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A PENALTY METHOD FOR THE TIME-DEPENDENT STOKES PROBLEM WITH THE SLIP BOUNDARY CONDITION AND ITS FINITE ELEMENT APPROXIMATION

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Abstract. We consider the finite element method for the time-dependent Stokes problem with the slip boundary condition in a smooth domain. To avoid a variational crime of numerical computation, a penalty method is introduced, which also facilitates the numerical implementation. For the continuous problem, the convergence of the penalty method is investigated. Then we study the fully discretized finite element approximations for the penalty method with the P1/P1-stabilization or P1b/P1 element. For the discretization of the penalty term, we propose reduced and non-reduced integration schemes, and obtain an error estimate for velocity and pressure. The theoretical results are verified by numerical experiments.

Keywords: penalty method; Stokes problem; finite element method; error estimate

MSC 2010: 65N30, 35Q30

1. Introduction

We consider the time-dependent Stokes problem in a smooth bounded domain $\Omega \subset \mathbb{R}^N$ $(N = 2, 3)$ with boundary $\partial\Omega = \gamma \cup \Gamma$, where $\overline{\gamma} \cap \overline{\Gamma} = \emptyset$ and γ has positive $(N-1)$-dim measure. The problem reads:

$$(1.1) \qquad (\mathbf{P}) \quad \begin{cases} u_t - \nu\Delta u + \nabla p = f, \quad \nabla \cdot u = 0 & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \gamma \times (0,T), \\ u \cdot n = 0, \quad (I - n \otimes n)\sigma(u,p)n = 0 & \text{on } \Gamma \times (0,T), \\ u(x,0) = u_0 & \text{in } \Omega, \end{cases}$$

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where $0 < T < \infty$, u and p denote the velocity and pressure of the fluid, respectively, ν denotes the viscosity constant, n is the unit outer normal vector to Γ, and $\sigma(u,p) = -pI + \nu(\nabla u + \nabla u^{\mathrm{T}})$ is the stress tensor.

The slip boundary condition $(1.1)_4$ has massive applications in the real flow problems (see [17], [15], [11], [19]). However, there exist some numerical difficulties to deal with the slip boundary condition when Γ is smooth. In the finite element method (FEM), Ω is usually approximated by a polygon or polyhedron Ω_h with the Dirichlet boundary γ_h and the slip boundary Γ_h. It is natural to discretize the slip boundary condition by $u_h \cdot n_h = 0$, where n_h is the unit outer normal vector to Γ_h. However, such discretization results in a variational crime and leads to the constraint $u_h = 0$ on Γ_h, because n_h is in general discontinuous at the vertices of Γ_h.

To overcome the variational crime, [22], [21] imposed $u_h \cdot n = 0$ at the nodes of Γ_h, where Ω is assumed to be a spherical shell and n is prescribed. Using the quadratic approximation, [1] proposed the discretization $u_h \cdot (n \circ G_h) = 0$ at all nodes and barycentres of the boundary elements on Γ_h, where G_h is an abstract transformation from Γ_h to Γ. However, in both methods, it is quite hard to compute G_h or n for a general domain. In addition, the implementations of $u_h \cdot n = 0$ and $u_h \cdot (n \circ G_h) = 0$ in finite element code require more advanced techniques than the Dirichlet boundary condition (see [1], [7]). Although one can use some approximation of n or n_h in the above schemes (see [2], [5]), a rigorous error analysis is difficult and some points still remain unclear in the literature.

On the other hand, a penalty method has also been proposed in order to avoid such numerical and theoretical difficulties. The penalty method is very simple and easy to implement by the popular FEM softwares, such as Freefem++ (see [9]) and FEniCS (see [16]). The idea of the penalty method is to replace the slip boundary condition by a Robin-type boundary condition (see $(2.6)_3$), which yields a penalty term in variational form, i.e., $\varepsilon^{-1} \int_\Gamma (u_\varepsilon \cdot n)(v \cdot n)\,\mathrm{d}\Gamma$ in (2.5) with a penalty parameter ε ($0 < \varepsilon \ll 1$).

In this paper, we consider a penalty method for the time-dependent Stokes problem. There exist a lot of works on the penalty method for stationary problems. However, to the best of our knowledge, there is no literature dealing with the time-dependent problem. The main contribution of the paper is to establish error estimates of the penalty method for such a problem. We emphasize that the error analysis cannot be obtained by a straightforward extension of the analysis in the stationary case and that there are indeed nontrivial difficulties in the proof, which is explained below.

Let us pay attention to the error estimate of the penalty method. For the stationary Stokes/Navier-Stokes problems, the sub-optimal error estimate of order $O(\sqrt{\varepsilon})$ is proved under *a priori* estimate of the traction tensor in the L^2 norm; whereas the optimal error estimate of order $O(\varepsilon)$ requires the boundedness of u and p in the H^2 and H^1 norms, respectively. To prove the optimal error estimate, the inf-sup conditions of pressure and Lagrange multiplier have been used (cf. [4], [6], [28]). However, these arguments are not applicable to the non-stationary problem. We explain the reasons in the following (see Section 3 for the detailed proof and discussion). First, owing to the loss of compatibility of the initial value and the boundary condition for $(\mathbf{P})$ and the penalty problem, we only obtain *a priori* estimates with weight $\sqrt{t}$ in front of u_{tt} and $u_{\varepsilon tt}$. Moreover, in the non-stationary case, we cannot use the inf-sup condition to get estimates of pressure and Lagrange multiplier depending only on velocity, because the time derivative of velocity is also involved.

As a result, we need to construct a new proof for error analysis. In this paper, we show *a priori* estimates of $(\mathbf{P})$ and the penalty problem under various regularity assumptions on given data, with help of which we derive the sub-optimal $O(\sqrt{\varepsilon})$ and quasi-optimal $O(\varepsilon|\log\varepsilon|)$ error estimates for the penalty method.

Now we turn our attention to the finite element approximation for the penalty problem. For the stationary Stokes/Navier-Stokes problem with the slip boundary condition, the FEM without penalty has been studied by Verfürth [25], [26], [27], Knobloch [14] and Bäncsh and Deckelnick [1], and the case of the penalty method has been investigated by Dione and Urquiza [6] and [12], [28]. The error estimates of all the above works become sub-optimal if the difference between n and n_h is carefully taken into account (see Introduction of [12] for a comprehensive description of these works). We mention that the error can be upgraded to optimal in the two-dimensional case by introducing a reduced integration for the penalty term (see [12], [28]).

All the above results are concerned with the stationary problem. In the present paper, we consider the P1/P1-stabilization (or P1b/P1) full-discrete finite element approximation for the time-dependent problem. Introducing the projection operators of velocity and pressure from [12], [28], we derive the error estimate $O(\tau + h + \sqrt{\varepsilon} + h/\sqrt{\varepsilon})$, where τ and h are the time and spatial discretization parameters. For the two-dimensional case with reduced integration for the penalty term, the error estimate is upgraded to $O(\tau + h + \sqrt{\varepsilon} + h^2/\sqrt{\varepsilon})$.

The paper is organized as follows. In Section 2, we introduce the penalty problem $(\mathbf{P}_\varepsilon)$, and derive *a priori* estimates for $(\mathbf{P})$ and $(\mathbf{P}_\varepsilon)$ under various regularity assumptions on the initial value and force. In Section 3, we deduce sub-optimal and quasi-optimal error estimates for the penalty method. Section 4 is devoted to the finite element scheme of the penalty method. Numerical experiments are presented in Section 5.

Notation. Throughout this paper, the norms of the Sobolev spaces $H^k(\omega)$ and $W^{k,p}(\omega)$ are denoted by $\|\cdot\|_{H^k(\omega)}$ and $\|\cdot\|_{W^{k,p}(\omega)}$, respectively. The inner product of $L^2(\omega)$ or $L^2(\omega)^N$ is denoted by $(\cdot,\cdot)_\omega$. We will use the abbreviation $L^m(H^k(\omega))$ to mean $L^m(0,T;H^k(\omega))$, $L^m(0,t;H^k(\omega))$, $L^m(0,t;H^k(\omega)^N)$ or $L^m(0,T;H^k(\omega)^N)$. Sometimes, we omit ω in the above notation when $\omega=\Omega$. We introduce the notation $v_n = v\cdot n$ and $v_T = (I - n\otimes n)v$ to represent the normal and tangential component of v on Γ, respectively. We use C to denote generic constants independent of ε, h, and τ. We also use $C(a,b)$ to emphasize that the constant is dependent on a and b. The volume and surface measures are denoted by $|\cdot|$.

2. The penalty problem and related estimates

2.1. Function spaces and bilinear forms. We introduce the function spaces

$$
\begin{aligned}
V &= \{v\in H^1(\Omega)^N;\ v=0 \text{ on } \gamma\}, \quad V_n = \{v\in V;\ v_n = 0 \text{ on } \Gamma\},\\
H^\sigma &= \{v\in L^2(\Omega)^N;\ \nabla\cdot v = 0 \text{ in weak sense}\},\\
H^\sigma_n &= \{v\in H^\sigma;\ v_n=0 \text{ holds weakly on } \Gamma\},\\
V^\sigma &= \{v\in V;\ \nabla\cdot v=0\}, \quad V_n^\sigma = V_n\cap V^\sigma, \quad Q = L^2(\Omega),\\
\mathring{Q} &= L^2_0(\Omega) = \{q\in L^2(\Omega);\ (q,1)=0\}, \quad \Lambda = H^{1/2}(\Gamma), \quad \Lambda^* = H^{-1/2}(\Gamma),
\end{aligned}
$$

where X^* denotes the dual space of a Banach space X.

For any $\omega\subset\mathbb{R}^N$, we define the bilinear forms

$$
\begin{aligned}
a_\omega(u,v) &:= \frac{\nu}{2}(\mathcal{E}(u),\mathcal{E}(v))_\omega, \quad \text{for } u,v\in H^1(\omega)^N,\\
b_\omega(v,p) &:= (-\nabla\cdot v, p)_\omega, \quad \text{for } v\in H^1(\omega)^N,\ p\in L^2(\omega),\\
c(\lambda,\mu) &:= (\lambda,\mu)_\Gamma, \quad \text{for } \lambda\in\Lambda, \mu\in\Lambda^*,
\end{aligned}
$$

where $\mathcal{E}(u) = \nabla u + \nabla u^{\mathrm{T}}$ and $(\cdot,\cdot)_\Gamma$ denotes the dual product between Λ and Λ^*. We introduce some inequalities for the above bilinear forms.

▷ Korn's inequality: there exists a constant C depending on Ω (note that $|\gamma|>0$) such that

$$
a_\Omega(v,v) \geqslant C\|v\|^2_{H^1} \quad \forall\, v\in V. \tag{2.1}
$$

▷ Inf-sup condition: there exists a constant C depending on Ω such that

$$
C\|q\|_{L^2} \leqslant \sup_{v\in H^1_0(\Omega)^N} \frac{b_\Omega(v,q)}{\|v\|_{H^1}} \quad \forall\, q\in\mathring{Q}, \tag{2.2}
$$

where $H^1_0(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to $\|\cdot\|_{H^1(\Omega)}$.

At this stage, let $f \in L^2(L^2(\Omega))$. Then the variational form of $(\mathbf{P})$ reads: Find $(u,p) \in (H^1(L^2) \cap L^2(V_n)) \times L^2(Q)$ with $u(0) = u_0$ such that for all $t \in (0,T)$,

$$\begin{cases} (u_t(t), v) + a_\Omega(u(t), v) + b_\Omega(v, p(t)) = (f(t), v) & \forall\, v \in V_n, \\ b_\Omega(u(t), q) = 0 & \forall\, q \in Q. \end{cases} \tag{2.3}$$

The unique existence of the weak solution of $(\mathbf{P})$ follows from the standard theory (see §1, Chapter 3 of [24]). In fact, given $u_0 \in H_n^\sigma$ and $f \in L^2(V_n^{\sigma *})$, there exists a unique weak solution $u \in C([0,T]; H_n^\sigma) \cap L^2(0,T; V_n^\sigma)$ to $(\mathbf{P})$, i.e., u satisfies: $u(x,0) = u_0$, and for all $t \in (0,T)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(u(t), v) + a_\Omega(u(t), v) = (f(t), v) \quad \forall\, v \in V_n^\sigma. \tag{2.4}$$

2.2. The penalty method. Let ε be the penalty parameter with $0 < \varepsilon \ll 1$, and let $u_{\varepsilon 0}$ be an initial value approximating u_0. The penalty problem in variational form reads: Find $(u_\varepsilon, p_\varepsilon) \in (H^1(L^2) \cap L^2(V)) \times L^2(Q)$ with $u_\varepsilon(0) = u_{\varepsilon 0}$ such that for all $t \in (0,T)$,

$$\begin{cases} (u_{\varepsilon t}(t), v) + a_\Omega(u_\varepsilon(t), v) + b_\Omega(v, p_\varepsilon(t)) + \varepsilon^{-1} c(u_{\varepsilon n}(t), v_n) = (f(t), v) & \forall\, v \in V, \\ b(u_\varepsilon(t), q) = 0 & \forall\, q \in Q. \end{cases} \tag{2.5}$$

The strong form of the penalty problem reads:

$$(\mathbf{P}_\varepsilon) \quad \begin{cases} u_{\varepsilon t} - \nu \Delta u_\varepsilon + \nabla p_\varepsilon = f, \quad \nabla \cdot u_\varepsilon = 0 & \text{in } \Omega \times (0,T), \\ u_\varepsilon = 0 & \text{on } \gamma \times (0,T), \\ \sigma(u_\varepsilon, p_\varepsilon) n + \varepsilon^{-1} u_{\varepsilon n} n = 0 & \text{on } \Gamma \times (0,T), \\ u_\varepsilon(x,0) = u_{\varepsilon 0} & \text{in } \Omega. \end{cases} \tag{2.6}$$

Proposition 2.1. *Given $u_{\varepsilon 0} \in H^\sigma$ and $f \in L^2(V^{\sigma *})$, there exists a unique weak solution $u_\varepsilon \in C([0,T]; H^\sigma) \cap L^2(V^\sigma)$ to $(\mathbf{P}_\varepsilon)$, i.e., u_ε satisfies $u_\varepsilon(x,0) = u_{\varepsilon 0}$ and for all $t \in (0,T)$,*

$$\frac{\mathrm{d}}{\mathrm{d}t}(u_\varepsilon(t), v) + a_\Omega(u_\varepsilon(t), v) + \varepsilon^{-1}(u_{\varepsilon n}(t), v_n)_\Gamma = (f(t), v) \quad \forall\, v \in V^\sigma.$$

Proof. In view of the coercivity $a_\Omega(v,v) + \varepsilon^{-1} c(v_n, v_n) \geqslant C\|v\|_{H^1}^2$, the unique existence follows from the standard argument (see §1, Chapter 3 of [24]). □

2.3. A priori estimates for (P) and $(\mathbf{P}_\varepsilon)$. To obtain error estimates of the penalty method, we need *a priori* estimates for $(\mathbf{P})$ and $(\mathbf{P}_\varepsilon)$.

2.3.1. A priori estimate for (P).

Proposition 2.2. *Let u be the solution of* $(\mathbf{P})$.

(1) *For $u_0 \in H^\sigma_n$ and $f \in L^2(V^{\sigma *}_n)$ we have:*

$$\|u\|^2_{L^\infty(L^2)} + \|u\|^2_{L^2(H^1)} \leqslant C(\|f\|^2_{L^2(V^{\sigma *}_n)} + \|u_0\|^2_{L^2}) =: C_1(f, u_0).$$

(2) *For $u_0 \in V^\sigma_n$ and $f \in L^2(L^2)$, we have:*

$$\|u_t\|^2_{L^2(L^2)} + \|u\|^2_{L^\infty(H^1)} \leqslant C(\|f\|^2_{L^2(L^2)} + \|u_0\|^2_{H^1}) =: C_2(f, u_0).$$

(3) *For $u_0 \in V^\sigma_n \cap H^2(\Omega)^N$, $f \in C([0,T]; L^2)$, and $f_t \in L^2(0,T; L^2)$ we have:*

$$\|u_t\|^2_{L^\infty(L^2)} + \|u_t\|^2_{L^2(H^1)} \leqslant C_{31}(f, u_0), \tag{2.7a}$$

$$\|\sqrt{t} u_{tt}\|^2_{L^2(L^2)} + \|\sqrt{t} u_t\|^2_{L^2(H^1)} \leqslant C\|\sqrt{t} f\|^2_{L^2(L^2)} + C_{31}(f, u_0), \tag{2.7b}$$

where $C_{31}(f, u_0) := C(\|f_t\|^2_{L^2(V^{\sigma *}_n)} + \|u_0\|^2_{H^2} + \|f\|^2_{C([0,t];L^2)})$. In addition, if $u_0 \in H^3(\Omega)^N$ and $f(0) \in H^1(\Omega)^N$, then we have:

$$\|u_{tt}\|^2_{L^2(L^2)} + \|u_t\|^2_{L^2(H^1)} \leqslant C(\|f_t\|^2_{L^2(L^2)} + \|u_0\|^2_{H^3} + \|f(0)\|^2_{H^1}) =: C_{32}(f, u_0). \tag{2.8}$$

The results of Proposition 2.2 have already been obtained by Heywood and Rannacher, Theorems 2.4 and 2.5 [10] for the Dirichlet boundary condition. By a similar argument, we can prove Proposition 2.2 for the slip boundary problem.

Remark 2.1 (Regularity of u). In a similar manner to Theorems 2.4 and 2.5 [10], we can show the regularity $\sup\limits_{0<t<T} t^{2n+m-2} \|D^n_t u\|^2_{H^m} < \infty$ when Ω and f are sufficiently smooth, which implies that one can obtain any regularity of u in (t_a, T) for $t_a > 0$.

Remark 2.2 (Regularity of p). Consider the stationary Stokes problem with the slip boundary condition:

$$\begin{cases} -\Delta u^* + \nabla p^* = f^*, \quad \nabla \cdot u^* = 0 \quad \text{in } \Omega, \\ u^* = 0 \quad \text{on } \gamma, \quad u^*_n = 0, \quad (I - n \otimes n)\sigma(u^*, p^*)n = 0 \quad \text{on } \Gamma. \end{cases}$$

For sufficiently smooth γ and Γ, we have $\|u^*\|_{H^{m+2}} + \|p^*\|_{H^{m+1}} \leqslant C\|f^*\|_{H^m}$ (cf. [18]). Hence, Proposition 2.2 (2) implies

$$\|u\|_{L^2(H^2)} + \|p\|_{L^2(H^1)} \leqslant C_2(f, u_0).$$

Moreover, it follows from (2.7) and (2.8) that

$$\|u\|_{C([0,T];H^2)} + \|p\|_{C([0,T];H^1)} \leqslant C_{31}(f, u_0), \tag{2.9a}$$

$$\|u_t\|_{L^2(H^2)} + \|p_t\|_{L^2(H^1)} \leqslant C_{32}(f, u_0). \tag{2.9b}$$

2.3.2. A priori estimate for $(\mathbf{P}_\varepsilon)$.

Proposition 2.3. *Let u_ε be the solution of $(\mathbf{P}_\varepsilon)$.*

(1) *For $u_{\varepsilon 0} \in H^\sigma$ and $f \in L^2(V^{\sigma *})$, we have:*

$$\|u_\varepsilon\|^2_{L^\infty(L^2)} + \|u_\varepsilon\|^2_{L^2(H^1)} + \varepsilon^{-1}\|u_{\varepsilon n}\|^2_{L^2(L^2(\Gamma))} \leqslant C_1(f, u_{\varepsilon 0}).$$

(2) *For $u_{\varepsilon 0} \in V^\sigma$ with $\|u_{\varepsilon 0} \cdot n\|_{L^2(\Gamma)} \leqslant C\sqrt{\varepsilon}$ and $f \in L^2(L^2)$, we have:*

$$\|u_{\varepsilon s}\|^2_{L^2(L^2)} + \|u_\varepsilon\|^2_{L^\infty(H^1)} + \varepsilon^{-1}\|u_{\varepsilon n}\|^2_{L^\infty(L^2(\Gamma))} \leqslant C_2(f, u_{\varepsilon 0}) + C\varepsilon^{-1}\|u_{\varepsilon 0}\|^2_{L^2(\Gamma)}.$$

(3) *For $u_{\varepsilon 0} \in V^\sigma \cap H^2(\Omega)^N$, $\|u_{\varepsilon 0} \cdot n\|_{H^{1/2}(\Gamma)} \leqslant C\varepsilon$, $f \in C([0,T]; L^2)$ and $f_t \in L^2(L^2)$, we have:*

$$\|u_{\varepsilon t}\|^2_{L^\infty(L^2)} + \|u_{\varepsilon t}\|^2_{L^2(H^1)} \leqslant C_{31}(f, u_{\varepsilon 0}) + C\|\varepsilon^{-1}u_{\varepsilon 0} \cdot n\|^2_{H^{1/2}(\Gamma)}, \tag{2.10a}$$

$$\|\sqrt{t}u_{\varepsilon tt}\|^2_{L^2(L^2)} + \|\sqrt{t}u_{\varepsilon t}\|^2_{L^\infty(H^1)} \leqslant C_{32}(f, u_{\varepsilon 0}) + C\|\varepsilon^{-1}u_{\varepsilon 0} \cdot n\|^2_{H^{1/2}(\Gamma)}. \tag{2.10b}$$

Proof. Substituting $v = u_\varepsilon$ and $v = u_{\varepsilon t}$ into (2.5) yields the *a priori* estimates (1) and (2), respectively.

In the following, we prove (3). There exists a $p_{\varepsilon 0} \in H^1(\Omega)$ satisfying

$$\begin{cases} (\nabla p_{\varepsilon 0}, \nabla q) = (f(0) + \Delta u_{\varepsilon 0}, \nabla q) \quad \forall\, q \in H^1_0(\Omega), \\ p_{\varepsilon 0} = \varepsilon^{-1}u_{\varepsilon 0} \cdot n + \mathcal{E}(u_{\varepsilon 0})n \cdot n \in H^{1/2}(\Gamma) \quad \text{on } \Gamma, \quad \nabla p_{\varepsilon 0} \cdot n = 0 \quad \text{on } \gamma. \end{cases} \tag{2.11}$$

Then $p_{\varepsilon 0}$ fulfills the estimate

$$\|p_{\varepsilon 0}\|_{H^1} \leqslant C(\varepsilon^{-1}\|u_{\varepsilon 0} \cdot n\|_{H^{1/2}(\Gamma)} + \|u_{\varepsilon 0}\|_{H^2}). \tag{2.12}$$

We define $\dot u_{\varepsilon 0} := f(0) + \Delta u_{\varepsilon 0} - \nabla p_{\varepsilon 0}$. By the definition of $p_{\varepsilon 0}$, it is easy to verify that $\nabla \cdot \dot u_{\varepsilon 0} = 0$ in weak sense, i.e., $\dot u_{\varepsilon 0} \in H^\sigma$. Then we have: for all $v \in V^\sigma$,

$$(\dot u_{\varepsilon 0}, v) + a_\Omega(u_{\varepsilon 0}, v) + \varepsilon^{-1}(u_{\varepsilon 0} \cdot n, v_n)_\Gamma = (f(0), v). \tag{2.13}$$

In fact, (2.12) yields

$$\|\dot u_{\varepsilon 0}\|_{L^2} \leqslant C(\varepsilon^{-1}\|u_{\varepsilon 0} \cdot n\|_{\{H^{1/2}(\Gamma)} + \|u_{\varepsilon 0}\|_{H^2} + \|f(0)\|_{L^2}).$$

By Proposition 2.1, there exists a unique weak solution $\dot u_\varepsilon \in C([0,T]; H^\sigma) \cap L^2(0,T;V^\sigma)$ such that

$$\begin{cases} (\dot u_{\varepsilon t}(t), v) + a_\Omega(\dot u_\varepsilon(t), v) \\ \qquad\qquad + \varepsilon^{-1}(\dot u_{\varepsilon n}(t), v_n)_\Gamma = (f_t(t), v) & \forall\, v \in V^\sigma,\ t \in (0,T), \\ \dot u_\varepsilon(x,0) = \dot u_{\varepsilon 0} & \text{in } \Omega, \end{cases} \tag{2.14}$$

satisfying

$$\|\dot u_\varepsilon\|^2_{L^\infty(L^2)} + \|\dot u_\varepsilon\|^2_{L^2(H^1)} \leqslant C_{31}(f, u_{\varepsilon 0}) + C\|\varepsilon^{-1} u_{\varepsilon 0} \cdot n\|^2_{H^{1/2}(\Gamma)}. \tag{2.15}$$

Define $U_\varepsilon(t) := u_{\varepsilon 0} + \int_0^t \dot u_\varepsilon(s)\,\mathrm{d}s$. Apparently, we have $U_\varepsilon(0) = u_{\varepsilon 0}$. Integrating (2.14) with respect to t and using (2.13), we obtain

$$(U_{\varepsilon t}(t), v) + a_\Omega(U_\varepsilon(t), v) + \varepsilon^{-1}(U_{\varepsilon n}(t), v_n)_\Gamma = (f(t), v) \quad \forall\, v \in V^\sigma,\ t \in (0,T).$$

By the uniqueness of the weak solution, we conclude $u_\varepsilon = U_\varepsilon$, $u_{\varepsilon t} = U_{\varepsilon t} = \dot u_\varepsilon$ and

$$\begin{aligned}(u_{\varepsilon tt}(t), v) + a_\Omega(u_{\varepsilon t}(t), v) &+ \varepsilon^{-1}(u_{\varepsilon t}(t) \cdot n, v_n)_\Gamma \\ &= (f_t(t), v) \quad \forall\, v \in V^\sigma,\ t \in (0,T).\end{aligned} \tag{2.16}$$

Obviously, (2.10a) follows from (2.15). Substituting $v = u_{\varepsilon tt}$ into (2.16), multiplying by t, integrating with respect to t, and combining the result with (2.15), we conclude (2.10b). □

R e m a r k 2.3 (Regularity of u_ε). By a similar argument to Theorems 2.4 and 2.5 [10], we can obtain any regularity of u_ε from $t = 0$. However, we have a breakdown of the regularity of u_ε on $\partial\Omega$ at $t = 0$. In order to derive $\|u_{\varepsilon tt}\|_{L^2(L^2)} \leqslant C$ (by substituting $v = u_{\varepsilon tt}$ into (2.16), and integrating with respect to t), we need $u_{\varepsilon t}(0) \in H^1(\Omega)^N$ and $\varepsilon^{-1}\|u_{\varepsilon t}(0) \cdot n\|_{L^2(\Gamma)} \leqslant C$, which cannot be realistically assumed. Hence, we only have $\sqrt{t} u_{\varepsilon tt} \in L^2(L^2)$.

Remark 2.4 (Regularity of p_ε). Consider the stationary Stokes problem with penalty:

$$\begin{cases} -\Delta u_\varepsilon^* + \nabla p_\varepsilon^* = f^*, \quad \nabla \cdot u_\varepsilon^* = 0 \quad \text{in } \Omega, \\ u_\varepsilon^* = 0 \quad \text{on } \gamma, \quad \sigma(u_\varepsilon^*, p_\varepsilon^*)n + \varepsilon^{-1} u_{\varepsilon n}^* n = 0 \quad \text{on } \Gamma. \end{cases}$$

For sufficiently smooth γ and Γ, given $f^* \in H^m(\Omega)^N$ $(m \in \mathbb{N})$, we have the regularity (cf. [28]): $\|u_\varepsilon^*\|_{H^{m+2}} + \|p_\varepsilon^*\|_{H^{m+1}} \leqslant C\|f^*\|_{H^m}$. Then it follows from (2.10) that

(2.17a) $\|u_\varepsilon\|_{C([0,T];H^2)} + \|p_\varepsilon\|_{C([0,T];H^1)} \leqslant C_{31}(f, u_0) + C\|\varepsilon^{-1} u_{\varepsilon 0} \cdot n\|_{H^{1/2}(\Gamma)}$,
(2.17b) $\|\sqrt{t} u_{\varepsilon t}\|_{L^2(H^2)} + \|\sqrt{t} p_{\varepsilon t}\|_{L^2(H^1)} \leqslant C_{32}(f, u_{\varepsilon 0}) + C\|\varepsilon^{-1} u_{\varepsilon 0} \cdot n\|_{H^{1/2}(\Gamma)}$.

3. The error estimate of the penalty method

In the previous section, we have derived variational forms for $(\mathbf{P})$ and $(\mathbf{P}_\varepsilon)$ in (2.3) and (2.5), respectively, and have proved their well-posedness and *a priori* estimates. However, the formulations (2.3) and (2.5) are not suitable for the derivation of an error estimate, which is the aim of this section, because the test function spaces involved are different. Therefore, we need other formulations for $(\mathbf{P})$ and $(\mathbf{P}_\varepsilon)$ which (u, p) and $(u_\varepsilon, p_\varepsilon)$ satisfy. To this end, we introduce Lagrange multipliers $\lambda = -\sigma(u, p)n \cdot n$ and $\lambda_\varepsilon = \varepsilon^{-1} u_{\varepsilon n}$ on Γ to find that (u, p, λ) satisfies: for all $t \in (0, T)$,

$$(3.1) \qquad \begin{cases} (u_t(t), v) + a_\Omega(u(t), v) + b_\Omega(v, p(t)) + c(\lambda(t), v_n) = (f(t), v) & \forall\, v \in V, \\ b_\Omega(u(t), q) = 0 & \forall\, q \in Q, \\ c(u_n(t), \mu) = 0 & \forall\, \mu \in \Lambda^*, \end{cases}$$

and that $(u_\varepsilon, p_\varepsilon, \lambda_\varepsilon)$ satisfies: for all $t \in (0, T)$,

$$(3.2) \qquad \begin{cases} (u_{\varepsilon t}(t), v) + a_\Omega(u_\varepsilon(t), v) + b_\Omega(v, p_\varepsilon(t)) \\ \qquad\qquad + c(\lambda_\varepsilon(t), v_n) = (f(t), v) & \forall\, v \in V, \\ b(u_\varepsilon(t), q) = 0 & \forall\, q \in Q, \\ c(u_{\varepsilon n}(t), \mu) = \varepsilon c(\lambda_\varepsilon(t), \mu) & \forall\, \mu \in \Lambda^*. \end{cases}$$

In the following, we establish error estimates between $(\mathbf{P})$ and $(\mathbf{P}_\varepsilon)$ based on (3.1) and (3.2). Since $p_\varepsilon(t) \notin \mathring{Q}$, we divide the pressure $p_\varepsilon(t)$ into a constant function $k_\varepsilon(t)$ and a zero-mean function $\mathring{p}_\varepsilon(t)$, where

$$k_\varepsilon(t) = \frac{1}{|\Omega|} \int_\Omega p_\varepsilon(t)\, \mathrm{d}x, \quad \mathring{p}_\varepsilon(t) = p_\varepsilon(t) - k_\varepsilon(t) \in \mathring{Q}.$$

Then we define errors for the velocity, pressure and Lagrange multiplier:

$$e_u(t) := u(t) - u_\varepsilon(t), \quad e_p(t) := p(t) - \mathring{p}_\varepsilon(t), \quad e_\lambda(t) := \lambda(t) - (\lambda_\varepsilon(t) - k_\varepsilon(t)).$$

Before beginning the detailed proof, we explain the main difference of the error analysis between the stationary and non-stationary cases. In the stationary case, the estimates of $\|e_p\|_{L^2}$ and $\|e_\lambda\|_{H^{-1/2}(\Gamma)}$ follow from the H^1-norm estimate of e_u by the inf-sup conditions of $b(\cdot,\cdot)$ and $c(\cdot,\cdot)$ (see [12], [28]). However, for the non-stationary case, we need to deal with the estimates of e_{ut}, e_p and e_λ at the same time, which makes the argument of the stationary case inapplicable. In this paper, we first prove sub-optimal error estimates $O(\sqrt{\varepsilon})$ of e_u and $\lambda - \lambda_\varepsilon$. Then we improve the error estimate to the quasi-optimal $O(\varepsilon|\log\varepsilon|)$, by dividing the estimate of e_u into three cases: (i) $0 < t < \varepsilon$, (ii) $\varepsilon < t < 1$ and (iii) $t > 1$. Case (i) follows from the energy estimate of e_u and the sub-optimal error estimates. In case (ii), owing to the *a priori* estimates with weight $\sqrt{t}$ and $\varepsilon < t < 1$, we get the error bound $O(\varepsilon|\log\varepsilon|)$. Moreover, this error bound can be extended to case (iii).

3.1. The sub-optimal error estimate.

Theorem 3.1. *Assume that $\|u_0 - u_{\varepsilon 0}\|_{L^2} \leqslant C_{i1}\sqrt{\varepsilon}$, $u_0 \in V_n^\sigma$ and $f \in L^2(L^2)$. Then we have*

$$\|e_u\|_{L^\infty(L^2)} + \|e_u\|_{L^2(H^1)} + \sqrt{\varepsilon}\|\lambda - \lambda_\varepsilon\|_{L^2(L^2(\Gamma))} \leqslant C\sqrt{\varepsilon}. \tag{3.3}$$

In addition, we assume that $\|u_0 - u_{\varepsilon 0}\|_{H^1} \leqslant C_{i1}\sqrt{\varepsilon}$, $\|u_{\varepsilon 0}\cdot n\|_{L^2(\Gamma)} \leqslant C\varepsilon$, $u_0 \in V_n^\sigma \cap H^3(\Omega)^N$, $f(0) \in H^1(\Omega)^N$, and $f_t \in L^2(L^2)$. Then we have

$$\|e_{ut}\|_{L^2(L^2)} + \|e_u\|_{L^\infty(H^1)} + \sqrt{\varepsilon}\|\lambda - \lambda_\varepsilon\|_{L^\infty(L^2(\Gamma))} \leqslant C\sqrt{\varepsilon}. \tag{3.4}$$

P r o o f. In view of

$$b_\Omega(v, p_\varepsilon(t)) + c(\lambda_\varepsilon(t), v_n) = b_\Omega(v, \mathring{p}_\varepsilon(t)) + c(\lambda_\varepsilon(t) - k_\varepsilon(t), v_n),$$

subtracting $(3.2)_1$ from $(3.1)_1$ we get:

$$(e_{ut}(t), v) + a_\Omega(e_u(t), v) + b_\Omega(v, e_p(t)) + c(e_\lambda(t), v_n) = 0 \quad \forall\, v \in V. \tag{3.5}$$

Substituting $v = e_u(t)$ into (3.5), by virtue of $e_u(t)\cdot n = u_n(t) - u_{\varepsilon n}(t) = 0 - \varepsilon\lambda_\varepsilon(t)$ on Γ we calculate

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|e_u(t)\|_{L^2}^2 + a_\Omega(e_u(t), e_u(t)) + 0 + c(e_\lambda(t), -\varepsilon\lambda_\varepsilon(t)) = 0. \tag{3.6}$$

Noting that $c(k_\varepsilon(t), \varepsilon\lambda_\varepsilon(t)) = k_\varepsilon(t)(u_{\varepsilon n}(t), 1)_\Gamma = k_\varepsilon(t)(\nabla\cdot u_\varepsilon(t), 1)_\Omega = 0$, we deduce

$$c(e_\lambda(t), -\varepsilon\lambda_\varepsilon(t)) = \varepsilon\|\lambda(t)-\lambda_\varepsilon(t)\|^2_{L^2(\Gamma)} - \varepsilon c(\lambda(t)-\lambda_\varepsilon(t), \lambda(t)). \tag{3.7}$$

Then (3.6) can be rewritten as

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|e_u\|^2_{L^2} + a_\Omega(e_u, e_u) + \varepsilon\|\lambda-\lambda_\varepsilon\|^2_{L^2(\Gamma)} = \varepsilon c(\lambda-\lambda_\varepsilon, \lambda). \tag{3.8}$$

Applying the Schwarz inequality to the right-hand side of (3.8), integrating with respect to t, and using Korn's inequality (2.1), we obtain

$$\begin{aligned}\|e_u(t)\|^2_{L^2} + \int_0^t \|e_u(s)\|^2_{H^1}\,\mathrm{d}s + \varepsilon\int_0^t \|\lambda(s)-\lambda_\varepsilon(s)\|^2_{L^2(\Gamma)}\,\mathrm{d}s \\ \leqslant C\varepsilon\int_0^t \|\lambda(s)\|^2_{L^2(\Gamma)}\,\mathrm{d}s + C\|e_u(0)\|^2_{L^2}.\end{aligned} \tag{3.9}$$

By Proposition 2.2 (2), Remark 2.2 and the trace theorem, the data $u_0 \in V_n^\sigma$ and $f \in L^2(L^2)$ imply the following regularity for λ:

$$\|\lambda\|_{L^2(L^2(\Gamma))} \leqslant C\|\lambda\|_{L^2(H^{1/2}(\Gamma))} \leqslant CC_2(u_0, f).$$

Together with (3.9) and the initial error $\|u_0 - u_{\varepsilon 0}\|_{L^2} \leqslant C_{i1}\sqrt{\varepsilon}$, we conclude (3.3).

Next, substituting $v = e_{ut}(t)$ into (3.5) yields (in view of $u_n = 0$ and $u_{\varepsilon n} = \varepsilon\lambda_\varepsilon$)

$$\|e_{ut}(t)\|^2_{L^2} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}a_\Omega(e_u(t), e_u(t)) + 0 + c(e_\lambda(t), -\varepsilon\lambda_{\varepsilon t}(t)) = 0. \tag{3.10}$$

Similarly to (3.7), we see that

$$c(e_\lambda(t), -\varepsilon\lambda_{\varepsilon t}(t)) = \frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\lambda(t)-\lambda_\varepsilon(t)\|^2_{L^2(\Gamma)} - \varepsilon c(\lambda(t)-\lambda_\varepsilon(t), \lambda_t(t)). \tag{3.11}$$

Integrating (3.10) with respect to t yields

$$\begin{aligned}\int_0^t \|e_{us}(s)\|^2_{L^2}\,\mathrm{d}s + \|e_u(t)\|^2_{H^1} + \varepsilon\|\lambda(t)-\lambda_\varepsilon(t)\|^2_{L^2(\Gamma)} \\ \leqslant C\varepsilon\int_0^t \|\lambda_s(s)\|^2_{L^2(\Gamma)}\,\mathrm{d}s + \|e_u(0)\|^2_{H^1} + \varepsilon\|\lambda(0)-\lambda_\varepsilon(0)\|^2_{L^2(\Gamma)}.\end{aligned} \tag{3.12}$$

Now we estimate the right-hand side of (3.12). The second term is the initial error bounded by $C_{i1}\sqrt{\varepsilon}$. To the third term we apply the triangle inequality and estimate $\|\lambda_\varepsilon(0)\|_{L^2(\Gamma)}$ and $\|\lambda(0)\|_{L^2(\Gamma)}$ separately. By assumption $\|u_{\varepsilon 0}\cdot n\|_{L^2(\Gamma)} \leqslant C\varepsilon$, we get

$\|\lambda_\varepsilon(0)\|_{L^2(\Gamma)} \leqslant C$. For $\|\lambda(0)\|_{L^2(\Gamma)}$, we see that $\lambda(0) = \sigma(u_0, p(0))n \cdot n$, where $p(0)$ is the solution to $\Delta p(0) = \nabla \cdot f(0)$ in Ω with the boundary condition $p(0) = \mathcal{E}(u_0)n \cdot n$ on Γ and $\nabla p(0) \cdot n = 0$ on γ. As a result, $\|p(0)\|_{H^1} \leqslant C(\|u_0\|_{H^2} + \|f(0)\|_{H^1})$, and it follows from the trace theorem that $\|\lambda(0)\|_{L^2(\Gamma)} \leqslant C(\|u_0\|_{H^2} + \|f(0)\|_{H^1})$. Thus the second term is bounded by $C\varepsilon$. By Proposition 2.2 (3), Remark 2.2, and the trace theorem, we have

$$\|\lambda_t\|_{L^2(L^2(\Gamma))} \leqslant C\|\lambda_t\|_{L^2(H^{1/2}(\Gamma))} \leqslant CC_{32}(u_0, f),$$

which implies that the first term is bounded by $C\varepsilon$. Hence, the right-hand side of (3.12) is bounded by $C\varepsilon$ and we conclude (3.4). □

3.2. The quasi-optimal error estimate. Under stronger assumptions than in Theorem 3.1, we prove the quasi-optimal error estimate.

Theorem 3.2. *We make the same assumption as in Theorem* 3.1. *Moreover, we assume that* $\|u_0 - u_{\varepsilon 0}\|_{L^2} \leqslant C_{i2}\varepsilon$, $\|u_{\varepsilon 0} \cdot n\|_{H^{1/2}(\Gamma)} \leqslant C\varepsilon$, *and* $f \in C([0,T]; L^2)$. *Then we have*

$$\|e_u\|_{L^\infty(L^2)} + \|e_u\|_{L^2(H^1)} + \|\sqrt{t}e_u\|_{L^\infty(H^1)} + \|\sqrt{t}e_{ut}\|_{L^2(L^2)} \leqslant C\varepsilon|\log\varepsilon|. \tag{3.13}$$

Remark 3.1. Because of the nonlocal compatibility condition, it is unrealistic to assume $\|u_{\varepsilon t}(0)\|_{H^1(\Omega)} \leqslant C$ and thus we only get an *a priori* estimate for $u_{\varepsilon tt}$ with weight $\sqrt{t}$ (see Proposition 2.3 (3)). Moreover, the initial error $\|\lambda(0) - \varepsilon^{-1}u_{\varepsilon 0} \cdot n + k_\varepsilon(0)\|_{L^2(\Gamma)} \leqslant C\sqrt{\varepsilon}$ seems non-trivial to ensure. For the above two reasons, we obtain the error estimate for e_{ut} with weight $\sqrt{t}$, and derive the error estimate $O(\varepsilon|\log\varepsilon|)$ instead of $O(\varepsilon)$.

Proof. Instead of (3.7) and (3.11), we deduce that

$$c(e_\lambda(t), -\varepsilon\lambda_\varepsilon(t)) = \varepsilon\|e_\lambda(t)\|^2_{L^2(\Gamma)} - \varepsilon c(e_\lambda(t), \lambda(t) + k_\varepsilon(t)), \tag{3.14a}$$

$$\begin{aligned} c(e_\lambda(t), -\varepsilon\lambda_{\varepsilon t}(t)) &= \varepsilon c(e_\lambda(t), e_{\lambda t}(t)) - \varepsilon c(e_\lambda(t), \lambda_t(t) + k_{\varepsilon t}(t)) \\ &= \frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|e_\lambda(t)\|^2_{L^2(\Gamma)} - \varepsilon c(e_\lambda(t), \lambda_t(t) + k_{\varepsilon t}(t)). \end{aligned} \tag{3.14b}$$

It follows from (3.6), (3.10) and (3.14) that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|e_u\|^2_{L^2} + a_\Omega(e_u(t), e_u(t)) + \varepsilon\|e_\lambda(t)\|^2_{L^2(\Gamma)} = \varepsilon c(e_\lambda(t), \lambda + k_\varepsilon(t)), \tag{3.15a}$$

$$\|e_{ut}(t)\|^2_{L^2} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}a_\Omega(e_u, e_u) + \frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|e_\lambda\|^2_{L^2(\Gamma)} = \varepsilon c(e_\lambda(t), \lambda_t(t) + k_{\varepsilon t}(t)). \tag{3.15b}$$

For u_0, $u_{\varepsilon 0}$ and f satisfying the assumptions, we have *a priori* estimates (2.9) and (2.17). By the trace theorem, we see that

$$\lambda \in C([0,T]; H^{1/2}(\Gamma)), \quad k_\varepsilon \in C([0,T]; \mathbb{R}), \tag{3.16a}$$

$$\sqrt{t}\lambda_t \in L^2(0,T; H^{1/2}(\Gamma)), \quad \sqrt{t}k_{\varepsilon t} \in L^2(0,T; \mathbb{R}). \tag{3.16b}$$

Owing to the weight $\sqrt{t}$ of (3.16b), we divide the estimate into three cases: (i) $0 \leqslant t \leqslant \varepsilon$, (ii) $\varepsilon \leqslant t \leqslant 1$, and (iii) $t > 1$.

(i) For $0 \leqslant t \leqslant \varepsilon$, the right-hand side of (3.15a) is bounded by

$$\varepsilon c(e_\lambda(t), \lambda(t) + k_\varepsilon(t)) \leqslant \frac{\varepsilon}{2}\|e_\lambda(t)\|^2_{L^2(\Gamma)} + \frac{\varepsilon}{2}\|\lambda(t) + k_\varepsilon(t)\|^2_{L^2(\Gamma)}. \tag{3.17}$$

It follows from (3.15a), (3.17), and Korn's inequality (2.1) that

$$\begin{aligned} \|e_u(t)\|^2_{L^2} + \int_0^t \|e_u(s)\|^2_{H^1} + \varepsilon\|e_\lambda(s)\|^2_{L^2(\Gamma)}\,\mathrm{d}s \\ \leqslant C\varepsilon \int_0^t \|\lambda(s) + k_\varepsilon(s)\|^2_{L^2(\Gamma)}\,\mathrm{d}s + \|u_0 - u_{\varepsilon 0}\|^2_{L^2} \\ \leqslant C\varepsilon^2 \quad (\text{by (3.16a) and } t \leqslant \varepsilon). \end{aligned} \tag{3.18}$$

In addition, by (3.4), we have $\|e_u(t)\|_{H^1} \leqslant C\sqrt{\varepsilon}$ for all $t \in (0, \varepsilon]$, which implies

$$\|\sqrt{t}e_u(t)\|_{H^1} \leqslant C\varepsilon|\log\varepsilon| \quad \forall\, t \in (0,\varepsilon].$$

(ii) For $\varepsilon \leqslant t \leqslant 1$, we need a function w whose trace equals $\lambda + k_\varepsilon$ on $\Gamma \times [0,T]$. To this end, we consider the elliptic problem

$$\Delta\varphi(t) = \frac{1}{|\Omega|}\int_\Gamma (\lambda(t) + k_\varepsilon(t))\,\mathrm{d}\Gamma \quad \text{in } \Omega, \quad \nabla\varphi(t)\cdot n = \lambda(t) + k_\varepsilon(t) \quad \text{on } \Gamma.$$

Setting $w = \nabla\varphi$, we see that

$$w_n(t) = \lambda + k_\varepsilon, \quad w_t \cdot n = \lambda_t + k_{\varepsilon t} \quad \text{on } \Gamma. \tag{3.19}$$

By (3.16), we have $\varphi \in C([0,T]; H^2)$ and $\sqrt{t}\varphi_t \in L^2(H^2)$, which implies

$$w \in C([0,T]; H^1), \quad \sqrt{t}w_t \in L^2(0,T; H^1). \tag{3.20}$$

Substituting $v = w$ and $v = w_t$ into (3.5), together with (3.19) and (3.20), we deduce that

$$\varepsilon c(e_\lambda, \lambda + k_\varepsilon) = -\varepsilon(u_t - u_{\varepsilon t}, w) - \varepsilon a_\Omega(u - u_\varepsilon, w), \tag{3.21a}$$

$$\varepsilon c(e_\lambda, \lambda_t + k_{\varepsilon t}) = -\varepsilon(u_t - u_{\varepsilon t}, w_t) - \varepsilon a_\Omega(u - u_\varepsilon, w_t). \tag{3.21b}$$

With the help of (3.21a) and Korn's inequality (2.1), integrating (3.15a) from ε to t yields

$$\begin{aligned}(3.22)\qquad \frac{1}{2}\|e_u(t)\|_{L^2}^2 &+ \int_\varepsilon^t (C\|e_u(s)\|_{H^1}^2 + \varepsilon\|e_\lambda(s)\|_{L^2(\Gamma)}^2)\,\mathrm{d}s \\ &\leqslant \frac{1}{2}\|e_u(\varepsilon)\|_{L^2}^2 - \varepsilon\int_\varepsilon^t [(u_s - u_{\varepsilon s}, w) + a_\Omega(u-u_\varepsilon, w)]\,\mathrm{d}s \\ &\leqslant \frac{1}{2}\|e_u(\varepsilon)\|_{L^2}^2 - \varepsilon(e_u(t), w(t)) + \varepsilon(e_u(\varepsilon), w(\varepsilon)) \\ &\quad + \varepsilon\int_\varepsilon^t (e_u, w_s)\,\mathrm{d}s - \varepsilon\int_\varepsilon^t a_\Omega(e_u, w)\,\mathrm{d}s,\end{aligned}$$

where we have applied integration by parts. By (3.18), the first and third terms in the right-hand side of (3.22) are bounded by

$$(3.23)\qquad \|e_u(\varepsilon)\|_{L^2}^2 \leqslant C\varepsilon^2, \quad \varepsilon|(e_u(\varepsilon), w(\varepsilon))| \leqslant C\varepsilon^2.$$

Applying the Schwarz inequality to the second and last terms gives

$$(3.24\text{a})\qquad |\varepsilon(e_u(t), w(t))| \leqslant \frac{1}{4}\|e_u(t)\|_{L^2}^2 + \varepsilon^2\|w(t)\|_{L^2}^2,$$

$$(3.24\text{b})\qquad \varepsilon\int_\varepsilon^t a_\Omega(e_u(s), w)\,\mathrm{d}s \leqslant \frac{C}{2}\int_\varepsilon^t \|e_u(s)\|_{H^1}^2\,\mathrm{d}s + \frac{\varepsilon^2}{2C}\int_\varepsilon^t \|w(s)\|_{H^1}^2\,\mathrm{d}s.$$

It remains to estimate $\varepsilon\int_\varepsilon^t (e_u, w_s)\,\mathrm{d}s$, which is bounded by

$$\begin{aligned}(3.25)\qquad \varepsilon\int_\varepsilon^t (e_u, w_s)\,\mathrm{d}s &\leqslant \varepsilon\int_\varepsilon^t \frac{1}{\sqrt{s}}\|e_u(s)\|_{L^2}\sqrt{s}\|w_s(s)\|_{L^2}\,\mathrm{d}s \\ &\leqslant \frac{1}{C}\frac{1}{|\log\varepsilon|^2}\int_\varepsilon^t \frac{1}{s}\|e_u(s)\|_{L^2}^2\,\mathrm{d}s + C\varepsilon^2|\log\varepsilon|^2\|\sqrt{t}w_t\|_{L^2(L^2)}^2.\end{aligned}$$

Since $e_u(s) = e_u(\varepsilon) + \int_\varepsilon^s \partial_r e_u(r)\,\mathrm{d}r$ for $s \in [\varepsilon, t]$, we calculate

$$\begin{aligned}\|e_u(s)\|_{L^2} &\leqslant \|e_u(\varepsilon)\|_{L^2} + \left\|\int_\varepsilon^s \partial_r e_u(r)\,\mathrm{d}r\right\|_{L^2} \\ &\leqslant \|e_u(\varepsilon)\|_{L^2} + \int_\varepsilon^s \|\partial_r e_u(r)\|_{L^2}\,\mathrm{d}r \\ &\leqslant \|e_u(\varepsilon)\|_{L^2} + \left(\int_\varepsilon^s \frac{1}{r}\,\mathrm{d}r\right)^{1/2}\left(\int_\varepsilon^s r\|\partial_r e_u(r)\|_{L^2}^2\,\mathrm{d}r\right)^{1/2} \\ &\leqslant C\varepsilon + C\Big(\log\frac{s}{\varepsilon}\Big)^{1/2}\left(\int_\varepsilon^s r\|\partial_r e_u(r)\|_{L^2}^2\,\mathrm{d}r\right)^{1/2}.\end{aligned}$$

By $\int_\varepsilon^t s^{-1}\log \varepsilon s^{-1}\,\mathrm{d}s = \frac{1}{2}(\log ts^{-1})^2$, $\varepsilon \leqslant e^{-1}$, and $0<\varepsilon\leqslant t\leqslant 1$, we deduce

$$\begin{aligned}(3.26)\qquad \int_\varepsilon^t \frac{1}{s}\|e_u(s)\|_{L^2}^2\,\mathrm{d}s &\leqslant C\varepsilon^2|\log\varepsilon| + \int_\varepsilon^t \frac{1}{s}\log\frac{s}{\varepsilon}\,\mathrm{d}s\int_\varepsilon^t r\|\partial_r e_u(r)\|_{L^2}^2\,\mathrm{d}r \\ &\leqslant C|\log\varepsilon|^2\bigg(\varepsilon^2 + \int_\varepsilon^t s\|\partial_s e_u(s)\|_{L^2}^2\,\mathrm{d}s\bigg).\end{aligned}$$

Putting together (3.26) and (3.25), we obtain

$$(3.27)\quad \varepsilon\int_\varepsilon^t (e_u, w_s)\,\mathrm{d}s \leqslant C\bigg(\varepsilon^2 + \int_\varepsilon^t s\|\partial_s e_u(s)\|_{L^2}^2\,\mathrm{d}s\bigg) + C\varepsilon^2|\log\varepsilon|^2\|\sqrt{s}w_s\|_{L^2(L^2)}^2.$$

Combining (3.23), (3.24a), (3.24b), and (3.27) with (3.22) yields

$$\begin{aligned}(3.28)\qquad \frac{1}{2}\|e_u(t)\|_{L^2}^2 &+ \int_\varepsilon^t (C\|e_u(s)\|_{H^1}^2 + \varepsilon\|e_\lambda(s)\|_{L^2(\Gamma)}^2)\,\mathrm{d}s \\ &\leqslant C\xi\bigg(\varepsilon^2 + \int_\varepsilon^s s\|\partial_s e_u(s)\|_{L^2}^2\,\mathrm{d}s\bigg) + C\xi^{-1}\varepsilon^2|\log\varepsilon|^2.\end{aligned}$$

Multiplying (3.15a) by t and integrating from 0 to t yields (by (3.21b), (2.1), and (3.19))

$$\begin{aligned} tC\|e_u(t)\|_{H^1}^2 &+ \varepsilon t\|e_\lambda(t)\|_{L^2(\Gamma)}^2 + \int_0^t s\|e_{us}(s)\|_{L^2}^2\,\mathrm{d}s \\ &\leqslant C\int_0^t (\|e_u(s)\|_{H^1}^2 + \varepsilon\|e_\lambda(s)\|_{L^2(\Gamma)}^2)\,\mathrm{d}s \\ &\quad - \varepsilon\int_0^t [s(e_{us}(s), w_s(s)) - a_\Omega(e_u(s), w_s(s))]\,\mathrm{d}s \\ &\leqslant C\int_0^t (\|e_u\|_{H^1}^2 + \varepsilon\|e_\lambda\|_{L^2(\Gamma)}^2)\,\mathrm{d}s + C\varepsilon\bigg(\int_0^t s\|e_{us}\|_{L^2}^2\,\mathrm{d}s\bigg)^{1/2} \\ &\quad + C\varepsilon\bigg(\int_0^t s\|e_u\|_{H^1}^2\,\mathrm{d}s\bigg)^{1/2}.\end{aligned}$$

This together with (3.18), (3.28) (with sufficiently small ξ) implies

$$\begin{aligned}\|e_u(t)\|_{L^2}^2 &+ \int_\varepsilon^t (\|e_u(s)\|_{H^1}^2 + \varepsilon\|e_\lambda(s)\|_{L^2(\Gamma)}^2)\,\mathrm{d}s \\ &+ \int_0^t s\|e_{us}(s)\|_{L^2}^2\,\mathrm{d}s + t\|e_u(t)\|_{H^1}^2 + \varepsilon t\|e_\lambda(t)\|_{L^2(\Gamma)}^2 \leqslant C\varepsilon^2|\log\varepsilon|^2.\end{aligned}$$

(iii) When $t>1$, according to Remarks 2.1 and 2.3, we have the regularity $\lambda_t \in L^2(1,T;H^{1/2}(\Gamma))$ and $k_{\varepsilon t}\in L^2(1,T;\mathbb{R})$, which yields $w_t \in L^2(1,T;H^1)$. Now, we see that

$$\varepsilon\int_1^t (e_u, w_s)\,\mathrm{d}s \leqslant \varepsilon\int_1^t \|e_u(s)\|_{L^2}\|w_s(s)\|_{L^2}\,\mathrm{d}s,$$

which is much simpler than (3.25). Hence, the argument is easier than that in case (ii) and we have

$$\|e_u(t)\|_{L^2}^2 + \int_\varepsilon^t (\|e_u(s)\|_{H^1}^2 + \varepsilon\|e_\lambda(s)\|_{L^2(\Gamma)}^2)\,\mathrm{d}s \\ + \int_0^t s\|e_{us}(s)\|_{L^2}^2\,\mathrm{d}s + t\|e_u(t)\|_{H^1}^2 + \varepsilon t\|e_\lambda(t)\|_{L^2(\Gamma)}^2 \leqslant C\varepsilon^2|\log\varepsilon|^2.$$

Combining the estimates obtained for the cases (i)–(iii), we conclude (3.13). □

4. The finite element approximation

We introduce a regular triangulation $\mathcal{T}_h$ to Ω_h, where $h := \max_{K\in\mathcal{T}_h} \mathrm{diam}(K)$ denotes the mesh size. In this paper, the P1/P1-stabilization (or P1b/P1) finite element approximation is considered. We set the finite element spaces for P1/P1 (or P1b/P1) element as follows:

$$V_h = \{v_h \in C(\overline{\Omega_h})^N;\ v_h \in P_1(K)^N \quad \forall K \in \mathcal{T}_h, v_h = 0 \text{ on } \gamma_h\} \text{ for P1/P1},$$
$$V_h = \{v_h \in C(\overline{\Omega_h})^N;\ v_h \in P_1(K)^N \oplus B(K)^N \quad \forall K \in \mathcal{T}_h, v_h = 0 \text{ on } \gamma_h\} \text{ for P1b/P1},$$
$$Q_h = \{q_h \in C(\overline{\Omega_h})^N;\ q_h \in P_1(K) \quad \forall K \in \mathcal{T}_h\}, \quad \mathring{Q}_h = Q_h \cap L^2_0(\Omega_h),$$

where $P_1(K)$ is the set of linear polynomials in a triangle K and $B(K)$ stands for the bubble function space on K. We denote by $\mathcal{S}_h$ the triangulation of Γ_h inherited from $\mathcal{T}_h$. The Dirichlet boundary condition $u|_\gamma = 0$ has been approximated by $u_h|_{\gamma_h} = 0$, the error of which has been well studied in the literature. In this paper, we focus on dealing with the slip boundary condition. For simplicity, we ignore the difference between γ and γ_h (namely, we assume $\gamma = \gamma_h$) in the following argument.

We consider the backward approximation for time differentiation. For an integer $M \in \mathbb{N}_+$ ($M \gg 1$), we denote by $\tau := T/M$ the time-step size. For $t_j = j\tau$ with $j = 0, 1, \ldots, M$, we set $(u^j, p^j) := (u(t_j), p(t_j))$, and use $\partial_\tau u^j := (u^j - u^{j-1})/\tau$ to denote the backward approximation. Given the initial value $u_{0h} \in V_h$, the finite element approximation problem reads

$$\text{(4.1)}\quad (\mathbf{P}_{\varepsilon,h})\quad \begin{cases} \text{find } (u_h^j, p_h^j) \in V_h \times Q_h,\ j = 1, \ldots, M, \text{ such that} \\ (\partial_\tau u_h^j, v_h)_{\Omega_h} + a_{\Omega_h}(u_h^j, v_h) + b_{\Omega_h}(v_h, p_h^j) \\ \qquad + \varepsilon^{-1} c_h(u_h^j \cdot n_h, v_h \cdot n_h) = (\tilde{f}^j, v_h)_{\Omega_h} & \forall v_h \in V_h, \\ b_{\Omega_h}(u_h^j, q_h) = \eta h^2 (\nabla p_h^j, \nabla q_h)_{\Omega_h} & \forall q_h \in Q_h, \end{cases}$$

where $\tilde{f}$ is a continuous extension of f to Ω_h (note that $\Omega \neq \Omega_h$) and η is a pressure stabilization parameter, which is set to be 0 for the P1b/P1 element and to be 1 for the P1/P1 element. We assume $f \in C([0,T]; L^2)$ so that $\tau \sum_{j=1}^{M} \|\tilde{f}^j\|^2_{L^2(\Omega_h)} \leqslant C$. The bilinear form $c_h(\cdot,\cdot)$ is defined below.

We consider two types of $c_h(\cdot,\cdot)$ to approximate $c(\cdot,\cdot)$: for any $\lambda_h, \mu_h \in \Lambda_h = \{v_h \cdot n_h \text{ on } \Gamma_h\,;\ v_h \in V_h\}$,

$$c_h(\lambda_h, \mu_h) = \begin{cases} c_h^N(\lambda_h, \mu_h) := (\lambda_h, \mu_h)_{\Gamma_h} & \text{(non-reduced integration)}, \\ c_h^R(\lambda_h, \mu_h) := \sum\limits_{S \in \mathcal{S}_h} |S| \lambda_h(m_S) \mu_h(m_S) & \text{(reduced integration)}, \end{cases}$$

where m_S denotes the barycentre of a boundary element S. We set $\|\mu_h\|^2_{c_h} := c_h(\mu_h, \mu_h)$. Note that $c_h^R(\cdot,\cdot)$ is the barycentre formula approximation to $c_h^N(\cdot,\cdot)$.

For the bilinear forms $a_{\Omega_h}(\cdot,\cdot)$ and $b_{\Omega_h}(\cdot,\cdot)$, the following inequalities hold:

▷ Korn's inequality (cf. [3], [13]): there exists a constant C such that

$$a_{\Omega_h}(v_h, v_h) \geqslant C \|v_h\|^2_{H^1(\Omega_h)} \quad \forall\, v_h \in V_h. \tag{4.2}$$

▷ Inf-sup condition (cf. [8], [20]): there exists a constant C such that

$$\sup_{v_h \in \mathring{V}_h} \frac{b_{\Omega_h}(v_h, q_h)}{\|v_h\|_{V_h}} + C\eta h \|\nabla q_h\|_{L^2(\Omega_h)} \geqslant C \|q_h\|_{L^2(\Omega_h)} \quad \forall\, q_h \in \mathring{Q}_h, \tag{4.3}$$

where $\mathring{V}_h := \{v_h \in V_h\,;\ v_h = 0 \text{ on } \Gamma_h\}$.

Proposition 4.1. *There exists a unique solution* $\{(u_h^m, p_h^m)\}_{m=1}^M \subset V_h \times Q_h$ *to* $(\mathbf{P}_{\varepsilon,h})$ *satisfying*

$$\begin{aligned} \|u_h^m\|^2_{L^2(\Omega_h)} &+ 2\tau \sum_{j=1}^{m} [\|u_h^j - u_h^{j-1}\|^2_{L^2(\Omega_h)} + \|u_h^j\|^2_{H^1(\Omega_h)} + \eta h^2 \|\nabla p_h^j\|^2_{L^2(\Omega_h)}] \\ &+ \varepsilon^{-1} 2\tau \sum_{j=1}^{m} \|u_h^j \cdot n_h\|^2_{c_h} \leqslant C \|u_h^0\|^2_{L^2(\Omega_h)} + C\tau \sum_{j=1}^{m} \|\tilde{f}^j\|^2_{L^2(\Omega_h)}. \end{aligned} \tag{4.4}$$

Assume that u_h^0 *satisfies* $\varepsilon^{-1}\|u_h^0 \cdot n_h\|^2_{c_h} \leqslant C$. *Moreover, for the* P1/P1 *element, we assume there exists a* $p_h^0 \in Q_h$ *such that* $b_{\Omega_h}(u_h^0, q_h) = \eta h^2 (\nabla p_h^0, \nabla q_h)_{\Omega_h}$ *for all* $q_h \in Q_h$. *For the* P1b/P1 *element, we assume* $b_{\Omega_h}(u_h^0, q_h) = 0$ *for all* $q_h \in Q_h$. *Then* *we have*

$$(4.5)\qquad \tau\sum_{j=1}^{m}\|\partial_\tau u_h^j\|^2_{L^2(\Omega_h)}+\|u_h^m\|_{H^1(\Omega_h)}+\varepsilon^{-1}\|u_h^m\cdot n_h\|^2_{c_h}+\eta h^2\|\nabla p_h^m\|^2_{L^2(\Omega_h)}$$
$$+\sum_{j=1}^{m}[\eta h^2\|\nabla(p_h^j-p_h^{j-1})\|^2_{L^2(\Omega_h)}+\varepsilon^{-1}\|(u_h^j-u_h^{j-1})\cdot n_h\|^2_{c_h}$$
$$+\|u_h^j-u_h^{j-1}\|_{H^1(\Omega_h)}]$$
$$\leqslant C\Big(\tau\sum_{j=1}^{m}\|\tilde f^j\|^2_{L^2(\Omega_h)}+\|u_h^0\|^2_{H^1(\Omega_h)}+\varepsilon^{-1}\|u_h^0\cdot n_h\|^2_{c_h}+\eta h^2\|\nabla p_h^0\|^2_{L^2(\Omega_h)}\Big).$$

Proof. Since $(\mathbf{P}_{\varepsilon,h})$ is a finite dimensional linear problem, it is sufficient to show that $u_h^0=0$ and $\tilde f^m=0$ for all m implies $(u_h^m,p_h^m)=(0,0)$. For $m=1$, $(\mathbf{P}_{\varepsilon,h})$ is equivalent to: for all $(v_h,q_h)\in V_h\times Q_h$,

$$(4.6)\qquad \frac{1}{\tau}(u_h^1,v_h)_{\Omega_h}+a_{\Omega_h}(u_h^1,v_h)+b_{\Omega_h}(v_h,p_h^1)-b_{\Omega_h}(u_h^1,q_h)$$
$$+\eta h^2(\nabla p_h^1,\nabla q_h)_{\Omega_h}+\varepsilon^{-1}c_h(u_h^1\cdot n_h,v_h\cdot n_h)=0.$$

We prove that (4.6) implies $(u_h^1,p_h^1)=(0,0)$. In fact, substituting $(v_h,q_h)=(u_h^1,q_h^1)$ into (4.6) yields (by Korn's inequality (4.2))

$$\frac{1}{\tau}\|u_h^1\|^2_{L^2(\Omega_h)}+C\|u_h^1\|^2_{H^1(\Omega_h)}+\eta h^2\|\nabla p_h^1\|^2_{L^2(\Omega_h)}+\varepsilon^{-1}\|u_h^1\cdot n_h\|^2_{c_h}\leqslant 0,$$

which implies $u_h^1=0$ and $\eta\nabla p_h^1=0$. It remains to prove $p_h=0$.

Case 1. For the P1/P1 element ($\eta=1$), $\nabla p_h^1=0$ means p_h^1 is a constant function, i.e., $p_h^1\equiv C$. Since $u_h^1=0$ and $\eta\nabla p_h^1=0$, we see that p_h^1 satisfies

$$0=b_{\Omega_h}(v_h,p_h^1)=C\int_{\Gamma_h}v_h\cdot n_h\,\mathrm{d}\Gamma_h\quad\forall\, v_h\in V_h,$$

which yields $C=0$. Therefore $(u_h^1,p_h^1)=(0,0)$.

Case 2. For the P1b/P1 element ($\eta=0$), it follows from $u_h^1=0$ that $0=b_{\Omega_h}(v_h,p_h^1)$ for all $v_h\in V_h$. By the inf-sup condition (4.3), we get $\|p_h^1\|_{L^2(\Omega_h)/\mathbb{R}}=0$, which means $p_h^1\equiv C$. Then, by an argument similar to Case 1, we have $C=0$. Thus, $(u_h^1,p_h^1)=0$.

We have proved $(u_h^1,p_h^1)=(0,0)$. By induction, it is not difficult to verify that $(u_h^m,p_h^m)=0$ for any m. Hence, we conclude the unique existence of the solution to $(\mathbf{P}_{\varepsilon,h})$.

Next, we show the *a priori* estimates (4.4) and (4.5). In view of

$$\left(\frac{a-b}{\tau}, a\right)_\omega = \frac{1}{2\tau}[(a,a)_\omega + (a-b,a-b)_\omega - (b,b)_\omega], \tag{4.7}$$

substituting $(v_h, q_h) = (u_h^j, p_h^j)$ into $(\mathbf{P}_{\varepsilon,h})$ and summing up with respect to j implies (4.4). Substituting $(v_h, q_h) = (\partial_\tau u_h^j, \partial_\tau p_h^j)$ into $(\mathbf{P}_{\varepsilon,h})$ and summing up with respect to j yields (with help of (4.7)):

$$\begin{aligned}
2\tau\sum_{j=1}^{m} &\|\partial_\tau u_h^j\|_{L^2(\Omega_h)}^2 + a_{\Omega_h}(u_h^m, u_h^m) \\
&+ \sum_{j=1}^{m} a_{\Omega_h}(u_h^j - u_h^{j-1}, u_h^j - u_h^{j-1}) + \eta h^2\|\nabla p_h^m\|_{L^2(\Omega_h)}^2 \\
&+ \eta h^2 \sum_{j=1}^{m} \|\nabla(p_h^j - p_h^{j-1})\|_{L^2(\Omega_h)}^2 + \varepsilon^{-1}\|u_h^m\cdot n_h\|_{c_h}^2 \\
&+ \varepsilon^{-1}\sum_{j=1}^{m}\|(u_h^j - u_h^{j-1})\cdot n_h\|_{c_h}^2 \\
&= 2\tau\sum_{j=1}^{m}(\tilde{f}^j, \partial_\tau u_h^j)_{\Omega_h} + a_{\Omega_h}(u_h^0, u_h^0) + \eta h^2\|\nabla p_h^0\|_{L^2(\Omega_h)}^2 + \varepsilon^{-1}\|u_h^0\cdot n_h\|_{c_h}^2.
\end{aligned}$$

Combining this with $(\tilde{f}^j, \partial_\tau u_h^j)_{\Omega_h} \leqslant \frac{1}{2}\|\partial_\tau u_h^j\|_{H^1(\Omega_h)} + \frac{1}{2}\|\tilde{f}^j\|_{L^2(\Omega_h)}^2$ and Korn's inequality (4.2), we obtain (4.5). □

Now we turn our attention to the error analysis of discretization. First, we introduce a projection lemma, which directly follows from [12], [28] for the stationary case.

Lemma 4.1 (Theorems 4.1 and 5.1 of [12])**.** *Let $(\tilde{u}^m, \tilde{p}^m)$ be a continuous extension of (u^m, p^m) to $\widetilde{\Omega} := \Omega \cup \Omega_h$ with $\tilde{f}^m = \tilde{u}_t^m - \nu\Delta\tilde{u}^m + \nabla\tilde{p}^m$ for $m = 1, \ldots, M$. There exists a unique $(P^u\tilde{u}^m, P^p\tilde{p}^m) \in V_h \times Q_h$ such that*

$$\begin{aligned}
a_{\Omega_h}(P^u\tilde{u}^m, v_h) + b_{\Omega_h}(v_h, P^p\tilde{p}^m) + \varepsilon^{-1}c_h(P^u\tilde{u}^m\cdot n_h, v_h\cdot n_h)& \\
= (\tilde{f}^m - \tilde{u}_t^m, v_h) \quad \forall\, v_h \in V_h&, \\
b_{\Omega_h}(P^u\tilde{u}^m, q_h) = \eta h^2(\nabla P^p\tilde{p}^m, \nabla q_h)_{\Omega_h} \quad \forall\, q_h \in Q_h&.
\end{aligned}$$

Moreover, the following error estimates hold:

▷ *For the non-reduced integration $c_h(\cdot,\cdot) = c_h^N(\cdot,\cdot)$,*

$$\|P^u\tilde{u}^m - \tilde{u}^m\|_{V_h} + \|P^p\tilde{p}^m - \tilde{p}^m\|_{Q_h/\mathbb{R}} + \eta h\|\nabla P^p\tilde{p}^m\|_{L^2(\Omega_h)} \leqslant C(h + \sqrt{\varepsilon} + h/\sqrt{\varepsilon}).$$

▷ *For the reduced integration* $c_h(\cdot,\cdot) = c_h^R(\cdot,\cdot)$,

$$\|P^u\tilde{u}^m - \tilde{u}^m\|_{V_h} + \|P^p\tilde{p}^m - \tilde{p}^m\|_{Q_h/\mathbb{R}} + \eta h\|\nabla P^p\tilde{p}^m\|_{L^2(\Omega_h)} \leqslant C(h + \sqrt{\varepsilon} + h^\beta/\sqrt{\varepsilon}),$$

where $\beta = 2$ *if* $N = 2$ *and* $\beta = 1$ *if* $N = 3$.

We make the following assumptions on (u,p) and the initial error $\|\tilde{u}_0 - u_h^0\|_{L^2(\Omega_h)}$:

($\mathbf{A_e1}$) $u \in C^2([0,T];L^2) \cap C^1([0,T];W^{2,r})$, where $r = \infty$ if $c_h(\cdot,\cdot) = c_h^R(\cdot,\cdot)$ with $N = 2$, otherwise $r = 2$.

($\mathbf{A_e2}$) $\|\tilde{u}_0 - u_h^0\|_{L^2(\Omega_h)} \leqslant Ch$. For the P1b/P1-element, $b_{\Omega_h}(u_h^0, q_h) = 0$ for all $q_h \in Q_h$.

Remark 4.1 (Regularity assumption for FEM). As stated in Remark 2.1, the assumption A_e1) requires nonlocal compatibility conditions for $f(0)$ and u_0. However, ($\mathbf{A_e1}$) can be satisfied in a time interval (t_a, T) for some $t_a > 0$ with smooth f and u_0. Analogously to [23], we assume ($\mathbf{A_e1}$) and deduce the error estimate for finite element discretization.

Defining the discretization errors of velocity and pressure by

$$e_{h,u}^m := u_h^m - \tilde{u}^m, \quad e_{h,p}^m := p_h^m - \tilde{p}^m,$$

where $(\tilde{u}^m, \tilde{p}^m)$ is stated in Lemma 4.1, we state the results of error estimate.

Theorem 4.1. *Under the assumptions* ($\mathbf{A_e1}$) *and* ($\mathbf{A_e2}$), *for* $1 \leqslant m \leqslant M$ *we have*

$$\|e_{h,u}^m\|_{L^2(\Omega_h)}^2 + \tau\sum_{j=1}^m \|e_{h,u}^j\|_{V_h}^2 \leqslant C(\tau + h + \sqrt{\varepsilon} + h^\beta/\sqrt{\varepsilon})^2, \tag{4.8a}$$

$$\begin{aligned}\tau\sum_{j=1}^m t_{j-1}\|\partial_\tau e_{h,u}^j\|_{L^2(\Omega_h)}^2 + t_{m-1}\|e_{h,u}^m\|_{V_h}^2 + \tau\sum_{j=1}^m t_{j-1}\|\partial_\tau e_{h,p}^j\|_{Q_h/\mathbb{R}}^2 \\ \leqslant C(\tau + h + \sqrt{\varepsilon} + h^\beta/\sqrt{\varepsilon})^2,\end{aligned} \tag{4.8b}$$

where $\beta = 1$ *for* $c_h(\cdot,\cdot) = c_h^N(\cdot,\cdot)$ *with* $N = 2, 3$, *and* $c_h(\cdot,\cdot) = c_h^R(\cdot,\cdot)$ *with* $N = 3$. *It can be improved to* $\beta = 2$ *when* $c_h(\cdot,\cdot) = c_h^R(\cdot,\cdot)$ *and* $N = 2$.

Proof. With the decomposition $e_{h,u}^j = u_h^j - P^u\tilde{u}^j + P^u\tilde{u}^j - \tilde{u}^j$ and $e_{h,p}^j = p_h^j - P^p\tilde{p}^j + P^p\tilde{p}^j - \tilde{p}^j$, and by virtue of Lemma 4.1, we only need to estimate $E_{h,u}^j := u_h^j - P^u\tilde{u}^j$ and $E_{h,p}^j := p_h^j - P^p\tilde{p}^j$.

Obviously, $\{(E^j_{h,u}, E^j_{h,p})\}_{j=1}^m$ satisfies: for all $(v_h, q_h) \in V_h \times Q_h$,

$$\begin{aligned}
&\text{(4.9a)} \quad (\partial_\tau E^j_{h,u}, v_h)_{\Omega_h} + a_{\Omega_h}(E^j_{h,u}, v_h) + b_{\Omega_h}(v_h, E^j_{h,p}) \\
&\qquad\qquad + \varepsilon^{-1} c_h(E^j_{h,u}\cdot n_h, v_h \cdot n_h) = (\tilde{u}^j_t - \partial_\tau P^u \tilde{u}^j, v_h)_{\Omega_h}, \\
&\text{(4.9b)} \quad b_{\Omega_h}(E^j_{h,p}, q_h) = \eta h^2 (\nabla E^j_{h,p}, \nabla q_h)_{\Omega_h}.
\end{aligned}$$

Substituting $v_h = E^j_{h,u}$ into (4.9) and summing up with respect to j, with help of (4.7) and Korn's inequality (4.2), we calculate:

$$\begin{aligned}
\|E^m_{h,u}\|^2_{L^2(\Omega_h)} &+ \sum_{j=1}^m \|E^j_{h,u} - E^{j-1}_{h,u}\|^2_{L^2(\Omega_h)} + 2\tau C \sum_{j=1}^m \|E^j_{h,u}\|^2_{H^1(\Omega_h)} \\
&+ 2\tau\eta h^2 \sum_{j=1}^m \|\nabla E^j_{h,p}\|^2_{L^2(\Omega)} + 2\tau\varepsilon^{-1} \sum_{j=1}^m \|E^j_{h,u}\cdot n_h\|^2_{c_h} \\
&\leqslant \|E^0_{h,u}\|^2_{L^2(\Omega_h)} + 2\tau \sum_{j=1}^m (\tilde{u}^j_t - \partial_\tau P^u \tilde{u}^j, E^j_{h,u})_{\Omega_h}.
\end{aligned} \tag{4.10}$$

The estimate of $\|E^0_{h,u}\|^2_{L^2(\Omega_h)}$ follows from ($\mathbf{A_e 2}$) and Lemma 4.1:

$$\|E^0_{h,u}\|^2_{L^2(\Omega_h)} \leqslant \|\tilde{u}_0 - u^0_h\|_{L^2(\Omega_h)} + \|\tilde{u}_0 - P^u \tilde{u}_0\|_{L^2(\Omega_h)} \leqslant Ch + C(h + \sqrt{\varepsilon} + h^\beta/\sqrt{\varepsilon}).$$

We divide $\tilde{u}^j_t - \partial_\tau P^u \tilde{u}^j$ into two parts:

$$\tilde{u}^j_t - \partial_\tau P^u \tilde{u}^j = (\tilde{u}^j_t - \partial_\tau \tilde{u}^j) + (\partial_\tau \tilde{u}^j - P^u \partial_\tau \tilde{u}^j) =: I^j_1 + I^j_2. \tag{4.11}$$

In view of $I^j_1 = \frac{1}{\tau}\int_{t_{j-1}}^{t_j} (t - t_{j-1}) \tilde{u}_{tt}(t)\,\mathrm{d}t$, we deduce that

$$\|I^j_1\|_{L^2(\Omega_h)} \leqslant C\tau \|\tilde{u}\|_{C^2([t_{j-1},t_j];L^2)}. \tag{4.12}$$

Lemma 4.1 yields the estimate of I^j_2:

$$\|I^j_2\|_{L^2(\Omega_h)} \leqslant C\|\partial_\tau \tilde{u}^j\|_{W^{2,r}}(h + \sqrt{\varepsilon} + h^\beta/\sqrt{\varepsilon}) \leqslant C(h + \sqrt{\varepsilon} + h^\beta/\sqrt{\varepsilon} + \tau).$$

Then, applying the Schwarz inequality to the last term of (4.10), and using the estimate of $\|I^j_1\|_{L^2(\Omega_h)}$ and $\|I^j_2\|_{L^2(\Omega_h)}$, we obtain the error estimate for $E^m_{h,u}$:

$$\begin{aligned}
\|E^m_{h,u}\|^2_{L^2(\Omega_h)} &+ 2\tau C \sum_{j=1}^m \|E^j_{h,u}\|^2_{H^1(\Omega_h)} + 2\tau\eta h^2 \sum_{j=1}^m \|\nabla E^j_{h,p}\|^2_{L^2(\Omega)} \\
&+ 2\tau\varepsilon^{-1} \sum_{j=1}^m \|E^m_{h,u}\cdot n_h\|^2_{c_h} \leqslant C(\tau + h + \sqrt{\varepsilon} + h^\beta/\sqrt{\varepsilon})^2.
\end{aligned} \tag{4.13}$$

Together with Lemma 4.1, we conclude (4.8a).

To prove (4.8b), substituting $v_h = \partial_\tau E^j_{h,u}$ into (4.9) and multiplying (4.9) by t_{j-1}, we have

$$\begin{aligned}
&t_{j-1}\|\partial_\tau E^j_{h,u}\|^2_{L^2(\Omega_h)} \\
&\quad + \frac{t_{j-1}}{2\tau}[\mathbf{D}(a_{\Omega_h}(E^j_{h,u}, E^j_{h,u})) + \eta h^2 \mathbf{D}(\|\nabla E^j_{h,p}\|^2_{L^2(\Omega_h)}) + \varepsilon^{-1}\mathbf{D}(\|E^j_{h,u}\cdot n_h\|^2_{c_h})] \\
&= t_{j-1}(\tilde{u}^j_t - \partial_\tau P^u \tilde{u}^j, \partial_\tau E^j_{h,u})_{\Omega_h},
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{D}(a_{\Omega_h}(E^j_{h,u}, E^j_{h,u})) &:= a_{\Omega_h}(E^j_{h,u}, E^j_{h,u}) + a_{\Omega_h}(E^j_{h,u} - E^{j-1}_{h,u}, E^j_{h,u} - E^{j-1}_{h,u}) \\
&\quad - a_{\Omega_h}(E^{j-1}_{h,u}, E^{j-1}_{h,u}), \\
\mathbf{D}(\|E^j\|^2) &:= \|E^j\|^2 + \|E^j - E^{j-1}\|^2 - \|E^{j-1}\|^2.
\end{aligned}$$

Summing up the above equality with respect to j gives (note that $t_0 = 0$)

$$\begin{aligned}
2\tau\sum_{j=1}^m & t_{j-1}\|\partial_\tau E^j_{h,u}\|^2_{L^2(\Omega_h)} + t_{m-1}\|\mathcal{E}(E^m_{h,u})\|^2_{L^2(\Omega_h)} \\
&+ \sum_{j=1}^m t_{j-1}\|\mathcal{E}(E^j_{h,u} - E^{j-1}_{h,u})\|^2_{L^2(\Omega_h)} \\
&+ \eta h^2 t_{m-1}\|\nabla E^m_{h,p}\|_{L^2(\Omega)} + 2\tau\eta h^2 \sum_{j=1}^m t_{j-1}\|\nabla(E^j_{h,p} - E^{j-1}_{h,p})\|^2_{L^2(\Omega)} \\
&+ \varepsilon^{-1} t_{m-1}\|E^m_{h,u}\cdot n_h\|^2_{c_h} + \varepsilon^{-1}\sum_{j=1}^m t_{j-1}\|(E^j_{h,u} - E^{j-1}_{h,p})\cdot n_h\|^2_{c_h} \\
\leqslant \tau & \sum_{j=1}^{m-1} a_{\Omega_h}(E^j_{h,u}, E^j_{h,u}) + \eta h^2\tau \sum_{j=1}^{m-1}\|\nabla E^j_{h,p}\|_{L^2(\Omega)} + \varepsilon^{-1}\tau\sum_{j=1}^{m-1}\|E^j_{h,u}\cdot n_h\|^2_{c_h} \\
&+ 2\tau\sum_{j=1}^m t_{j-1}(\tilde{u}^j_t - \partial_\tau P^u\tilde{u}^j, \partial_\tau E^j_{h,u})_{\Omega_h}.
\end{aligned}$$

Noting that $C\|E^j_{h,u}\|^2_{H^1(\Omega_h)} \leqslant \|E^j_{h,u}\|^2_{L^2(\Omega_h)} \leqslant C_1\|E^j_{h,u}\|^2_{H^1(\Omega_h)}$ and applying the Schwarz inequality to the last term, we obtain (using (4.11)–(4.13))

$$\begin{aligned}
(4.14)\qquad 2\tau\sum_{j=1}^m & t_{j-1}\|\partial_\tau E^j_{h,u}\|^2_{L^2(\Omega_h)} \\
&+ t_{m-1}[\|E^m_{h,u}\|^2_{H^1(\Omega_h)} + \eta h^2\|\nabla E^m_{h,p}\|^2_{L^2(\Omega)} + \varepsilon^{-1}\|E^m_{h,u}\cdot n_h\|^2_{c_h}] \\
&\leqslant CT(\tau + h + \sqrt{\varepsilon} + h^\beta/\sqrt{\varepsilon})^2.
\end{aligned}$$

By inf-sup condition (4.3) and (4.9a), we derive the error estimate of pressure (note that $v_h = 0$ on Γ for $v_h \in \mathring{V}_h$):

$$
\begin{aligned}
&\|E^m_{h,p}\|_{L^2(\Omega_h)/\mathbb{R}} \\
&\quad \leqslant \sup_{v_h \in \mathring{V}_h} \big((\tilde{u}^m_t - \partial_\tau P^u \tilde{u}^m, v_h)_{\Omega_h} - (\partial_\tau E^m_{h,u}, v_h)_{\Omega_h} - a_{\Omega_h}(E^m_{h,u}, v_h)\big)/\|v_h\|_{H^1(\Omega_h)} \\
&\qquad + \eta Ch\|\nabla E^m_{h,p}\|_{L^2(\Omega_h)} \\
&\quad \leqslant C(\|\partial_\tau E^m_{h,u}\|_{L^2(\Omega)} + \|E^m_{h,u}\|_{H^1(\Omega)} + \|\tilde{u}^m_t - \partial_\tau P^u \tilde{u}^m\|_{L^2(\Omega)}) + \eta Ch\|\nabla E^m_{h,p}\|_{L^2(\Omega_h)}.
\end{aligned}
$$

Then, applying (4.13) and (4.14) to the right-hand side, we find that

$$
\tau \sum_{j=1}^{m} t_{j-1} \|E^j_{h,p}\|^2_{L^2(\Omega_h)/\mathbb{R}} \leqslant CT(\tau + h + \sqrt{\varepsilon} + h^\beta/\sqrt{\varepsilon})^2.
$$

Together with (4.14) and Lemma 4.1, we conclude (4.8b). □

R e m a r k 4.2. The error estimates (4.8a) and (4.8b) indicate the optimal choice of ε and h, which is stated as follows

- ▷ For the non-reduced integration ($c_h(\cdot,\cdot) = c^N_h(\cdot,\cdot)$), we choose $\varepsilon = Ch$ and have the error $O(\sqrt{h}+\tau)$.
- ▷ For the reduced integration ($c_h(\cdot,\cdot) = c^R_h(\cdot,\cdot)$), when $N = 3$ we choose $\varepsilon = Ch$ and obtain the error $O(\sqrt{h}+\tau)$. When $N = 2$, setting $\varepsilon = Ch^2$ the error is upgraded to $O(h+\tau)$.

5. The numerical experiment

We consider (**P**) in an annular domain $\Omega = \{(x,y);\ 1 \leqslant x^2 + y^2 < 4\}$ with boundaries $\Gamma = \{(x,y);\ x^2 + y^2 = 4\}$ and $\gamma = \{(x,y);\ x^2 + y^2 = 1\}$. Here, f and u_0 are chosen so that the exact solution is given by

$$
u(x,y,t) = ((t^2+1)y(x^2+y^2-1), -(t^2+1)x(x^2+y^2-1)), \quad p(x,y,t) = (t^2+1)xy.
$$

We easily see that $n = \frac{1}{2}(x,y)^T$ and $u_n = 0$ on Γ. Since $g := (I - n \otimes n)\sigma(u,p)n \neq 0$ on Γ, we need to add $\int_\Gamma g \cdot v_T \,\mathrm{d}\Gamma$ to the right-hand sides of $(3.1)_1$ and $(3.2)_1$. Correspondingly, we add $\int_{\Gamma_h} \tilde{g}^m \cdot v_{hT} \,\mathrm{d}\Gamma_h$ to the right-hand side of $(4.1)_1$, where $\tilde{g}^m := (I - n_h \otimes n_h)\sigma(u(t_m), p(t_m))n_h$ is an approximation of $g(t_m)$.

We solve (**P**) by the penalty method with finite element approximation, and test both the non-reduced ($c^N(\cdot,\cdot)$) and reduced ($c^R(\cdot,\cdot)$) integration schemes for the penalty term. In the following, we show the errors of numerical solutions for the case of the P1/P1 element. The numerical results of the P1b/P1 element are not shown, because they are almost identical with those of the P1/P1 element.

First, fixing h and τ, we plot the errors of the non-reduced and reduced schemes in Figure 1, where N and R stand for the non-reduced and reduced scheme, respectively. From this, we can observe that the orders of the convergence of both the schemes are almost $O(\varepsilon)$, which verifies our theoretical results (see Theorem 3.2). Note that the error saturates as ε decreases because h and τ are fixed. Moreover, we observe that the non-reduced integration scheme fails to converge for $\varepsilon \ll h$, which does not occur for the reduced integration scheme. It suggests that the reduced scheme is more stable for small ε than the non-reduced one.

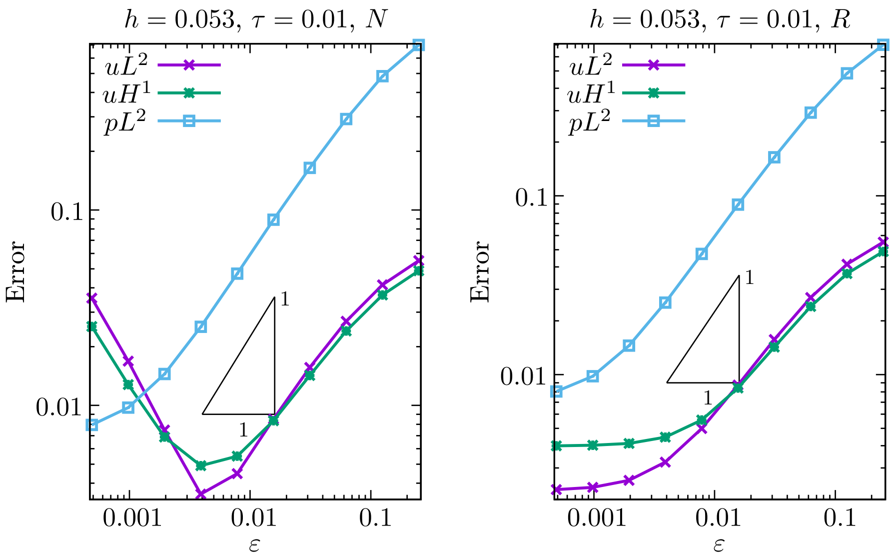

Figure 1. The errors of velocity in the L^2 and H^1 norms and pressure in the L^2 norm (denoted by uL^2, uH^1 and pL^2, respectively) are plotted for different ε with h and τ fixed. The slopes of the triangles represent the order $O(\varepsilon)$.

Next, we plot the errors depending on h in Figure 2. According to Theorem 4.1 and Remark 4.2, the optimal choice is to let $\varepsilon = Ch$ for the reduced scheme ($N = 3$) and the non-reduced scheme ($N = 2, 3$) and $\varepsilon = Ch^2$ for the reduced scheme ($N = 2$). We observe that the convergence orders of the non-reduced scheme are $O(h)$, which is better than our theoretical result $O(\sqrt{h})$ (see Remark 4.2). For the reduced scheme, we see that the convergence order of the velocity in the H^1 norm is $O(h)$, which corresponds to our theoretical result (see Remark 4.2). Moreover, the numerical experiment shows the convergence order of the velocity in the L^2 norm is $O(h^2)$. It is noted that the L^2 error of the velocity saturates as h decreases in the graph on the right of Figure 2, because we have fixed $\tau = 0.01$.

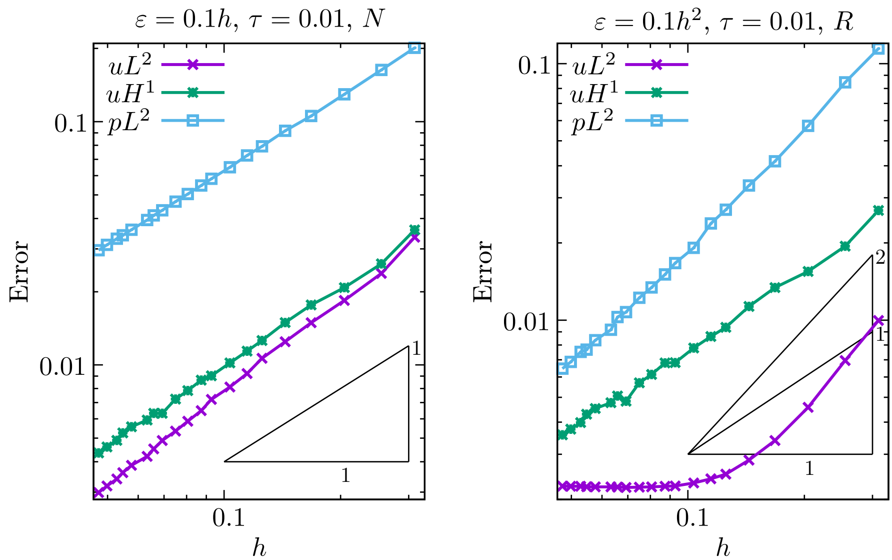

Figure 2. The relative errors are plotted for different h. We set $\varepsilon = 0.1h$ for the non-reduced scheme and $\varepsilon = 0.1h^2$ for the reduced scheme and fix $\tau = 0.01$. The slope in the left figure represents the order $O(h)$. The lower slope in the right figure represents the order $O(h)$, the higher one represents $O(h^2)$.

Finally, we verify the errors depending on τ. Theorem 4.1 shows that for fixed ε and h, the convergence orders are estimated to be $O(\tau)$, which is confirmed by our numerical examples, see Figure 3.

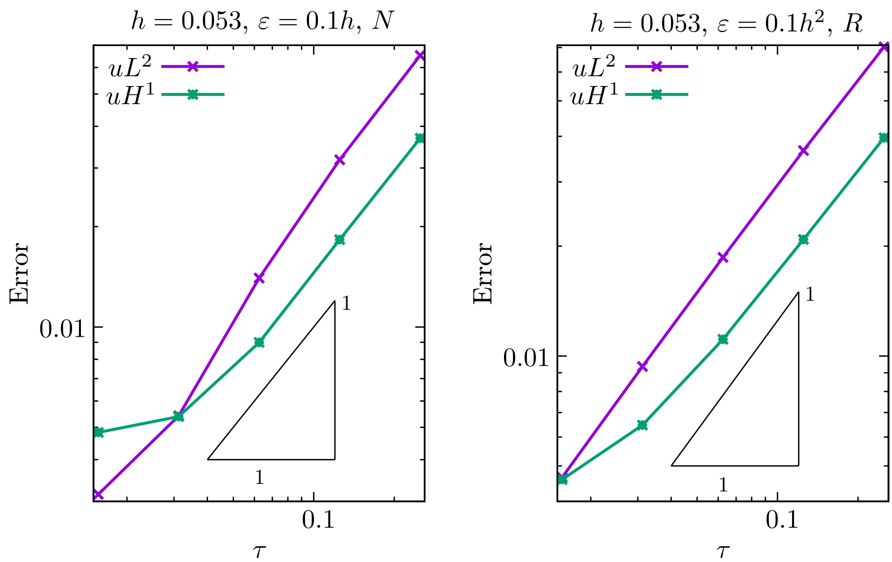

Figure 3. The errors are plotted for different τ with h and ε fixed. The slopes represent the order $O(\tau)$.

References

[1] *E. Bänsch, K. Deckelnick*: Optimal error estimates for the Stokes and Navier-Stokes equations with slip-boundary condition. M2AN, Math. Model. Numer. Anal. *33* (1999), 923–938. zbl MR doi

[2] *E. Bänsch, B. Höhn*: Numerical treatment of the Navier-Stokes equations with slip boundary condition. SIAM J. Sci. Comput. *21* (2000), 2144–2162. zbl MR doi

[3] *S. C. Brenner, L. R. Scott*: The Mathematical Theory of Finite Element Methods. Texts in Applied Mathematics 15, Springer, New York, 2002. zbl MR doi

[4] *A. Çağlar*: Weak imposition of boundary conditions for the Navier-Stokes equations. Appl. Math. Comput. *149* (2004), 119–145. zbl MR doi

[5] *I. Dione, C. Tibirna, J. Urquiza*: Stokes equations with penalised slip boundary conditions. Int. J. Comput. Fluid Dyn. *27* (2013), 283–296. MR doi

[6] *I. Dione, J. M. Urquiza*: Penalty: finite element approximation of Stokes equations with slip boundary conditions. Numer. Math. *129* (2015), 587–610. zbl MR doi

[7] *C. Geuzaine, J.-F. Remacle*: Gmsh: A 3-D finite element mesh generator with built-in pre- and post-processing facilities. Int. J. Numer. Methods Eng. *79* (2009), 1309–1331. zbl MR doi

[8] *V. Girault, P.-A. Raviart*: Finite Element Methods for Navier-Stokes Equations. Theory and Algorithms. Springer Series in Computational Mathematics 5, Springer, Berlin, 1986. zbl MR doi

[9] *F. Hecht*: New development in freefem++. J. Numer. Math. *20* (2012), 251–265. zbl MR doi

[10] *J. G. Heywood, R. Rannacher*: Finite element approximation of the nonstationary Navier-Stokes problem. I. Regularity of solutions and second-order error estimates for spatial discretization. SIAM J. Numer. Anal. *19* (1982), 275–311. zbl MR doi

[11] *V. John*: Slip with friction and penetration with resistance boundary conditions for the Navier-Stokes equations—numerical tests and aspects of the implementation. J. Comput. Appl. Math. *147* (2002), 287–300. zbl MR doi

[12] *T. Kashiwabara, I. Oikawa, G. Zhou*: Penalty method with P1/P1 finite element approximation for the Stokes equations under the slip boundary condition. Numer. Math. *134* (2016), 705–740. zbl MR doi

[13] *P. Knobloch*: Discrete Friedrich's and Korn's inequalities in two and three dimensions. East-West J. Numer. Math. *4* (1996), 35–51. zbl MR

[14] *P. Knobloch*: Variational crimes in a finite element discretization of 3D Stokes equations with nonstandard boundary conditions. East-West J. Numer. Math. *7* (1999), 133–158. zbl MR

[15] *W. Layton*: Weak imposition of "no-slip" conditions in finite element methods. Comput. Math. Appl. *38* (1999), 129–142. zbl MR doi

[16] *A. Logg, K.-A. Mardal, G. Wells,* Editors: Automated Solution of Differential Equations by the Finite Element Method. The FEniCS Bbook. Lecture Notes in Computational Science and Engineering 84, Springer, Heidelberg, 2012. zbl MR doi

[17] *H. Saito, L. E. Scriven*: Study of coating flow by the finite element method. J. Comput. Phys. *42* (1981), 53–76. zbl doi

[18] *Y. Shibata, R. Shimada*: On a generalized resolvent estimate for the Stokes system with Robin boundary condition. J. Math. Soc. Japan *59* (2007), 469–519. zbl MR doi

[19] *Y. Stokes, G. Carey*: On generalized penalty approaches for slip, free surface and related boundary conditions in viscous flow simulation. Int. J. Numer. Meth. Heat and Fluid Flow *21* (2011), 668–702. doi

[20] *M. Tabata*: Uniform solvability of finite element solutions in approximate domains. Japan J. Ind. Appl. Math. *18* (2001), 567–585. zbl MR doi

[21] *M. Tabata*: Finite element approximation to infinite Prandtl number Boussinesq equations with temperature-dependent coefficients—Thermal convection problems in a spherical shell. Future Gener. Comput. Syst. *22* (2006), 521–531. doi

[22] *M. Tabata, A. Suzuki*: A stabilized finite element method for the Rayleigh-Bénard equations with infinite Prandtl number in a spherical shell. Comput. Methods Appl. Mech. Eng. *190* (2000), 387–402. zbl MR doi

[23] *M. Tabata, D. Tagami*: Error estimates for finite element approximations of drag and lift in nonstationary Navier-Stokes flows. Japan J. Ind. Appl. Math. *17* (2000), 371–389. zbl MR doi

[24] *R. Temam*: Navier-Stokes Equations. Theory and Numerical Analysis. AMS Chelsea Publishing, Providence, 2001. zbl MR doi

[25] *R. Verfürth*: Finite element approximation of steady Navier-Stokes equations with mixed boundary conditions. RAIRO, Modélisation Math. Anal. Numér. *19* (1985), 461–475. zbl MR doi

[26] *R. Verfürth*: Finite element approximation of incompressible Navier-Stokes equations with slip boundary condition. Numer. Math. *50* (1987), 697–721. zbl MR doi

[27] *R. Verfürth*: Finite element approximation of incompressible Navier-Stokes equations with slip boundary condition. II. Numer. Math. *59* (1991), 615–636. zbl MR doi

[28] *G. Zhou, T. Kashiwabara, I. Oikawa*: Penalty method for the stationary Navier-Stokes problems under the slip boundary condition. J. Sci. Comput. *68* (2016), 339–374. zbl MR doi

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