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BOUNDS FOR THE NUMBER OF MEETING EDGES IN GRAPH PARTITIONING

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Abstract. Let G be a weighted hypergraph with edges of size at most 2. Bollobás and Scott conjectured that G admits a bipartition such that each vertex class meets edges of total weight at least $(w_1-\Delta_1)/2+2w_2/3$, where w_i is the total weight of edges of size i and Δ_1 is the maximum weight of an edge of size 1. In this paper, for positive integer weighted hypergraph G (i.e., multi-hypergraph), we show that there exists a bipartition of G such that each vertex class meets edges of total weight at least $(w_0-1)/6+(w_1-\Delta_1)/3+2w_2/3$, where w_0 is the number of edges of size 1. This generalizes a result of Haslegrave. Based on this result, we show that every graph with m edges, except for K_2 and $K_{1,3}$, admits a tripartition such that each vertex class meets at least $\lceil 2m/5\rceil$ edges, which establishes a special case of a more general conjecture of Bollobás and Scott.

Keywords: graph; weighted hypergraph; partition; judicious partition

MSC 2010: 05C35, 05C75

1. Introduction

Let $G=(V,E)$ be a graph. For subsets S and T of V, $e_G(S,T)$ is the number of edges of G with one end in S and the other end in T, and $e_G(S)$ is the number of edges of G with both ends in S. By $d_G(S)$, we mean the number of edges of G meeting S (i.e., containing at least one vertex of S). For a weighted graph (or hypergraph) G with weight function w, denote by $d_G^w(S)$ the total weight of edges of G meeting S. If $S=\{v\}$, then we write $e_G(v,T)$, $d_G(v)$ and $d_G^w(v)$ for $e_G(\{v\},T)$, $d_G(\{v\})$ and $d_G^w(\{v\})$, respectively. When understood, the reference to G in the subscript will be dropped. Additionally, we write $\overline{S}$ for $V\setminus S$, $[t]$ for $\{1,\ldots,t\}$ and $\binom{S}{j}$ for the set of all j-element subsets of S.

Classical graph or hypergraph partitioning problems often ask for partitioning the vertex set of a graph or hypergraph into pairwise disjoint subsets that opti-

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mize a single quantity. For example, the well-known Max-Cut problem asks for a maximum bipartite subgraph of a graph, i.e., a bipartition V_1, V_2 of a given graph with m edges maximizing the number of edges between V_1 and V_2. Edwards in [6], [7] proved the essentially best possible result: a bipartite subgraph with at least $m/2 + (\sqrt{2m+1/4} - 1/2)/4$ edges. An extension of Edwards' bound for partitions into more than two parts was proved in [4].

In practice, one often needs to find a partition of a given graph or hypergraph to optimize several quantities simultaneously. Such problems are called *Judicious partitioning problems* by Bollobás and Scott in [5]. The *Bottleneck bipartition problem* is a judicious partition problem: Find a partition V_1, V_2 of $V(G)$ that minimizes $\max\{e(V_1), e(V_2)\}$. Bollobás and Scott in [2] showed that every graph with m edges admits a bipartition such that each vertex class spans at most

$$\frac{m}{4} + \frac{\sqrt{2m+\frac{1}{4}} - \frac{1}{2}}{8} \text{ edges.}$$

The bound is tight for the complete graph K_{2n+1}. In the same paper, the authors also extended the result for partitions into more than two parts. For more about judicious partitioning problems, we refer the reader to [1], [8], [9], [11], [12], [13], [14], [15], [18], [19]. For survey articles, see [5], [16].

In this paper, we consider another type of judicious partitioning problems about graphs with requirement on edges as well as on vertices, and such problems are called *mixed partitioning problems*. We follow Bollobás and Scott [5] in using the term "*hypergraph with edges of size at most 2*". Note that a *hypergraph* $G = (V, E)$ consists of a finite set $V := V(G)$ of vertices and a set $E := E(G)$ of edges, where each edge is a subset of V. For each edge $e \in E$, if e contains at most two elements of V, then G is a hypergraph with edges of size at most 2.

Let G be a weighted hypergraph with edges of size at most 2. Denote by Δ_1 the maximum weight of an edge of size 1 and by w_i the total weight of edges of size i for $i = 1, 2$. Bollobás and Scott in [5] gave the following conjecture.

Conjecture 1.1 (Bollobás and Scott [5]). *Every weighted hypergraph G admits a bipartition such that each vertex class meets edges of total weight at least*

$$\frac{w_1 - \Delta_1}{2} + \frac{2w_2}{3}.$$

Recently, Xu et al. in [17] established a weaker version of the conjecture. For weighted hypergraphs G with weight function $w\colon E \to \mathbb{N}^+$, Haslegrave in [10] confirmed the conjecture for the case $\Delta_1 \leqslant 1$.

Theorem 1.2 (Haslegrave [10]). *For $\Delta_1 \leqslant 1$, the weighted hypergraph G admits a bipartition V_1, V_2 such that for $i = 1, 2$*

$$d^w(V_i) \geqslant \frac{w_1 - \Delta_1}{2} + \frac{2w_2}{3}.$$

By using a different method, we generalize the result of Haslegrave and show

Theorem 1.3. *The weighted hypergraph G has a bipartition V_1, V_2 such that for $i = 1, 2$*

$$d^w(V_i) \geqslant \frac{w_0 - 1}{6} + \frac{w_1 - \Delta_1}{3} + \frac{2w_2}{3},$$

where w_0 is the number of edges of size 1.

Remark. Since the bound of Theorem 1.2 is easy to obtain when $\Delta_1 = 0$, we can always assume that $\Delta_1 \geqslant 1$ in our theorem. Note that $w_0 = w_1$ provided $\Delta_1 = 1$. Thus, our result generalizes Theorem 1.2.

Bollobás and Scott in [5] noted that mixed partitioning problems are useful in proving results about uniform hypergraphs. Particularly, we establish a special case of another conjecture of Bollobás and Scott for graphs based on the $\Delta_1 = 2$ case of Theorem 1.3.

Conjecture 1.4 ([3], [16]). *For every integer $k \geqslant 2$, every graph with m edges has a partition into k sets, each of which meets at least*

$$\frac{2m}{2k-1} \text{ edges.}$$

If true, the complete graph K_{2k-1} shows that the bound should be sharp. Actually, in [16], the author also assumes that $m \geqslant \binom{k}{2}$ to avoid the trivial cases such as K_{k-1}. Ma et al. in [14] solved the conjecture for very large m (in terms of k). In this paper, we confirm the case $k = 3$.

Theorem 1.5. *Let G be a graph with m edges. Suppose that G is not isomorphic to K_2 and $K_{1,3}$ (modulo isolated vertices). Then there exists a tripartition V_1, V_2, V_3 of G such that for $i = 1, 2, 3$*

$$d(V_i) \geqslant \left\lceil \frac{2m}{5} \right\rceil.$$

2. Bipartitions of weighted hypergraphs

In this section, we consider the bipartitions of weighted hypergraphs and give the proof of Theorem 1.3. Before proving, we present the following algorithm and lemmas.

Let $G = (V, E(G))$ be a weighted hypergraph with edges of size at most 2 and let $w\colon E \to \mathbb{N}^+$ be its weight function. First, we construct a weighted complete graph $G_1 = (V, E(G_1))$ from G. Let $w^1\colon V \cup E(G_1) \to \mathbb{N}$ be the weight function of G_1 such that for each $v \in V$ and $e \in E(G_1)$

$$w^1(v) = \begin{cases} w(\{v\}) & \text{if } \{v\} \in E(G),\\ 0 & \text{otherwise,}\end{cases} \quad \text{and} \quad w^1(e) = \begin{cases} w(e) & \text{if } e \in E(G),\\ 0 & \text{otherwise.}\end{cases}$$

Let Δ_1 be the maximum weight of an edge of size 1 of G . Clearly, by the construction, Δ_1 is also the maximum weight of a vertex in G_1.

Now, we construct a graph sequence $\mathcal{G} = (G_i)_{i\geqslant 1}$ consisting of weighted complete graphs $G_i = (V, E(G_i))$ with weight function $w^i\colon V \cup E(G_i) \to \mathbb{N}$ according to the following procedure, which we will call the $\mathcal{G}$ *algorithm*: set $i = 1$ and $s_1 = |\{v \in V\colon w^1(v) > 1\}|$. Repeat the following steps until $s_i \leqslant 1$.

- ▷ Set $s_i = |\{v \in V\colon w^i(v) > 1\}|$. If $s_i = 0$, then stop; otherwise, set $\delta_i = \min\{w^i(v) > 1\colon v \in V\}$ and $\Delta_i = \max\{w^i(v)\colon v \in V\}$.
- ▷ If $s_i = 1$ and $v \in V$ is the unique vertex satisfying $w^i(v) > 1$, then set $w^i(v) = 1$, and stop.
- ▷ If $s_i > 1$, then choose an edge $e = uv$ arbitrarily from G_i satisfying $w^i(u) = \delta_i$ and $w^i(v) = \Delta_i$. Set $w^{i+1}(u) = 1$, $w^{i+1}(v) = \Delta_i - \delta_i + 1$ and $w^{i+1}(e) = w^i(e) + \delta_i - 1$. For each $x \in V\backslash\{u, v\}$ and $f \in \binom{V}{2}\backslash\{e\}$, set $w^{i+1}(v) = w^i(v)$ and $w^{i+1}(f) = w^i(f)$. Increment i.

Let t be the length of the resulting sequence $\mathcal{G}$. Clearly, $1 \leqslant t \leqslant |V|$. For each $S \subseteq V$ and $i \in [t]$, define $\tau^{w^i}(S) = \sum\limits_{v\in S} w^i(v)$. By the construction, we immediately have the following two lemmas.

Lemma 2.1. *For each $S \subseteq V$ and $1 \leqslant i \leqslant j \leqslant t$,*

$$\tau^{w^i}(S) + d^{w^i}(S) \geqslant \tau^{w^j}(S) + d^{w^j}(S).$$

Proof. According to the $\mathcal{G}$ algorithm, for each $v \in V$ we have

$$(1) \qquad w^i(v) + d^{w^i}(v) \geqslant w^j(v) + d^{w^j}(v).$$

In fact, the equality holds for each $j \leqslant t-1$ and, if $j = t$, it holds for at least $|V|-1$ vertices. Summing over all $v \in S$ in (1) yields

$$\sum_{v\in S} w^i(v) + \sum_{\substack{e\in E(G_i)\\ |e\cap S|=1}} w^i(e) + 2\sum_{\substack{e\in E(G_i)\\ |e\cap S|=2}} w^i(e) \geqslant \sum_{v\in S} w^j(v) + \sum_{\substack{e\in E(G_j)\\ |e\cap S|=1}} w^j(e) + 2\sum_{\substack{e\in E(G_j)\\ |e\cap S|=2}} w^j(e),$$

which is equivalent to

$$\tau^{w^i}(S) + d^{w^i}(S) \geqslant \tau^{w^j}(S) + d^{w^j}(S) + \sum_{e\in\binom{S}{2}} (w^j(e) - w^i(e)).$$

The inequality follows from the fact that G_i is a complete graph on V for each $i \in [t]$. Note that $w^j(e) \geqslant w^i(e)$ for each $e \in \binom{S}{2}$. Thus, we have

$$\tau^{w^i}(S) + d^{w^i}(S) \geqslant \tau^{w^j}(S) + d^{w^j}(S),$$

as required. □

For each $i \in [t]$, let $w_1^i = \sum_{v\in V} w^i(v)$ and $w_2^i = \sum_{e\in E(G_i)} w^i(e)$. The next lemma shows that G_t has a 'good' judicious partition.

Lemma 2.2. *Every weighted graph G_t admits a bipartition V_1, V_2 such that for $j = 1, 2$*

$$\tau^{w^t}(V_j) + d^{w^t}(V_j) \geqslant \frac{w_1^t - 1}{6} + \frac{w_1^1 - \Delta_t}{3} + \frac{2w_2^1}{3}.$$

P r o o f. Note that the difference $w_1^1 - w_1^t$ is the total weight of vertices decreasing in the process of the $\mathcal{G}$ algorithm. Similarly, the difference $w_2^t - w_2^1$ is the total weight of edges increasing in the process of the $\mathcal{G}$ algorithm. If $s_t = 0$, by the construction, we immediately have $w_1^1 - w_1^t = 2(w_2^t - w_2^1)$. If $s_t = 1$, similarly, we have $w_1^1 - w_1^t - (\Delta_t - 1) = 2(w_2^t - w_2^1)$. With help of the preceding two equalities, we conclude

$$w_1^1 - w_1^t - (\Delta_t - 1) \leqslant 2(w_2^t - w_2^1). \tag{2}$$

Now, we view G_t as a weighted hypergraph with edges of size at most 2. Note that $w^t(v) \leqslant 1$ for each $v \in V$ by the construction. Clearly, each edge of size 1 of G_t

has weight at most 1. Thus, by Theorem 1.2, there exists a bipartition V_1, V_2 of G_t such that for $j = 1, 2$

$$\tau^{w^t}(V_j) + d^{w^t}(V_j) \geqslant \frac{w_1^t - 1}{2} + \frac{2w_2^t}{3},$$

which together with (2) implies the desired result. □

We can now complete the proof of our main result.

Proof of Theorem 1.3. Note that $d^w(S) = \tau^{w^1}(S) + d^{w^1}(S)$ by the construction of G_1. By Lemma 2.1, for each $S \subseteq V$ we have

$$d^w(S) \geqslant \tau^{w^t}(S) + d^{w^t}(S).$$

It follows from Lemma 2.2 that G admits a bipartition V_1, V_2 such that for $j = 1, 2$

$$d^w(V_j) \geqslant \frac{w_1^t - 1}{6} + \frac{w_1^1 - \Delta_t}{3} + \frac{2w_2^1}{3}. \tag{3}$$

Again, by the construction of G_1, we have $w_1^1 = w_1$ and $w_2^1 = w_2$. In addition, the $\mathcal{G}$ algorithm implies that $w_1^t = w_0$ and $\Delta_t \leqslant \Delta_1$. Now, the result follows immediately from inequality (3). □

3. Tripartitions of graphs

In this section, we consider the tripartitions of graphs and prove Theorem 1.5. First, we introduce some definitions and lemmas.

Let $G = (V, E)$ be a graph. For a partition V_1, V_2, V_3 of G, define the degree of V_1, V_2, V_3 as $d(V_1, V_2, V_3) = \sum_{i=1}^{3} d(V_i)$. We call the partition *optimal* if $d(V_1, V_2, V_3)$ is as large as possible over partitions $V = V_1 \cup V_2 \cup V_3$, and *semi-optimal* if this degree cannot be increased by moving a vertex into V_3. Note that semi-optimality depends on the order of the sets in our partition. We shall always take the last set, V_3, to be the exceptional one. Trivially, every optimal partition is also semi-optimal. In the following, for every semi-optimal partition we show that the degree $d(V_1, V_2, V_3)$ can be lower bounded.

Lemma 3.1. *Let G be a graph with m edges. Suppose that V_1, V_2, V_3 is a semi-optimal partition of G. Then*

$$d(V_1, V_2, V_3) \geqslant 2m - d(V_3).$$

Proof. Since V_1, V_2, V_3 is semi-optimal, for each $v \in V_i$ and $i = 1, 2$ we have

$$e(v, \overline{V_3}) \leqslant e(v, \overline{V_i}). \tag{4}$$

Otherwise, $e(v, \overline{V_3}) > e(v, \overline{V_i})$. Let $X_i = V_i \setminus \{v\}$, $X_{3-i} = V_{3-i}$ and $X_3 = V_3 \cup \{v\}$. Clearly, we have $d(X_i) = d(V_i) - e(v, \overline{V_i})$, $d(X_{3-i}) = d(V_{3-i})$ and $d(X_3) = d(V_3) + e(v, \overline{V_3})$. This implies that $d(X_1, X_2, X_3) > d(V_1, V_2, V_3)$, a contradiction with the choice of V_1, V_2, V_3.

By (4), for each $v \in V_i$ and $i = 1, 2$ we deduce

$$e(v, V_i) \leqslant e(v, V_3).$$

Summing over all $v \in V_i$ yields $2e(V_i) \leqslant e(V_i, V_3)$, giving that

$$2(e(V_1) + e(V_2)) \leqslant e(V_1, V_3) + e(V_2, V_3) = d(V_3) - e(V_3).$$

This establishes that

$$\sum_{i=1}^{3} e(V_i) \leqslant \frac{d(V_3)}{2} + \frac{e(V_3)}{2} \leqslant d(V_3).$$

Noting that $d(V_1, V_2, V_3) + \sum\limits_{i=1}^{3} e(V_i) = 2m$, we obtain

$$d(V_1, V_2, V_3) = 2m - \sum_{i=1}^{3} e(V_i) \geqslant 2m - d(V_3),$$

as desired. $\square$

Next, we show that the semi-optimality of a partition V_1, V_2, V_3 of G is preserved if we move vertices into V_3.

Lemma 3.2. *Let V_1, V_2, V_3 be a semi-optimal partition of a graph G, and let U_1, U_2, U_3 be another partition of G with $U_1 \subseteq V_1$, $U_2 \subseteq V_2$ and $U_3 \supseteq V_3$. Then U_1, U_2, U_3 is also semi-optimal.*

Proof. For each $v \in U_i$ and $i = 1, 2$, let $U_i' = U_i \setminus \{v\}$, $U_{3-i}' = U_{3-i}$ and $U_3' = U_3 \cup \{v\}$. Similarly, let $V_i' = V_i \setminus \{v\}$, $V_{3-i}' = V_{3-i}$ and $V_3' = V_3 \cup \{v\}$. Then

$$d(U_i) - d(U_i') = e(v, \overline{U_i}) \geqslant e(v, \overline{V_i}) = d(V_i) - d(V_i')$$

and

$$d(U_3') - d(U_3) = e(v, \overline{U_3}) \leqslant e(v, \overline{V_3}) = d(V_3') - d(V_3).$$

Thus, we have

$$d(U_i') + d(U_3') \leqslant d(U_i) + d(U_3) + (d(V_i') + d(V_3') - d(V_i) - d(V_3)),$$

which is equivalent to

$$d(U_1', U_2', U_3') \leqslant d(U_1, U_2, U_3) + (d(V_1', V_2', V_3') - d(V_1, V_2, V_3)). \tag{5}$$

Since $d(V_1, V_2, V_3)$ cannot be increased by moving a vertex into V_3, we have $d(V_1', V_2', V_3') \leqslant d(V_1, V_2, V_3)$. It follows from (5) that U_1, U_2, U_3 is also a semi-optimal partition of G as claimed. □

Now, we are ready to give the proof of Theorem 1.5.

Proof of Theorem 1.5. Since isolated vertices contribute nothing to the meeting edges, we may assume that G contains no isolated vertices. It is easy to check that the result holds for $m \leqslant 3$, except when G is isomorphic to K_2 or $K_{1,3}$. Assume that $m \geqslant 4$. Let Δ be the maximum degree of G and $l = \lceil 2m/5 \rceil$. We proceed by showing the following several claims.

Claim 1. $\Delta < l$. Otherwise, let v be a vertex in G with degree $\Delta \geqslant l$. Consider the graph H_1 induced by $V \setminus \{v\}$. We view H_1 as a weighted hypergraph with m edges, of which Δ have size 1 and $m - \Delta$ have size 2. Let w be the weight function of H_1. For each $f \in E(H)$, we define $w(f) = 1$. Now, we use Theorem 1.2 setting $\Delta_1 = 1$, $w_1 = \Delta$ and $w_2 = m - \Delta$. Thus, there exists a bipartition U_1, U_2 of H_1 such that for $i = 1, 2$

$$d(U_i) \geqslant \frac{w_1 - 1}{2} + \frac{2w_2}{3} = \frac{2m}{3} - \frac{\Delta}{6} - \frac{1}{2} > l - 1.$$

The last inequality holds because $\Delta \leqslant m$ and $m \geqslant 4$. By the integrality of $d(U_i)$, we have $d(U_i) \geqslant l$. Set $U_3 = \{v\}$. Clearly, U_1, U_2, U_3 will do for our tripartition. This completes the proof of Claim 1.

Let V_1, V_2, V_3 be an optimal partition of G, ordered so that $d(V_1) \geqslant d(V_2) \geqslant d(V_3)$. If $d(V_3) \geqslant l$, we are done. Suppose that $d(V_3) \leqslant l-1$.

Claim 2. $d(V_2) \geqslant l$. Let H_2 be the graph induced by $\overline{V_3}$. By the maximality of $d(V_1, V_2, V_3)$, we know that

$$(*) \qquad V_1, V_2 \textit{ is a bipartition of } H_2 \textit{ minimizing } e_{H_2}(V_1) + e_{H_2}(V_2).$$

Otherwise, let V_1', V_2' be another bipartition of H_2 such that

$$e_{H_2}(V_1') + e_{H_2}(V_2') < e_{H_2}(V_1) + e_{H_2}(V_2).$$

Note that $d(V_1, V_2, V_3) = 2m - \sum_{i=1}^{3} e(V_i)$ and $e_{H_2}(S) = e(S)$ for each $S \subseteq \overline{V_3}$. Clearly, V_1', V_2', V_3 is another partition of G satisfying $d(V_1', V_2', V_3) > d(V_1, V_2, V_3)$, contradicting the choice of V_1, V_2, V_3.

For each $v \in V_i$ and $i = 1, 2$, it follows from $(*)$ that

$$e(v, V_i) \leqslant e(v, V_{3-i}).$$

Summing over all $v \in V_i$ gives that $2e(V_i) \leqslant e(V_1, V_2)$. Observing that $e(\overline{V_3}) = e(V_i) + e(V_{3-i}) + e(V_1, V_2)$, we have

$$3e(V_i) + e(V_{3-i}) \leqslant e(\overline{V_3}).$$

It follows that $e(V_i) \leqslant e(\overline{V_3})/3$ for $i = 1, 2$. Note that $d(V_2) \geqslant e(\overline{V_3}) - e(V_1)$ and $d(V_3) \leqslant l-1$ by our assumption. Thus,

$$d(V_2) \geqslant \frac{2e(\overline{V_3})}{3} = \frac{2(m - d(V_3))}{3} > l - 1,$$

i.e., $d(V_2) \geqslant l$ by integrality, completing the proof of Claim 2.

For $i = 1, 2$, let X_i be a minimal subset of V_i satisfying $d(X_i) \geqslant l$, and $X_3 = V \setminus (X_1 \cup X_2)$. If $d(X_3) \geqslant l$, then X_1, X_2, X_3 is a suitable tripartition. Suppose that $d(X_3) \leqslant l-1$. Without loss of generality, we may assume that $d(X_1) \geqslant d(X_2)$.

Claim 3. $|X_1| = 2$. Since V_1, V_2, V_3 is an optimal partition of G, by Lemma 3.2, X_1, X_2, X_3 is a semi-optimal partition of G. It follows from Lemma 3.1 that

$$d(X_1) + d(X_2) \geqslant 2(m - d(X_3)) \geqslant 2(m - l + 1).$$

Clearly, we have $d(X_1) \geqslant m - l + 1$. By the minimality of X_1, for each vertex $x \in X_1$ there are at least $m - 2(l-1)$ edges meeting X_1 only at x. Since otherwise

$$d(X_1 \setminus \{x\}) = d(X_1) - e(x, \overline{X_1}) \geqslant (m - l + 1) - (m - 2(l-1) - 1) = l,$$

a contradiction. Note that $m-2(l-1)\geqslant l/2$. Hence, any two vertices of X_1 meet at least l edges, and so two vertices cannot be a proper subset of X_1. Thus, $|X_1|\leqslant 2$. Since $d(X_1)\geqslant l$, by Claim 1 we have $|X_1|=2$. Thus, we complete the proof of Claim 3.

Let $X_1=\{x_1,x_2\}$ and $\theta=|N(x_1)\cap N(x_2)|$. Since $\Delta<l$, we may assume that

$$d(x_1)+d(x_2)=2(l-1)-r, \tag{6}$$

where $r\geqslant 0$ is an integer. Write $e=x_1x_2$ and define the indicator variable $\mathbf{1}_e=1$ if and only if $e\in E(G)$, otherwise $\mathbf{1}_e=0$. Now, we may write

$$\theta=l-1-\mathbf{1}_e-s, \tag{7}$$

where $s\geqslant 0$ is an integer. Let $\mathbf{g}=(m,r,s,\mathbf{1}_e)$, $\mathbf{g}_1=(6,0,0,1)$, $\mathbf{g}_2=(8,0,0,1)$, $\mathbf{g}_3=(8,0,0,0)$ and $\mathbf{g}_4=(8,0,1,1)$.

Claim 4. $\mathbf{g}\in\{\mathbf{g}_i\colon 1\leqslant i\leqslant 4\}$. Consider the graph H_3 induced by $X_2\cup X_3$ and view H_3 as a weighted hypergraph with weight function w. Let $N_1=N(x_1)\cup N(x_2)$ and $N_2=N(x_1)\cap N(x_2)$. For each $x\in N_1\setminus\{x_1,x_2\}$, let $\{x\}$ be the edge of size 1 of H_3. If also $x\in N_2$, define $w(\{x\})=2$; otherwise, set $w(\{x\})=1$. For each edge f of G contained in $X_2\cup X_3$, let f be the edge of size 2 of H_3 and define $w(f)=1$. Now, we apply Theorem 1.3 setting $\Delta_1=2$, $w_1=d(x_1)+d(x_2)-2\cdot\mathbf{1}_e$, $w_2=m-w_1-\mathbf{1}_e$ and $w_0=w_1-\theta$. Thus, there exists a bipartition X_2',X_3' of H_3 such that for $i=2,3$

$$d(X_i')\geqslant\frac{w_0-1}{6}+\frac{w_1-\Delta_1}{3}+\frac{2w_2}{3}=\frac{2m}{3}-\frac{l}{2}+\frac{r+s-2-\mathbf{1}_e}{6}.$$

The last equality follows from (6) and (7). In the following, we aim at showing that X_1, X_2', X_3' is a suitable tripartition of G. By integrality, it suffices to show that

$$\frac{2m}{3}-\frac{l}{2}+\frac{r+s-2-\mathbf{1}_e}{6}>l-1,$$

which is equivalent to proving that

$$4m+r+s+3\geqslant 9l+\mathbf{1}_e. \tag{8}$$

Clearly, if $r+s\geqslant 1+\mathbf{1}_e$, (8) follows immediately from the fact $m\geqslant 4$. Note that, if $r\geqslant 1$, we have $\min\{d(x_1),d(x_2)\}\leqslant l-2$ by (6). Since $\theta\leqslant\min\{d(x_1),d(x_2)\}-\mathbf{1}_e$, by (7) we know that $s\geqslant 1$ provided $r\geqslant 1$. Thus, we may assume that $r=0$ and $s\leqslant 1$. Now, it is easy to check that (8) holds except when $\mathbf{g}=\mathbf{g}_i$, where $i=1,2,3,4$. This completes the proof of Claim 4.

Since $\Delta < l$ and $r = 0$, we have $d(x_1) = d(x_2) = l-1$ by (6). Therefore, $\Delta = l-1$. Let G_i be the graph G satisfying $\mathbf{g} = \mathbf{g}_i$ for each $i = 1, 2, 3, 4$. Note that G_i contains at least $\lceil 2E(G_i)/\Delta(G_i)\rceil$ vertices.

Claim 5. For each $1 \leqslant i \leqslant 4$, G_i admits a tripartition such that each vertex class meets at least l edges.

If $\mathbf{g} = \mathbf{g}_1$, then $l = 3$, $\theta = 1$ and $d(x_1) = d(x_2) = 2$. Suppose that $N(x_1) = \{x_2, x_3\}$ and $N(x_2) = \{x_1, x_3\}$. Since G_1 contains at least 6 vertices, assume that $\{x_1, \ldots, x_6\} \subseteq V(G_1)$. Let Z_1, Z_2, Z_3 be a partition of G_1 satisfying $\{x_1, x_4\} \subseteq Z_1$, $\{x_2, x_5\} \subseteq Z_2$ and $\{x_3, x_6\} \subseteq Z_3$. Clearly, Z_1, Z_2, Z_3 will do for our tripartition.

If $\mathbf{g} = \mathbf{g}_i$ for $i = 2, 3$, then $l = 4$ and $d(x_1) = d(x_2) = 3$. Moreover, if $\mathbf{g} = \mathbf{g}_2$, then $\theta = 2$. Set $N(x_1) = \{x_2, x_3, x_4\}$ and $N(x_2) = \{x_1, x_3, x_4\}$. If $\mathbf{g} = \mathbf{g}_3$, then $\theta = 3$. Set $N(x_1) = N(x_2) = \{x_3, x_4, x_5\}$. Again, G_i contains at least 6 vertices, say $\{x_1, \ldots, x_6\} \subseteq V(G_i)$. Let Z_1, Z_2, Z_3 be a partition of G_i satisfying $\{x_1, x_6\} \subseteq Z_1$, $\{x_2, x_5\} \subseteq Z_2$ and $\{x_3, x_4\} \subseteq Z_3$. In either case, Z_1, Z_2, Z_3 will do for our tripartition.

If $\mathbf{g} = \mathbf{g}_4$, then $l = 4$, $\theta = 1$ and $d(x_1) = d(x_2) = 3$. Let $N(x_1) = \{x_2, x_3, x_4\}$ and $N(x_2) = \{x_1, x_3, x_5\}$. Suppose that G_4 contains at least 7 vertices, say $\{x_1, \ldots, x_7\} \subseteq V(G_4)$. Let Z_1, Z_2, Z_3 be a partition of G_4 satisfying $\{x_1, x_6\} \subseteq Z_1$, $\{x_2, x_7\} \subseteq Z_2$ and $\{x_3, x_4, x_5\} \subseteq Z_3$. Clearly, Z_1, Z_2, Z_3 is a desired tripartition. Thus, G_4 contains exactly 6 vertices, say $x_1, \ldots, x_6$. Note that $\sum_{i=1}^{6} d(x_i) = 2m = 16$. If $d(x_6) = 1$, then $d(x_i) = 3$ for each $1 \leqslant i \leqslant 5$. Clearly, $\{x_1, x_6\}$, $\{x_2, x_5\}$ and $\{x_3, x_4\}$ is a desired tripartition. Hence, $d(x_6) \geqslant 2$. Now, $\{x_1, x_5\}$, $\{x_2, x_4\}$ and $\{x_3, x_6\}$ will do for our tripartition as required.

Thus, we complete the proof of Theorem 1.5. $\square$

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