must be divisible by some element of $I^{t-1}_{n,d}$. Hence $(I^t_{n,d} : uv) \subseteq I^{t-1}_{n,d}$. It follows that $(I^t_{n,d} : uv) = I^{t-1}_{n,d}$.

We get $\operatorname{sdepth}(S/(I^t_{n,d} : uv)) = \operatorname{sdepth}(S/I^{t-1}_{n,d}) = n - \frac{n-k}{d-k}$ by induction hypothesis on $t$. Similarly, we prove that $\operatorname{depth}(S/(I^t_{n,d} : uv)) = n - \frac{n-k}{d-k}$.

Since $u$ divides any element of $G(I^t_{n,d})$ which is divided by $a_{(n-k)/(d-k)}$ or $a_{(n-d)/(d-k)}$, we get $((I^t_{n,d} : v), u) = (I^t_{n-2d+2k,d}S, u)$. Notice that $u$ is regular on $S/I^t_{n-2d+2k,d}S$, hence the induction hypothesis on $n$ and [8], Lemma 3.6, imply that

$$\begin{aligned} \operatorname{sdepth}_S(S/((I^t_{n,d} : v), u)) &= \operatorname{sdepth}_{S_{n-2d+2k}}\Big(\frac{S_{n-2d+2k}}{I^t_{n-2d+2k,d}}\Big) + (2d - 2k) - 1 \\ &= (n - 2d + 2k) - \frac{(n - 2d + 2k) - k}{d - k} + 2d - 2k - 1 \\ &= n - \frac{n - k}{d - k} + 1, \end{aligned}$$

where $S_{n-2d+2k} = K[x_1, \dots, x_{n-2d+2k}]$. Similarly, $\operatorname{depth}(S/((I^t_{n,d} : v), u)) = n - \frac{n-k}{d-k} + 1$. Then we have $\operatorname{sdepth}(S/(I^t_{n,d} : v)) \geqslant n - \frac{n-k}{d-k}$ and $\operatorname{depth}(S/(I^t_{n,d} : v)) = n - \frac{n-k}{d-k}$ by Lemma 2.1 and Lemma 2.2.

Since $v$ divides any element of $G(I^t_{n,d})$ which is divided by $a_{(n-k)/(d-k)}$, $(I^t_{n,d}, v) = (I^t_{n-d+k,d}S, v)$. Noting that $v$ is regular on $S/I^t_{n-d+k,d}S$, by induction hypothesis on $n$ and [8], Lemma 3.6, we get

$$\begin{aligned}\operatorname{sdepth}_S(S/(I_{n,d}^t, v)) &= \operatorname{sdepth}_{S_{n-d+k}}\Big(\frac{S_{n-d+k}}{I_{n-d+k,d}^t}\Big) + (d-k) - 1 \\ &= (n-d+k) - \frac{(n-d+k)-k}{d-k} + d - k - 1 \\ &= n - \frac{n-k}{d-k},\end{aligned}$$

where $S_{n-d+k} = K[x_1, \ldots, x_{n-d+k}]$. Similarly, we get $\operatorname{depth}(S/(I_{n,d}^t, v)) = n - \frac{n-k}{d-k}$. By applying Lemmas 2.1 and 2.2 to the short exact sequence

$$0 \to \frac{S}{(I_{n,d}^t : v)} \to \frac{S}{I_{n,d}^t} \to \frac{S}{(I_{n,d}^t, v)} \to 0,$$

we obtain $\operatorname{sdepth}(S/I_{n,d}^t) \geqslant n - \frac{n-k}{d-k}$ and $\operatorname{depth}(S/I_{n,d}^t) = n - \frac{n-k}{d-k}$.

From Lemma 2.3, $P = (x_{k+1}, x_{k+1+(d-k)}, \ldots, x_{n-d+k+1}) \in \operatorname{Min}(S/I_{n,d}) = \operatorname{Min}(S/I_{n,d}^t) \subseteq \operatorname{Ass}(S/I_{n,d}^t)$ for all $t \geqslant 1$, and $\operatorname{ht}(P) = \frac{n-k}{d-k}$. Then $\operatorname{sdepth}(S/I_{n,d}^t) \leqslant \dim(S/P) = n - \frac{n-k}{d-k}$ by [8], Proposition 1.3. This completes the proof. □

**Proposition 2.8.** $\operatorname{bight}(J_{n,d}) = \operatorname{pd}(S/J_{n,d}) = \operatorname{ara}(J_{n,d}) = |G(J_{n,d})| = \frac{n}{d-k}$.

Proof. We have $P = (x_{k+1}, x_{k+1+(d-k)}, x_{k+1+2(d-k)}, \ldots, x_{n-d+2k+1}) \in \operatorname{Min}(S/J_{n,d})$ and $\operatorname{ht}(P) = \frac{n}{d-k}$ by Lemma 2.3. Then $\frac{n}{d-k} \leqslant \operatorname{bight}(J_{n,d}) \leqslant \operatorname{pd}(S/J_{n,d}) \leqslant \operatorname{ara}(J_{n,d}) \leqslant |G(J_{n,d})| = \frac{n}{d-k}$. The proof is completed. □

**Theorem 2.9.** $\operatorname{sdepth}(S/J_{n,d}) = \operatorname{depth}(S/J_{n,d}) = n - \frac{n}{d-k}$.

Proof. Since $|G(J_{n,d})| = \frac{n}{d-k}$, by [4], Proposition 1.2, we have $\operatorname{sdepth}(S/J_{n,d}) \geqslant n - \frac{n}{d-k}$. On the other hand, there exists a prime ideal $P \in \operatorname{Ass}(S/J_{n,d})$ such that $\operatorname{ht}(P) = \frac{n}{d-k}$ by Lemma 2.3. Then $\operatorname{sdepth}(S/J_{n,d}) \leqslant n - \frac{n}{d-k}$ by [8], Proposition 1.3. By the Auslander-Buchsbaum formula and Proposition 2.8, we obtain $\operatorname{depth}(S/J_{n,d}) = n - \operatorname{pd}(S/J_{n,d}) = n - \frac{n}{d-k}$. □

The following corollary implies that the Stanley inequality holds for $J_{n,d}$.

**Corollary 2.10.** $\operatorname{sdepth}(J_{n,d}) \geqslant n - \lfloor \frac{n}{2(d-k)} \rfloor > \operatorname{depth}(J_{n,d})$.

Proof. Since $|G(J_{n,d})| = \frac{n}{d-k}$, we have $\operatorname{sdepth}(J_{n,d}) \geqslant \max\{1, n - \lfloor \frac{1}{2}|G(J_{n,d})| \rfloor\} = n - \lfloor \frac{n}{2(d-k)} \rfloor > n - \frac{n}{d-k} + 1 = \operatorname{depth}(J_{n,d})$ by [10], Theorem 2.3, and Theorem 2.9. □

**2.2. The case $d = 2k$.** Let $\mathcal{P} \subset 2^{[n]}$ be a poset. If $F, G \subset [n]$, the interval $[F, G]$ consists of all subsets $X$ of $[n]$ such that $F \subset X \subset G$. Let $\mathbf{P}: \mathcal{P} = \bigcup_{i=1}^{r} [F_i, G_i]$ be a partition of $\mathcal{P}$, i.e. $[F_i, G_i] \cap [F_j, G_j] = \emptyset$ for all $i \neq j$. We denote $\operatorname{sdepth}(\mathbf{P}) = \min_{i\in[r]}\{|G_i|\}$. Also, we define the Stanley depth of $\mathcal{P}$ to be the number

$$\operatorname{sdepth}(\mathcal{P}) = \max\{\operatorname{sdepth}(\mathbf{P})\colon \mathbf{P} \text{ is a partition of } \mathcal{P}\}.$$

For $\alpha \in \mathbb{N}$ and $\sigma \in \mathcal{P}$, we put

$$\mathcal{P}_\alpha = \{\tau \in \mathcal{P}\colon |\tau| = \alpha\}, \qquad \mathcal{P}_{\alpha,\sigma} = \{\tau \in \mathcal{P}_\alpha\colon \sigma \subset \tau\}.$$

From the proof of [5], Theorem 1.9, we see that if $\sigma \in \mathcal{P}$ is such that $\mathcal{P}_{\alpha,\sigma} = \emptyset$, then $\operatorname{sdepth}(\mathcal{P}) < \alpha$. We recall the method of Herzog, Vladoiu and Zheng in [8] for computing the Stanley depth of $S/I$ and $I$, where $I$ is a squarefree monomial ideal. Let $G(I) = \{u_1, \ldots, u_s\}$ be the set of minimal monomial generators of $I$. We define the following two posets:

$$\mathcal{P}_I = \Big\{\sigma \subset [n]\colon u_i \mid x_\sigma = \prod_{j\in\sigma} x_j \text{ for some } i\Big\} \quad \text{and} \quad \mathcal{P}_{S/I} = 2^{[n]} \setminus \mathcal{P}_I.$$

From [8], Corollary 2.2, it follows that $\operatorname{sdepth}(I) = \operatorname{sdepth}(\mathcal{P}_I)$ and $\operatorname{sdepth}(S/I) = \operatorname{sdepth}(\mathcal{P}_{S/I})$.

Now, we give another main result of this article.

**Theorem 2.11.** $\operatorname{sdepth}(S/I_{n,d}) = \operatorname{depth}(S/I_{n,d}) = \frac{(d-2)n}{d} + \lceil \frac{2n}{3d} \rceil$.

P r o o f. First, we show that $\operatorname{sdepth}(S/I_{n,d}) \geqslant \operatorname{depth}(S/I_{n,d}) = \frac{(d-2)n}{d} + \lceil \frac{2n}{3d} \rceil$ by induction on $n$. If $n = d$, then $I_{n,d} = (x_1 \ldots x_d)$ is principal. Thus $\operatorname{sdepth}(S/I_{n,d}) = \operatorname{depth}(S/I_{n,d}) = d - 1$. If $n = d + k$, we denote $u := x_{k+1} \ldots x_d$ and consider the short exact sequence

$$0 \to S/(I_{n,d} : u) \to S/I_{n,d} \to S/(I_{n,d}, u) \to 0.$$

Note that $(I_{n,d} : u) = (x_1 \ldots x_k, x_{d+1} \ldots x_{d+k})$, $(I_{n,d}, u) = (u)$, and they both are complete intersections. Thus $\operatorname{sdepth}(S/(I_{n,d} : u)) = \operatorname{depth}(S/(I_{n,d} : u)) = n - 2$ and $\operatorname{sdepth}(S/(I_{n,d}, u)) = \operatorname{depth}(S/(I_{n,d}, u)) = n - 1$. Then we get $\operatorname{sdepth}(S/I_{n,d}) \geqslant \operatorname{depth}(S/I_{n,d}) = n - 2$ by Lemmas 2.1 and 2.2.

Suppose that $n \geqslant d + 2k$ and consider the short exact sequence

$$0 \to \frac{S}{(I_{n,d} : x_{n-d+1} \ldots x_{n-d+k})} \to \frac{S}{I_{n,d}} \to \frac{S}{(I_{n,d}, x_{n-d+1} \ldots x_{n-d+k})} \to 0.$$

Then $(I_{n,d} : x_{n-d+1} \dots x_{n-d+k}) = (I_{n-3k,d}S, x_{n-3k+1} \dots x_{n-d}, x_{n-d+k+1} \dots x_n) := I'$. Since $x_{n-3k+1} \dots x_{n-d}, x_{n-d+k+1} \dots x_n$ is a regular sequence on $S/I_{n-3k,d}S$, by induction hypothesis and [8], Lemma 3.6 we get

$$\begin{aligned} \operatorname{depth}_S(S/I') &= \operatorname{depth}_{S_{n-3k}}(S_{n-3k}/I_{n-3k,d}) + 3k - 2 \\ &= \frac{(d-2)(n-3k)}{d} + \left\lceil \frac{2(n-3k)}{3d} \right\rceil + 3k - 2 \\ &= \frac{(d-2)n}{d} + \left\lceil \frac{2n}{3d} \right\rceil, \end{aligned}$$

where $S_{n-3k} = K[x_1, \dots, x_{n-3k}]$. Similarly, $\operatorname{sdepth}(S/I') \geqslant \frac{(d-2)n}{d} + \lceil \frac{2n}{3d} \rceil$.

Also, we have $(I_{n,d}, x_{n-d+1} \dots x_{n-d+k}) = (I_{n-d,d}S, x_{n-d+1} \dots x_{n-d+k}) := I''$, and $x_{n-d+1} \dots x_{n-d+k}$ is regular on $S/I_{n-d,d}S$. We deduce that

$$\begin{aligned} \operatorname{depth}_S(S/I'') &= \operatorname{depth}_{S_{n-d}}(S_{n-d}/I_{n-d,d}) + d - 1 \\ &= \frac{(d-2)(n-d)}{d} + \left\lceil \frac{2(n-d)}{3d} \right\rceil + d - 1 \\ &= \frac{(d-2)n}{d} + \left\lceil \frac{2n+d}{3d} \right\rceil \end{aligned}$$

by induction hypothesis and [8], Lemma 3.6, where $S_{n-d} = K[x_1, \dots, x_{n-d}]$. Similarly, $\operatorname{sdepth}(S/I'') \geqslant \frac{(d-2)n}{d} + \lceil \frac{2n+d}{3d} \rceil$. Then $\operatorname{sdepth}(S/I_{n,d}) \geqslant \operatorname{depth}(S/I_{n,d}) = \frac{(d-2)n}{d} + \lceil \frac{2n}{3d} \rceil$ by Lemmas 2.1 and 2.2.

Next we only need to show that $\operatorname{sdepth}(S/I_{n,d}) \leqslant \frac{(d-2)n}{d} + \lceil \frac{2n}{3d} \rceil$.

Let $\mathcal{P} = \mathcal{P}_{S/I_{n,d}}$, $t = \lceil \frac{n}{3k} \rceil$ and $\alpha = \frac{(d-2)n}{d} + \lceil \frac{2n}{3d} \rceil$. We consider the following two cases.

1. If $\frac{n}{k} \equiv 1 \pmod 3$ and $\sigma = \{k+1, \dots, 2k, k+1+3k, \dots, 2k+3k, \dots, k+1+(t-2)3k, \dots, 2k+(t-2)3k, k+1+(t-2)3k+2k, \dots, 2k+(t-2)3k+2k\}$, then $\mathcal{P}_{\alpha+1,\sigma} = \emptyset$. Thus $\operatorname{sdepth}(S/I_{n,d}) = \operatorname{sdepth}(\mathcal{P}) \leqslant \alpha$.

2. If $\frac{n}{k} \equiv 0 \pmod 3$ or $\frac{n}{k} \equiv 2 \pmod 3$, and $\sigma = \{k+1, \dots, 2k, k+1+3k, \dots, 2k+3k, \dots, k+1+(t-1)3k, \dots, 2k+(t-1)3k\}$, then $\mathcal{P}_{\alpha+1,\sigma} = \emptyset$. Thus $\operatorname{sdepth}(S/I_{n,d}) = \operatorname{sdepth}(\mathcal{P}) \leqslant \alpha$.

It follows that $\operatorname{sdepth}(S/I_{n,d}) \leqslant \alpha = \frac{(d-2)n}{d} + \lceil \frac{2n}{3d} \rceil$. □

**Remark 2.12.** Set $d = 2, k = 1$ in Theorem 2.11. Then we get $\operatorname{depth}(S/I_{n,2}) = \operatorname{sdepth}(S/I_{n,2}) = \lceil \frac{1}{3}n \rceil$, so our results generalize [9], Lemma 2.8, and [14], Lemma 4. On the other hand, by the Auslander-Buchsbaum formula, we have $\operatorname{pd}(S/I_{n,2}) = n - \lceil \frac{1}{3}n \rceil$, which coincides with [2], Proposition 3.1.1 (1).

As a consequence of Theorem 2.11, we get the following corollary.

**Corollary 2.13.** $\operatorname{sdepth}(I_{n,d}) \geqslant n - \lfloor \frac{n-k}{2k} \rfloor \geqslant \operatorname{depth}(I_{n,d})$.

Proof. Note that $|G(I_{n,d})| = \frac{n}{k} - 1$, which implies that $\operatorname{sdepth}(I_{n,d}) \geqslant \max\{1, n - \lfloor \frac{1}{2}|G(I_{n,d})| \rfloor\} = n - \lfloor \frac{n-k}{2k} \rfloor \geqslant \operatorname{depth}(I_{n,d})$ by [10], Theorem 2.3 and Theorem 2.11. □

The following proposition generalizes [9], Proposition 3.2, (where $d = 2$).

**Proposition 2.14.** $\operatorname{depth}(S/I^t_{n,d}) \geqslant \max\{1, \frac{(d-2)n}{d} + \lceil \frac{2n-dt+d}{3d} \rceil\}$ *for all* $t \geqslant 1$, *and* $\operatorname{sdepth}(S/I^t_{n,d}) \geqslant \max\{1, \frac{(d-2)n}{d} + \lceil \frac{2n-dt+d}{3d} \rceil\}$ *for all* $t \geqslant 1$.

Proof. Notice that $(x_1, \dots, x_n) \notin \operatorname{Ass}(S/I^t_{n,d})$, hence $\operatorname{depth}(S/I^t_{n,d}) \geqslant 1$ for all $t \geqslant 1$. By [4], Theorem 2.1, we also get $\operatorname{sdepth}(S/I^t_{n,d}) \geqslant 1$ for all $t \geqslant 1$. Then it remains to show that $\operatorname{depth}(S/I^t_{n,d}) \geqslant \frac{(d-2)n}{d} + \lceil \frac{2n-dt+d}{3d} \rceil$ and $\operatorname{sdepth}(S/I^t_{n,d}) \geqslant \frac{(d-2)n}{d} + \lceil \frac{2n-dt+d}{3d} \rceil$.

The proof is by induction on $n$ and $t$. If $n = d$, then $I^t_{n,d} = (x^t_1 \dots x^t_d)$ is principal. Thus $\operatorname{sdepth}(S/I^t_{n,d}) = \operatorname{depth}(S/I^t_{n,d}) = d - 1 \geqslant \frac{(d-2)d}{d} + \lceil \frac{2d-dt+d}{3d} \rceil$ for all $t \geqslant 1$. If $n = d + k$, then $I_{n,d} = (x_1 \dots x_d, x_{k+1} \dots x_{3k})$. Next we use induction on $j$ to show that $\operatorname{depth}(S/I^j_{n,d}) \geqslant 3(k-1) + \lceil \frac{4-j}{3} \rceil$ and $\operatorname{sdepth}(S/I^j_{n,d}) \geqslant 3(k-1) + \lceil \frac{4-j}{3} \rceil$ for all $j \geqslant 1$.

If $j = 1$, then the results hold by Theorem 2.11. Let $j \geqslant 2$. We denote $w_1 := x_{k+1} \dots x_d$, $w_2 := x_{d+1} \dots x_{3k}$ and consider the short exact sequence

$$0 \to \frac{S}{(I^j_{n,d} : w_1 w_2)} \to \frac{S}{(I^j_{n,d} : w_2)} \to \frac{S}{((I^j_{n,d} : w_2), w_1)} \to 0.$$

Note that $(I^j_{n,d} : w_1 w_2) = I^{j-1}_{n,d}$ from the proof of Theorem 2.7. We get $\operatorname{depth}(S/(I^j_{n,d} : w_1 w_2)) \geqslant 3(k-1) + \lceil \frac{4-j}{3} \rceil$ and $\operatorname{sdepth}(S/(I^j_{n,d} : w_1 w_2)) \geqslant 3(k-1) + \lceil \frac{4-j}{3} \rceil$ by induction hypothesis. Also, $((I^j_{n,d} : w_2), w_1) = (w_1)$ is principal. Thus $\operatorname{depth}(S/((I^j_{n,d} : w_2), w_1)) = \operatorname{sdepth}(S/((I^j_{n,d} : w_2), w_1)) = 3k - 1$. Then $\operatorname{depth}(S/(I^j_{n,d} : w_2)) \geqslant 3(k-1) + \lceil \frac{4-j}{3} \rceil$ and $\operatorname{sdepth}(S/(I^j_{n,d} : w_2)) \geqslant 3(k-1) + \lceil \frac{4-j}{3} \rceil$ by Lemmas 2.1 and 2.2. We consider another short exact sequence

$$0 \to \frac{S}{(I^j_{n,d} : w_2)} \to \frac{S}{I^j_{n,d}} \to \frac{S}{(I^j_{n,d}, w_2)} \to 0.$$

Since $(I^j_{n,d}, w_2) = (x^j_1 \dots x^j_d, w_2)$ is a complete intersection, $\operatorname{depth}(S/(I^j_{n,d}, w_2)) = \operatorname{sdepth}(S/(I^j_{n,d}, w_2)) = 3k - 2$. Then we get $\operatorname{depth}(S/I^j_{n,d}) \geqslant 3(k-1) + \lceil \frac{4-j}{3} \rceil$ and $\operatorname{sdepth}(S/I^j_{n,d}) \geqslant 3(k-1) + \lceil \frac{4-j}{3} \rceil$ by Lemmas 2.1 and 2.2.

Assume that $n \geqslant d + 2k$. If $t = 1$, by Theorem 2.11 the results hold for all $n$. Now let $t \geqslant 2$. We denote $u := x_{n-d+1} \dots x_{n-d+k}$, $v := x_{n-d+k+1} \dots x_n$ and consider the short exact sequence

$$(*) \qquad 0 \to \frac{S}{(I^t_{n,d} : uv)} \to \frac{S}{(I^t_{n,d} : u)} \to \frac{S}{((I^t_{n,d} : u), v)} \to 0.$$

Let $G(I_{n,d}) = \{a_1, a_2, \ldots, a_{(n-k)/(d-k)}\}$, the same as in the proof of Lemma 2.3. Note that $(I_{n,d}^t : uv) = I_{n,d}^{t-1}$ from the proof of Theorem 2.7. By induction hypothesis on $t$,

$$\begin{aligned}\operatorname{depth}(S/(I_{n,d}^t : uv)) &= \operatorname{depth}(S/I_{n,d}^{t-1}) \\ &\geqslant \frac{(d-2)n}{d} + \left\lceil \frac{2n - d(t-1) + d}{3d} \right\rceil \\ &= \frac{(d-2)n}{d} + \left\lceil \frac{2n - dt + 2d}{3d} \right\rceil.\end{aligned}$$

Similarly, $\operatorname{sdepth}(S/(I_{n,d}^t : uv)) \geqslant \frac{(d-2)n}{d} + \lceil \frac{2n-dt+2d}{3d} \rceil$.

Since $v$ divides any element of $G(I_{n,d}^t)$ which is divided by $a_{(n-k)/(d-k)}$, we have $((I_{n,d}^t : u), v) = ((I_{n-k,d}^t S : u), v)$. Noting that $v$ is regular on $S/(I_{n-k,d}^t S : u)$, by [8], Lemma 3.6, we get $\operatorname{depth}_S(S/((I_{n,d}^t : u), v)) = \operatorname{depth}_{S_{n-k}}(S_{n-k}/(I_{n-k,d}^t : u)) + k - 1$ and $\operatorname{sdepth}_S(S/((I_{n,d}^t : u), v)) = \operatorname{sdepth}_{S_{n-k}}(S_{n-k}/(I_{n-k,d}^t : u)) + k - 1$, where $S_{n-k} = K[x_1, \ldots, x_{n-k}]$. We denote $w := x_{n-3k+1} \ldots x_{n-d}$ and consider another short exact sequence

$$0 \to \frac{S_{n-k}}{(I_{n-k,d}^t : wu)} \to \frac{S_{n-k}}{(I_{n-k,d}^t : u)} \to \frac{S_{n-k}}{((I_{n-k,d}^t : u), w)} \to 0.$$

From the proof of Theorem 2.7, $(I_{n-k,d}^t : wu) = I_{n-k,d}^{t-1}$. By induction hypothesis,

$$\begin{aligned}\operatorname{depth}(S_{n-k}/(I_{n-k,d}^t : wu)) &= \operatorname{depth}(S_{n-k}/I_{n-k,d}^{t-1}) \\ &\geqslant \frac{(d-2)(n-k)}{d} + \left\lceil \frac{2(n-k) - d(t-1) + d}{3d} \right\rceil \\ &= \frac{(d-2)n}{d} + \left\lceil \frac{2n - dt + d}{3d} \right\rceil - (k-1).\end{aligned}$$

Similarly, $\operatorname{sdepth}(S_{n-k}/(I_{n-k,d}^t : wu)) \geqslant \frac{(d-2)n}{d} + \lceil \frac{2n-dt+d}{3d} \rceil - (k-1)$.

We see that $((I_{n-k,d}^t : u), w) = (I_{n-3k,d}^t S_{n-k}, w) := I'$, since $w$ divides any element of $G(I_{n-k,d}^t)$ which is divided by $a_{(n-d)/(d-k)}$ or $a_{(n-3k)/(d-k)}$. Noticing that $w$ is regular on $S_{n-k}/I_{n-3k,d}^t S_{n-k}$, by induction hypothesis on $n$ and [8], Lemma 3.6, we get

$$\begin{aligned}\operatorname{depth}_{S_{n-k}}(S_{n-k}/I') &= \operatorname{depth}_{S_{n-3k}}(S_{n-3k}/I_{n-3k,d}^t) + 2k - 1 \\ &\geqslant \frac{(d-2)(n-3k)}{d} + \left\lceil \frac{2(n-3k) - dt + d}{3d} \right\rceil + 2k - 1 \\ &= \frac{(d-2)n}{d} + \left\lceil \frac{2n - dt + d}{3d} \right\rceil - (k-1),\end{aligned}$$

where $S_{n-3k} = K[x_1, \ldots, x_{n-3k}]$. Similarly, $\operatorname{sdepth}(S_{n-k}/I') \geqslant \frac{(d-2)n}{d} + \lceil \frac{2n-dt+d}{3d} \rceil - (k-1)$. Thus we have $\operatorname{depth}(S_{n-k}/(I_{n-k,d}^t : u)) \geqslant \frac{(d-2)n}{d} + \lceil \frac{2n-dt+d}{3d} \rceil - (k-1)$ and

$\operatorname{sdepth}(S_{n-k}/(I_{n-k,d}^t : u)) \geqslant \frac{(d-2)n}{d} + \lceil \frac{2n-dt+d}{3d} \rceil - (k-1)$ by Lemmas 2.1 and 2.2. It follows that $\operatorname{depth}(S/((I_{n,d}^t : u), v)) \geqslant \frac{(d-2)n}{d} + \lceil \frac{2n-dt+d}{3d} \rceil$ and

$$\operatorname{sdepth}(S/((I_{n,d}^t : u), v)) \geqslant \frac{(d-2)n}{d} + \left\lceil \frac{2n-dt+d}{3d} \right\rceil.$$

By applying Lemmas 2.1 and 2.2 to the sequence $(*)$, we get $\operatorname{depth}(S/(I_{n,d}^t : u)) \geqslant \frac{(d-2)n}{d} + \lceil \frac{2n-dt+d}{3d} \rceil$ and $\operatorname{sdepth}(S/(I_{n,d}^t : u)) \geqslant \frac{(d-2)n}{d} + \lceil \frac{2n-dt+d}{3d} \rceil$.

Finally, we consider the following short exact sequence

$$0 \to \frac{S}{(I_{n,d}^t : u)} \to \frac{S}{I_{n,d}^t} \to \frac{S}{(I_{n,d}^t, u)} \to 0.$$

Since $u$ divides any element of $G(I_{n,d}^t)$ which is divided by element $a_{(n-k)/(d-k)}$ or $a_{(n-d)/(d-k)}$, we get $(I_{n,d}^t, u) = (I_{n-d,d}^t S, u)$. Noting that $u$ is regular on $S/I_{n-d,d}^t S$, by induction hypothesis on $n$ and [8], Lemma 3.6, we obtain

$$\begin{aligned} \operatorname{depth}_S(S/(I_{n,d}^t, u)) &= \operatorname{depth}_{S_{n-d}}(S_{n-d}/I_{n-d,d}^t) + d - 1 \\ &\geqslant \frac{(d-2)(n-d)}{d} + \left\lceil \frac{2(n-d)-dt+d}{3d} \right\rceil + d - 1 \\ &= \frac{(d-2)n}{d} + \left\lceil \frac{2n-dt+2d}{3d} \right\rceil, \end{aligned}$$

where $S_{n-d} = K[x_1, \ldots, x_{n-d}]$. Similarly, $\operatorname{sdepth}(S/(I_{n,d}^t, u)) \geqslant \frac{(d-2)n}{d} + \lceil \frac{2n-dt+2d}{3d} \rceil$. Now the results follow from Lemmas 2.1 and 2.2. $\square$

**Example 2.15.** Let $\Delta_{6,4}$ be the simplicial complex with the set of facets $\mathcal{F}(\Delta_{6,4}) = \{\{1,2,3,4\},\{3,4,5,6\}\}$. Then $I_{6,4} = (x_1x_2x_3x_4, x_3x_4x_5x_6)$ and we compute that $\operatorname{depth}(S/I_{6,4}^4) = 4$, while in this case $\frac{(d-2)n}{d} + \lceil \frac{2n-dt+d}{3d} \rceil = 3$. So the bound for the depth given by Proposition 2.14 is not necessarily sharp.

Now we consider the depth and Stanley depth of $S/J_{n,d}$. The next proposition generalizes [5], Propositions 1.3 and 1.8 (where $d = 2$, $k = 1$).

**Proposition 2.16.** $\operatorname{sdepth}(S/J_{n,d}) \geqslant \operatorname{depth}(S/J_{n,d}) = \frac{(d-2)n}{d} + \lceil \frac{2n-d}{3d} \rceil$.

Proof. We use induction on $n$. If $n = 3k$, then we have $J_{n,d} = (x_1 \ldots x_{2k}, x_{k+1} \ldots x_{3k}, x_{2k+1} \ldots x_{3k} x_1 \ldots x_k)$. We denote $u := x_{2k+1} \ldots x_{3k}$ and consider the short exact sequence

$$0 \to S/(J_{n,d} : u) \to S/J_{n,d} \to S/(J_{n,d}, u) \to 0.$$

Note that $(J_{n,d} : u) = (x_1 \ldots x_k, x_{k+1} \ldots x_{2k})$, $(J_{n,d}, u) = (x_1 \ldots x_{2k}, u)$ and they are both complete intersections. Thus $\operatorname{sdepth}(S/(J_{n,d} : u)) = \operatorname{depth}(S/(J_{n,d} : u)) =$

$3k-2$ and $\operatorname{sdepth}(S/(J_{n,d},u)) = \operatorname{depth}(S/(J_{n,d},u)) = 3k-2$. Hence $\operatorname{sdepth}(S/J_{n,d}) \geqslant \operatorname{depth}(S/J_{n,d}) = 3k-2$ by Lemmas 2.1 and 2.2.

Suppose that $n \geqslant 4k$ and consider the short exact sequence

$$0 \to \frac{S}{(J_{n,d} : x_{n-k+1}\dots x_n)} \to \frac{S}{J_{n,d}} \to \frac{S}{(J_{n,d}, x_{n-k+1}\dots x_n)} \to 0.$$

We denote $w := x_{n-k+1}\dots x_n$. Then $(J_{n,d} : w) = (x_1\dots x_k, x_{n-2k+1}\dots x_{n-k}, I'S)$, where $I' := (x_{k+1}\dots x_{3k}, \dots, x_{n-4k+1}\dots x_{n-2k}) \subset S' := K[x_{k+1},\dots,x_{n-2k}]$, and $x_1\dots x_k, x_{n-2k+1}\dots x_{n-k}$ is a regular sequence on $S/I'S$. By [8], Lemma 3.6, and Theorem 2.11, we have

$$\begin{aligned}
\operatorname{depth}_S(S/(J_{n,d} : w)) &= \operatorname{depth}_{S'}(S'/I') + 3k - 2\\
&= \operatorname{depth}_{S_{n-3k}}(S_{n-3k}/I_{n-3k,d}) + 3k - 2\\
&= \frac{(d-2)(n-3k)}{d} + \left\lceil \frac{2(n-3k)}{3d} \right\rceil + 3k - 2\\
&= \frac{(d-2)n}{d} + \left\lceil \frac{2n}{3d} \right\rceil,
\end{aligned}$$

where $S_{n-3k} = K[x_1,\dots,x_{n-3k}]$. Similarly, $\operatorname{sdepth}(S/(J_{n,d} : w)) = \frac{(d-2)n}{d} + \lceil \frac{2n}{3d} \rceil \geqslant \frac{(d-2)n}{d} + \lceil \frac{2n-d}{3d} \rceil$.

Also, $(J_{n,d}, w) = (I_{n-k,d}S, w)$ and $w$ is regular on $S/I_{n-k,d}S$. By [8], Lemma 3.6, and Theorem 2.11, we get

$$\begin{aligned}
\operatorname{depth}_S(S/(J_{n,d}, w)) &= \operatorname{depth}_{S_{n-k}}(S_{n-k}/I_{n-k,d}) + k - 1\\
&= \frac{(d-2)(n-k)}{d} + \left\lceil \frac{2(n-k)}{3d} \right\rceil + k - 1\\
&= \frac{(d-2)n}{d} + \left\lceil \frac{2n-d}{3d} \right\rceil,
\end{aligned}$$

where $S_{n-k} = K[x_1,\dots,x_{n-k}]$. Similarly, $\operatorname{sdepth}(S/(J_{n,d}, w)) = \frac{(d-2)n}{d} + \lceil \frac{2n-d}{3d} \rceil$, thus $\operatorname{sdepth}(S/J_{n,d}) \geqslant \frac{(d-2)n}{d} + \lceil \frac{2n-d}{3d} \rceil$ by Lemma 2.2.

Now we consider the depth. If $\frac{n}{k} \equiv 0 \pmod 3$ or $\frac{n}{k} \equiv 2 \pmod 3$, then $\frac{(d-2)n}{d} + \lceil \frac{2n}{3d} \rceil = \frac{(d-2)n}{d} + \lceil \frac{2n-d}{3d} \rceil$, so $\operatorname{depth}(S/J_{n,d}) = \frac{(d-2)n}{d} + \lceil \frac{2n-d}{3d} \rceil$ by Lemma 2.1.

Assume that $\frac{n}{k} \equiv 1 \pmod 3$. We have

$$\frac{(J_{n,d} : x_{n-k+1}\dots x_n)}{J_{n,d}} \cong x_{n-2k+1}\dots x_{n-k}(R/Q)[x_{n-2k+1},\dots,x_{n-k}]$$

$$\bigoplus_{i=1}^{k} x_1\dots x_k \cdot x_{n-k+2-i}\dots x_{n-k}(R_i/Q_1)[x_1,\dots,x_k,x_{n-k+2-i},\dots,x_{n-k}],$$

where $R = K[x_1, \ldots, x_{n-2k}, x_{n-k+1}, \ldots, x_n]$, $Q = (x_{n-3k+1} \ldots x_{n-2k}, x_{n-k+1} \ldots x_n, I_{n-3k,d}S)$, $R_i = K[x_{k+1}, \ldots, x_{n-2k}, x_{n-2k+1}, \ldots, x_{n-k-i}, x_{n-k+1}, \ldots, x_n]$, $1 \leqslant i \leqslant k$, and $Q_1 = (x_{k+1} \ldots x_{2k}, x_{n-k+1} \ldots x_n, x_{2k+1} \ldots x_{4k}, \ldots, x_{n-4k+1} \ldots x_{n-2k})$. Using the isomorphism and Theorem 2.11, we obtain that

$$\begin{aligned}\operatorname{depth}\Bigl(\frac{(J_{n,d} : w)}{J_{n,d}}\Bigr) &= \min_i\{\operatorname{depth}_R(R/Q) + k, \operatorname{depth}_{R_i}(R_i/Q_1) + (k+i-1)\}\\ &= \frac{(d-2)(n-4k)}{d} + \Bigl\lceil \frac{2(n-4k)}{3d} \Bigr\rceil + (3k-i-2) + (k+i-1)\\ &= \frac{(d-2)n}{d} + \Bigl\lceil \frac{2n-d}{3d} \Bigr\rceil.\end{aligned}$$

Now, applying Lemma 2.1 to the short exact sequence

$$0 \to \frac{(J_{n,d} : w)}{J_{n,d}} \to \frac{S}{J_{n,d}} \to \frac{S}{(J_{n,d} : w)} \to 0,$$

the proof is completed. □

**Corollary 2.17.** $\operatorname{sdepth}(J_{n,d}) \geqslant n - \lfloor \frac{n}{2k} \rfloor \geqslant \operatorname{depth}(J_{n,d})$.

P r o o f. Since $|G(J_{n,d})| = \frac{n}{k}$, we have $\operatorname{sdepth}(J_{n,d}) \geqslant \max\{1, n - \lfloor \frac{1}{2}|G(J_{n,d})| \rfloor\} = n - \lfloor \frac{n}{2k} \rfloor \geqslant \operatorname{depth}(J_{n,d})$ by [10], Theorem 2.3, and Proposition 2.16. □

The next theorem generalizes [5], Theorem 1.9.

**Theorem 2.18.** (1) $\operatorname{sdepth}(S/J_{n,d}) = \frac{(d-2)n}{d} + \lceil \frac{2n-d}{3d} \rceil$ *for* $\frac{n}{k} \equiv 0 \pmod 3$ *and* $\frac{n}{k} \equiv 2 \pmod 3$. (2) $\operatorname{sdepth}(S/J_{n,d}) \leqslant \frac{(d-2)n}{d} + \lceil \frac{2n}{3d} \rceil$ *for* $\frac{n}{k} \equiv 1 \pmod 3$.

P r o o f. Let $\mathcal{P} = \mathcal{P}_{S/J_{n,d}}$, $t = \lceil \frac{n}{3k} \rceil$ and $\alpha = \frac{(d-2)n}{d} + \lceil \frac{2n-d}{3d} \rceil$. We consider the following two cases.

1. If $\frac{n}{k} \equiv 0 \pmod 3$ or $\frac{n}{k} \equiv 2 \pmod 3$, and $\sigma = \{1, \ldots, k, 1+3k, \ldots, k+3k, \ldots, 1+(t-1)3k, \ldots, k+(t-1)3k\}$, then $\mathcal{P}_{\alpha+1,\sigma} = \emptyset$. Thus $\operatorname{sdepth}(S/J_{n,d}) = \operatorname{sdepth}(\mathcal{P}) \leqslant \alpha = \frac{(d-2)n}{d} + \lceil \frac{2n-d}{3d} \rceil$.

2. If $\frac{n}{k} \equiv 1 \pmod 3$ and $\sigma = \{1, \ldots, k, 1+3k, \ldots, k+3k, \ldots, 1+(t-2)3k, \ldots, k+(t-2)3k, 1+(t-2)3k+2k, \ldots, k+(t-2)3k+2k\}$, then $\mathcal{P}_{\alpha+2,\sigma} = \emptyset$. Thus $\operatorname{sdepth}(S/J_{n,d}) = \operatorname{sdepth}(\mathcal{P}) \leqslant \alpha + 1 = \frac{(d-2)n}{d} + \lceil \frac{2n+2d}{3d} \rceil = \frac{(d-2)n}{d} + \lceil \frac{2n}{3d} \rceil$.

Then the results follow from Proposition 2.16. □

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*Authors' address*: Xiaoqi Wei, Yan Gu (corresponding author), School of Mathematical Sciences, Soochow University, No. 1 Shizi Street, Suzhou 215006, Jiangsu, P. R. China, e-mail: `weixiaoqi1989@sina.com`, `guyan@suda.edu.cn`.