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IMPROVING THE PERFORMANCE OF SEMIGLOBAL OUTPUT CONTROLLERS FOR NONLINEAR SYSTEMS

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For a large class of nonlinear control systems, the main drawback of a semiglobal stabilizing output feedback controllers $(\mathcal{U}_R)_{R>0}$ with increasing regions of attraction $(\Omega_R)_{R>0}$ is that, when the region of attraction Ω_R is large, the convergence of solutions of the closed-loop system to the origin becomes slow. To improve the performance of a semiglobal controller, we look for a new feedback control law that preserves the semiglobal stability of the nonlinear system under consideration and that is equal to some "fast" controller $\mathcal{U}_{R_0}$ on a neighborhood of the origin. Under an input-output-to-state stability (IOSS) assumption, we propose a new semiglobal stabilizing hybrid feedback controller that unifies a "slow" controller that has a large region of attraction with a "fast" controller having a small region of attraction. This unification is inspired from the elegant hybrid unification of a local controller with a global one given in [21]. Moreover, this unification is different from the recent result [24], since in the cited paper the objective is just the stabilization; whereas in our study, the objective is the stabilization with *high performance*. Finally, we illustrate our main result by means of two numerical examples.

Keywords: nonlinear system, hybrid output feedback, semiglobal output stabilization, local performance

Classification: 93C10, 93D15

1. INTRODUCTION

Stabilization of nonlinear systems is central to control theory. There are numerous tools for global stabilization by state feedback controllers, such as backstepping, forwarding, passivity, control Lyapunov function and feedback linearization (see for instance the textbooks [6, 27] and [10] and the references therein). However, the stabilization techniques do not take into account the performance issue. As pointed out in [21, 24] and [30], by hybrid unification of a high-performance local controller (obtained for example by linearization) with global controller can ameliorate the performance.

However, for a large class of nonlinear control systems global state or output stabilization fails for many nonlinear control systems. For global output stabilization, the majority of existing results deal with a specific class of systems such as triangular systems (see for example [15, 18] and [22]).

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On the other hand, from a practical point of view and in many situations, we do not need to build a global controller, but it is only sufficient to stabilize the system under consideration locally around the origin with a region of attraction arbitrarily large. This is the notion of semiglobal stabilization. For a large class of systems, through high-gain observer (see [7]), it is possible to build an output feedback controller that solves the problem of semiglobal output stabilization (see [13, 19] and [12]). In this paper, we address the performance issue of output semiglobal stabilizers for nonlinear control systems. The proposed strategy consists in unifying a local continuous output controller with high performance with a second continuous output controller having a large region of attraction. The new obtained hybrid output controller stabilizes semiglobally the nonlinear system and ameliorates the performance.

It is well known now, that hybrid feedback is an efficient tool for robust stabilization of nonlinear control systems [20]. It removes some classical restrictions imposed by continuous feedback (see [20] and references therein). Recently, many important results of stabilization are established (see for example [1, 2, 26] and [25]).

To the best of our knowledge, the first unification of two output local controllers is solved in [24]. Unfortunately, such strategy of unification does not ameliorate the performance as we will show by a numerical examples. In this paper, we modify the hybrid unification introduced in [24] in order to accelerate the convergence.

To motivate the problem that we want to solve, we consider the following example.

Example 1.1. Consider the linear control system,

$$\dot{x} = Ax + B\ sat(u), \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}$ is the control input, A and B are constant real matrices with appropriate dimensions and $sat(.)$ is the symmetric saturation function defined as

$$\forall z \in \mathbb{R},\quad sat(z) = sign(z) \min\left\{|z|, \bar{u}\right\}, \tag{2}$$

where $\bar{u}$ is a positive constant. It is well known (see the textbook [29] page 57) that, if there exist a positive definite matrix $P_\lambda \in \mathbb{R}^{n\times n}$ and matrices $X, Y \in \mathbb{R}^{1\times n}$ that satisfy the following linear matrix inequalities (LMI)

$$AP_\lambda + P_\lambda A^T + B\Gamma_j^+ Y + Y^T\Gamma_j^+ B^T + B\Gamma_j^- X + X^T\Gamma_j^- B^T \leq -2\lambda P_\lambda,$$

and

$$\begin{pmatrix} P_\lambda & X^T \\ X & \eta\bar{u}^2 \end{pmatrix} \geq 0,$$

where $\Gamma_1^+ = 1, \Gamma_2^+ = 0$, and $\Gamma_j^- = 1 - \Gamma_j^+$, for $j = 1, 2$, then the origin of system (1) with linear control $u = Kx = YP_\lambda^{-1}x$ is exponentially stable with a decay rate $\lambda > 0$ and a region of attraction that contains the ellipsoid

$$\Omega_\lambda(P_\lambda, \eta) = \{x \in \mathbb{R}^2,\ x^T P_\lambda^{-1} x \leq \eta^{-1}\},$$

where η is a strictly positive real number.

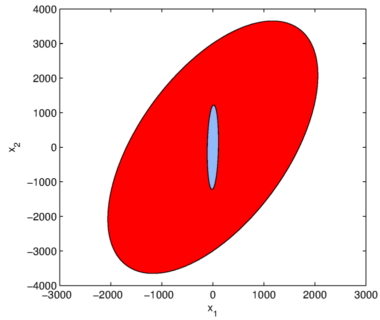


Fig. 1. Regions of attraction of system (1) with $\bar{u} = 1000$, $\lambda = \lambda_0 = 10$ and $\lambda = \lambda_1 = 1$. The size of region of attraction is inversely proportional to the decay rate λ.

For $\bar{u} = 1000$, $A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, Figure 1 shows that the ellipsoid Ω_{10} in blue color (i.e. with $\lambda = 10$) is included in the ellipsoid Ω_1 in red color (i.e. with $\lambda = 1$).

As shown in Figure 1, for system (1), the size of a region of attraction Ω_λ is inversely proportional to the decay rate λ. In other words, for a small prescribed region of attraction, we can choose a sufficiently large decay rate λ, while for a large prescribed region of attraction, we are forced to choose a relatively small decay rate λ. Such situation occurs in most observable and controllable linear systems of the form

$$\begin{cases} \dot{x} = Ax + Bu, \\ y = Cx, \end{cases} \tag{3}$$

with dynamic output linear controller

$$\begin{cases} \dot{\xi} = A_c\xi + B_c y, \\ u = sat(K_0\xi + K_1 y). \end{cases} \tag{4}$$

This motivates us to improve the performance of the semiglobal output controllers (4). To do this, we exploit the idea of hybrid unification of two output controllers introduced in [21] and generalized recently in [24].

In this paper, we consider a nonlinear control system for which we know a family of output controllers $(\mathcal{U}_R)_{R>0}$ with regions of attraction $(\Omega_R)_{R>0}$, such that $\Omega_{R_1} \subset \Omega_{R_2}$, for all $R_1 < R_2$. We assume that when we use the controller $\mathcal{U}_R$:

- for large values of R, the solutions of the closed loop system converge *slowly*, and
- for some value R_0, they converge *quickly* to the origin.

We are looking for a new output feedback that preserves the semiglobal stability of the origin of nonlinear system under consideration such that, for a given region of attraction we use a slow controller $\mathcal{U}_R$ for some large real R to steer the system trajectories to a neighborhood of the origin, and consequently, we apply the fast local controller $\mathcal{U}_{R_0}$ to converge rapidly to the origin. The strategy of combining two controllers has been well used in the literature [4, 5, 17, 30] and [23]. To apply this strategy, there are two main difficulties.

1. The first difficulty is that, since we cannot measure all the components of the system state, we do not know if the trajectories enter or not the region of attraction Ω_{R_0}. Thus, we do not know when we switch from the slower controller to the faster one.

2. The second difficulty is that, when switching between the slower controller and the faster one, the solution can leave the two regions of attraction Ω_{R_0} and Ω_R. Hence, the solution cannot converge to the origin.

As in [21], the first difficulty is overcome by means of a norm estimator (For more details, see [14]). For the second difficulty, we impose that when we apply the slower controller $\mathcal{U}_R$, we cannot switch to the faster controller $\mathcal{U}_{R_0}$ before some positive time τ^*. In our study, the objective of the unification is different from the one solved in [24]. In the cited paper, the objective is to stabilize the system at any price, while in our work, the objective is the stabilization with a *high performance*. As a consequence, in [24] the time trigger τ^* is chosen sufficiently large. This means that they use the slow controller frequently and this is not good from the performance point of view.

In our work, we prove that for all *arbitrarily small positive time trigger* τ^* there exists $\gamma(R)$ (depending on τ^* see (30)) such as we can unify $\mathcal{U}_{\gamma(R)}$ with $\mathcal{U}_{R_0}$. The obtained controller has a region of attraction that contains the region of attraction of the system under consideration in closed loop with the controller $\mathcal{U}_R$.

We point out that hybrid feedback can achieve asymptotic stabilization that is robust to small measurement noise, actuator errors, and external disturbances (see [20]). For these reasons, we consider the unification based on hybrid feedback.

The paper is organized as follows. In section 2, we present the problem under consideration and we introduce the new hybrid output controller that solves it. Moreover, we present our main result, which is summarized in Theorem 2.3. Section 3 is devoted to the proof of the main Theorem. In section 4, we give two examples that illustrate the improvement of the performance of our new hybrid output controller and we compare this performance with the initial continuous controller and with the hybrid controller of [24]. Section 5 is dedicated to a discussion about some drawbacks of the given hybrid controller.

2. PROBLEM FORMULATION AND THE MAIN RESULT

Consider the nonlinear system

$$\left\{ \begin{array}{rcl} \dot{x} & = & f(x,u), \\ y & = & h(x), \end{array} \right. \tag{5}$$

where $x \in \mathbb{R}^{n_p}$ is the system state, $u \in \mathbb{R}^m$ is the control input and $y \in \mathbb{R}^p$ is the measured output. We assume that the function $f : \mathbb{R}^{n_p} \times \mathbb{R}^m \to \mathbb{R}^{n_p}$ is locally Lipschitz with $f(0,0) = 0$ and the function $h : \mathbb{R}^{n_p} \to \mathbb{R}^p$ is continuously differentiable with $h(0) = 0$.

We denote by $|x|$ the Euclidean norm of vector x, and for $R > 0$ and $n \in \mathbb{N}^*$ (where $\mathbb{N}^*$ is the set of strictly positive integers), the closed ball of $\mathbb{R}^n$ of radius R and centered at the origin is denoted by $B_n(0,R)$ and defined as follows

$$B_n(0,R) = \{x \in \mathbb{R}^n,\ |x| \le R\}.$$

A continuous function $\alpha : [0,a[\to [0,+\infty[$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. A continuous function $\alpha : [0,+\infty[\to [0,+\infty[$ is said to belong to class $\mathcal{K}_{+\infty}$ if it is a class $\mathcal{K}$ function and $\lim_{r\to+\infty} \alpha(r) = +\infty$.

2.1. Assumptions and objective

For system (5), we consider the following assumptions.

Assumption 1. There exist a positive integer l_1 and a family of output feedback controllers $(\mathcal{U}_R)_{R>0} = (\alpha_R, \varphi_R)_{R>0}$ where, $\alpha_R : \mathbb{R}^{l_1} \times \mathbb{R}^p \to \mathbb{R}^m$, and $\varphi_R : \mathbb{R}^{l_1} \times \mathbb{R}^p \to \mathbb{R}^{l_1}$ are continuous functions vanishing at the origin such that, the origin of the closed loop system

$$(\mathcal{S}_R) : \quad \begin{cases} \dot{x} = f(x, \alpha_R(\xi_1, y)), \\ \dot{\xi}_1 = \varphi_R(\xi_1, y), \end{cases} \tag{6}$$

is asymptotically stable with region of attraction containing the invariant set $B_{n_p}(0,R) \times B_{l_1}(0,R)$.

Assumption 2. The functions α_R and φ_R are uniformly bounded with respect the parameter R, i.e. there exists a class $\mathcal{K}$ function θ such that

$$\max\{|\varphi_R(\xi_1,y)|, |\alpha_R(\xi_1,y)|\} \le \theta\big(|\xi_1| + |y|\big), \ \ \forall (\xi_1, y) \in \mathbb{R}^{l_1} \times \mathbb{R}^p.$$

Assumption 3. System (5) is input-output-to-state stable (IOSS).

A discussion about the class of system under consideration and assumptions are given in the following remarks.

Observation 2.1. Assumption 2 is not restrictive since asymptotic stabilization of nonlinear system is not a result of the magnitude of the feedback control but it is a result of the "way" of stabilization. For example in [16], the class of considered system is globally stabilizable by an arbitrarily small state feedback (see Assumption A2 page 3). Moreover, as explained by Mazenc in [11] that Assumption A2 is not restrictive since feedback stabilization is the result of the "way" of stabilization and not of feedback magnitude. In this work, we claim that Assumption 2 can be canceled. Precisely, we conjectured that if there exists a family of output feedbacks $(\mathcal{U}_R)_{R>0} = (\alpha_R, \varphi_R)_{R>0}$ satisfying Assumption 1, then we can build a new family of output feedbacks $(\tilde{\mathcal{U}}_R)_{R>0} = (\tilde{\alpha}_R, \tilde{\varphi}_R)_{R>0}$ satisfying Assumptions 1 and 2.

On the other hand, as in [24], Assumption 3 can be relaxed to require output-to-state stability (OSS) of the closed loop system with the feedback controller $\mathcal{U}_R$ instead of the IOSS property. Since OSS is equivalent to the norm observability (see [28]), we believe that the OSS property is the minimal assumption required to unify two output feedbacks.

Observation 2.2. Note that, in Assumption 1 we can define the invariant sets $B_{n_p}(0,R)\times B_{l_1}(0,R)$, $R>0$ by using a Lyapunov function $W(x,\xi_1)$ that satisfies

$$\omega_1(|(x,\xi_1)|) \leq W(x,\xi_1) \leq \omega_2(|(x,\xi_1)|),$$

and the derivative of W along the solutions of (6) satisfies

$$\dot{W}(x,\xi_1) < 0,$$

for all (x,ξ_1) in a neighborhood of $(0,0)$, where ω_1 and ω_2 are class $\mathcal{K}$ functions. In this setting, the invariant sets have the form

$$\Omega_c = \{(x,\xi_1) \in \mathbb{R}^n \times \mathbb{R}^{l_1}, W(x,\xi_1) \leq c\},$$

for sufficiently small positive real c. This generates technical difficulties in the proof of Theorem 2.3. That is why we assume that each invariant set of the system (6) contains a set of the form $B_{n_p}(0,R) \times B_{l_1}(0,R)$, for some positive real number R. From a topological point of view, there is no difference between the two settings since the norm defined by the Lyapunov function $W(x,\xi_1)$ and the Euclidian norm on $\mathbb{R}^{n_p} \times \mathbb{R}^{l_1}$ are equivalent.

Furthermore, suppose that there exists a positive real $R_0 > 0$ such that the solutions of system (5) in closed loop with the output controller $\mathcal{U}_{R_0}$,

$$(\mathcal{S}_{R_0}) : \quad \begin{cases} \dot{x} = f(x, \alpha_{R_0}(\xi_0, y)), \\ \dot{\xi}_0 = \varphi_{R_0}(\xi_0, y), \end{cases} \tag{7}$$

converge quickly to the origin and for all $R > R_0$, the convergence of solutions of system $(\mathcal{S}_{R_0})$ is faster than the solutions of system $(\mathcal{S}_R)$. In other words, the output controller $\mathcal{U}_{R_0}$ is faster than any output controller $\mathcal{U}_R$, $R \geq R_0$. We say that $\mathcal{U}_{R_0}$ is the *fast controller* and $\mathcal{U}_R$ is a *slow controller.*

Our objective is to construct a new semiglobal output feedback controller for system (5) that improves the performance locally. The main idea of the solution to this problem is to use a slow controller to steer the trajectory in the region of attraction of the fast controller and then we switch to the fast one.

This work is inspired from [21], where a hybrid output feedback controller solving the problem of uniting local and global output feedback controllers has been constructed. The previous work has been generalized in [24], for hybrid output feedbacks instead of continuous output feedback and for output to state stable (OSS) systems instead of IOSS systems. Here the problem is more challenging since we are interested in the stabilization with high performance and not just the stabilization as in [24]. To build our new semiglobal output hybrid feedback with high performance, as in [24], we impose that when we apply the slow controller $\mathcal{U}_R$, we cannot switch to the fast controller $\mathcal{U}_{R_0}$ before some positive trigger time $\tau^* > 0$. In contrast to [24], where the trigger time τ^* is chosen sufficiently large; in our work, the trigger time τ^* can be *chosen arbitrarily small*. This has been established using the adequate estimation (25) which induces the estimation (26) in the proof of the main Theorem. This leads to the balance (30) between the trigger time τ^* and $\gamma(R)$.

To present our hybrid controller, we introduce some basic concepts about hybrid systems.

2.2. Basic concepts of hybrid system

A hybrid system $(\mathcal{H})$ is governed by a continuous dynamic

$$\text{if } x \in C, \;\; \dot{x} = f_0(x),$$

and a discrete dynamic

$$\text{if } x \in D, \;\; x^+ = g_0(x),$$

where, $f_0 : \mathbb{R}^n \to \mathbb{R}^n$ and $g_0 : \mathbb{R}^n \to \mathbb{R}^n$ are outer semi continuous and locally bounded functions, C and D are two closed sets of $\mathbb{R}^n$, C is the flow set and D is the jump set with $C \cup D = \mathbb{R}^n$. The set $S \subset \mathbb{R}_+ \times \mathbb{N}$ is a compact hybrid time domain if $S = \bigcup_{j=0}^{J-1}([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \ldots \leq t_J$. The set S is a hybrid time domain if for all $(T, J) \in S, S \cap ([0, T] \times \{0, 1 \ldots, J\})$ is a compact hybrid time domain.

A hybrid arc x is a function defined on a hybrid time domain $dom(x)$ such that for all $j \in \mathbb{N}, t \mapsto x(t, j)$ is locally absolutely continuous on $dom(x) \cap ([0, +\infty[\times\{j\})$. A hybrid arc x is a solution (or a trajectory) of the hybrid system $(\mathcal{H})$ if,

1. for all $j \in \mathbb{N}$, and for almost t such that $(t, j) \in dom(x)$,

$$x(t, j) \in C \quad \text{and} \quad \dot{x}(t, j) = f_0(x(t, j)),$$

2. for all $(t, j) \in dom(x)$, such that $(t, j + 1) \in dom(x)$,

$$x(t, j) \in D \text{ and } \; x(t, j + 1) = g_0(x(t, j)).$$

For more details about the existence of solutions of hybrid systems, see the recent textbook [8].

For all $R > 0$, we consider a dynamic hybrid output feedback controller (C, D, u, v, w) where, for a given integer l, $C \subset \mathbb{R}^l$, $D \subset \mathbb{R}^l$ are closed sets, $u : \mathbb{R}^p \times C \to \mathbb{R}^m$, $v : \mathbb{R}^p \times C \to \mathbb{R}^l$ and $w : \mathbb{R}^p \times D \to \mathbb{R}^l$ are continuous functions. System (5) in closed loop with the dynamic hybrid output feedback controller $(C, D, u, v, w)_{R>0}$ indexed by a real parameter R is defined as the hybrid system

$$\begin{aligned}
&\text{if } \;\xi \in C, \quad \left\{ \begin{array}{l} \dot{x} = f(x, u(h(x), \xi)), \\ \dot{\xi} = v(h(x), \xi), \end{array} \right. \\
&\text{if } \;\xi \in D, \quad \left\{ \begin{array}{l} x^+ = x, \\ \xi^+ = w(h(x), \xi). \end{array} \right.
\end{aligned} \tag{8}$$

The origin of the parameterized dynamic hybrid system (8) is said to be semiglobal asymptotically stable, if

(1) *Local stability* : for all $\varepsilon > 0$, and all $R > 0$, there exists $\delta > 0$, such that for all initial conditions (x^0, ξ^0) in $\mathbb{R}^{n_p} \times (C \cup D)$ and $|(x^0, \xi^0)| < \delta$, for all trajectories (x, ξ) of (8) starting from (x^0, ξ^0), we have

$$|(x(t,j), \xi(t,j))| < \varepsilon, \ \forall\ (t,j) \in dom(x, \xi).$$

(2) *semiglobal attractivity* : for all compact $K \subset \mathbb{R}^{n_p} \times (C \cup D)$, there exists a parameter $R_K > 0$, such that for all $(x^0, \xi^0) \in K$, for all trajectories (x, ξ) of (8) with parameter R_K starting from (x^0, ξ^0), we have

$$\lim_{t+j \to \infty} |(x(t,j), \xi(t,j))| = 0.$$

A family of hybrid controller $(C, D, u, v, w)_{R>0}$ stabilizes semiglobally the origin of system (5), if the origin of the closed loop system (8) is semiglobal asymptotically stable.

2.3. Problem formulation

Now, we can present in a precise way the problem that we want to solve.

Problem. Given any family of output controllers $(\mathcal{U}_R)_{R \geq R_0}$ that stabilizes semiglobally the origin of system (5), find a hybrid output feedback controller

$$\mathfrak{U}_R = (C_R, D_R, u_R, v_R, w_R),$$

such that,

(I1) The family of controllers $(\mathfrak{U}_R)_{R \geq R_0}$ stabilizes semiglobally the origin of system (5).

(I2) There exist a positive real δ and a matrix $M \in \mathbb{R}^{l_1 \times l}$, such that for all initial state (x^0, ξ^0), $|(x^0, \xi^0)| < \delta$, then $(x, M\xi)$ is a trajectory of system $(\mathcal{S}_{R_0})$, where (x, ξ) is a trajectory of system (5) in closed loop with the hybrid controller $\mathfrak{U}_R$.

Note that item (I2) is equivalent to say that, for any $R \geq R_0$, the hybrid controller $\mathfrak{U}_R$ is locally equal to the fast controller $\mathcal{U}_{R_0}$. As we will see in the proof of Theorem 2.3, the new hybrid output feedback $\mathfrak{U}_R$ is a hybrid combination of certain slow controller $\mathcal{U}_{\gamma(R)}$ with the fast controller $\mathcal{U}_{R_0}$. Then, item (I2) says that the hybrid controller $\mathfrak{U}_R$ has the highest performance of the fast continuous controller $\mathcal{U}_{R_0}$ locally.

To solve the above problem, using Assumption 3, we introduce two norm estimators of the system state.

2.4. Two norm estimators

The IOSS property of system (5) (See [14]) given by Assumption 3 is equivalent to the existence of an IOSS-Lyapunov function V_1 that satisfies

$$\kappa_1(|x|) \leq V_1(x) \leq \kappa_2(|x|), \tag{9}$$

where κ_1, κ_2 are class $\mathcal{K}_\infty$ functions, and

$$\dot{V}_1(x) = \nabla V_1(x).f(x,u) \leq -V_1(x) + \sigma_1(|u|) + \sigma_2(|y|), \tag{10}$$

for all $(x, y, u) \in \mathbb{R}^{n_p} \times \mathbb{R}^p \times \mathbb{R}^m$, and for some class $\mathcal{K}_\infty$ functions σ_1 and σ_2. We point out that, for IOSS-nonlinear system there is no a constructive method to build an IOSS-Lyapunov function satisfying (10).

Estimation (10) can be used to construct the following norm estimator for system (5),

$$\dot{z}_1 = -z_1 + \rho_1(u, h(x)), \tag{11}$$

with $\rho_1(u,y) = \sigma_1(|u|) + \sigma_2(|y|)$. By taking the difference between (10) and (11), we obtain

$$\dot{V}_1(x) - \dot{z}_1 \leq -(V_1(x) - z_1). \tag{12}$$

Integrating (12), it yields

$$\begin{array}{rcl} V_1(x(t)) - z_1(t) & \leq & (V_1(x^0) - z_1^0)e^{-t} \\ & \leq & (V_1(|x^0|) + |z_1^0|)e^{-t}. \end{array}$$

Then,

$$V_1(x(t)) \leq z_1(t) + (V_1(x^0) + |z_1^0|)e^{-t}, \ \forall t \in [0, T_{\sup}(x, z_1)), \tag{13}$$

for all initial conditions (x^0, z_1^0) in $\mathbb{R}^{n_p} \times \mathbb{R}_+$, and all piecewise continuous signal $u(t)$, where $x(t)$ and $z_1(t)$ are the solutions of systems (5) and (11), respectively.

We use the norm estimator $z_1(t)$ in the hybrid controller in the following way. When we apply the slow controller and after a large time t if $z_1(t)$ is small, i. e. $z_1(t) \leq \varepsilon_{1a}$, by estimation (13) we deduce that $V_1(x(t))$ becomes small, i. e. $V_1(x(t)) \leq \varepsilon_{1b}$ (see the proof of Lemma 3 page 23). Then, we conclude that $x(t)$ enters the region of attraction of the fast controller and then we can switch to it.

Moreover, we construct a "local" norm estimator for the state of the closed loop system with hybrid controller. Observe that the Lyapunov function

$$V_0(x, \xi_0) = V_1(x) + \frac{1}{2}|\xi_0|^2,$$

is an IOSS-Lyapunov function for system (7)

$$\left\{ \begin{array}{l} \dot{x} = f(x,u), \\ \dot{\xi}_0 = v, \end{array} \right. \tag{14}$$

with input (u, v) and output $(h(x), \xi_0)$. Indeed, the derivative of V_0 along the solutions of system (14) is bounded as

$$\dot{V}_0(x, \xi_0) \leq -V_0(x, \xi_0) + \rho_0(h(x), \xi_0, u, v), \tag{15}$$

where

$$\rho_0(h(x), \xi_0, u, v) = \sigma_1(|u|) + \frac{1}{2}|v|^2 + \sigma_2(|y|) + |\xi_0|^2.$$

In view of (15), the norm estimator

$$\dot{z}_0 = -z_0 + \rho_0(h(x), \xi_0, u, v), \tag{16}$$

satisfies,

$$V_0(x(t), \xi_0(t)) \le z_0(t) + (V_0(x^0, \xi_0^0) + |z_0^0|)e^{-t}, \tag{17}$$

for all initial conditions (x^0, ξ_0^0, z_0^0), and all piecewise continuous signals $u : \mathbb{R}_+ \to \mathbb{R}^m$, $v : \mathbb{R}_+ \to \mathbb{R}^{l_1}$, and $t \in [0, T_{\sup}(x, \xi_0, z_0))$, where $x(t)$ and $z_0(t)$ are the solutions of the systems (14) and (16), respectively.

Note that, when we apply the fast controller and when $V_0(x(t), \xi_0(t))$ grows with respect to time t; by estimation (17), we deduce that $z_0(t)$ becomes large enough to conclude that $(x(t), \xi_0(t))$ is not in the region of attraction of the fast controller. And then, we must switch to the slow controller.

2.5. The hybrid controller and the closed loop system

As in [21], and by continuity of the functions V_1, ρ_0, ρ_1 and h, for all $R > R_0$, there exist a positive constants $\varepsilon_{0a} < \varepsilon_{0bR}$, $\varepsilon_{1a} < \varepsilon_{1b}$ and ε_2, such that the following items are satisfied :

($\mathcal{I}_1$) The set

$$\{(x, \xi_0) \in \mathbb{R}^{n_p} \times \mathbb{R}^{l_1},\ V_0(x, \xi_0) \le \varepsilon_{0bR}\},$$

is included in $B_{n_p}(0, R) \times B_{l_1}(0, R)$.

($\mathcal{I}_2$) Since the trajectories of system (6) converge to the origin, then for all $(x^0, \xi_1^0) \in B_{n_p}(0, R) \times B_{l_1}(0, R)$, there exists a time t_0, such that

$$V_1(x(t)) + |\xi_1(t)| < \varepsilon_2,\ \forall t \in [t_0, T_{\sup}(x^0, \xi_1^0)[. \tag{18}$$

($\mathcal{I}_3$) For each trajectory of system $(\mathcal{S}_R)$ starting from $\{(x, \xi_0), V_1(x) \le \varepsilon_{1b}, \xi_0 = 0\}$, we have $\rho_0(h(x(t)), \xi_0(t), 0, \alpha_R(.), \varphi_R(.), 0) < \varepsilon_{0a}, \forall t \ge 0$.

($\mathcal{I}_4$) For all initial conditions (x^0, ξ_1^0) in the set $\{(x, \xi_1) \in \mathbb{R}^{n_p} \times \mathbb{R}^{l_1},\ V_1(x) + |\xi_1| \le \varepsilon_2\}$, the trajectory of system $(\mathcal{S}_R)$ satisfies $\rho_1(\alpha_R(h(x(t), \xi_1(t)), h(x(t)) < \varepsilon_{1a}, \forall t \ge 0$.

Note that, ε_{0bR} can be chosen sufficiently large when R tends to $+\infty$. Since $R > R_0$, ε_{1b} and ε_2 are chosen independent from R, and by Assumption 2, $\varepsilon_{0a}, \varepsilon_{1a}$ are independent from R.

Now, using ($\mathcal{I}_1$) to ($\mathcal{I}_4$) we can give an explicit expression of the new hybrid output controller $(\mathfrak{U}_R)_{R > R_0}$. Let $\tau^* > 0$ be an arbitrarily strictly positive real time, and $R > R_0$. Denote $l = 2l_1 + 4$, and decomposing $\xi \in \mathbb{R}^l$ as $\xi = (\xi_0, \xi_1, z_0, z_1, s, q) \in \mathbb{R}^{l_1} \times \mathbb{R}^{l_1} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. Consider the hybrid output controller $\mathfrak{U}_R = (C_R, D_R, u_R, v_R, w_R)$ defined as follows :

- $C_R = C_{0R} \cup C_{1R}, \quad D_R = D_{0R} \cup D_{1R}$,
- $u_R : \mathbb{R}^p \times C_R \to \mathbb{R}^p,\ (y, \xi) \mapsto (1 - q)\alpha_{R_0}(y, \xi_0) + q\alpha_{\gamma(R)}(y, \xi_1)$,

- $v_R : \mathbb{R}^p \times C_R \to \mathbb{R}^l$
  $(y,\xi) \mapsto \Big((1-q)\varphi_{R_0}(y,\xi_0), q\varphi_{\gamma(R)}(y,\xi_1), (1-q)(-z_0 + \rho_{0R_0}(y,\xi_0)),$
  $- z_1 + \rho_1\big((1-q)\alpha_{R_0}(y,\xi_0) + q\alpha_{\gamma(R)}(y,\xi_1), y\big), q, 0\Big),$
- $w_R : \mathbb{R}^p \times D_R \to \mathbb{R}^l$
  $(y,\xi) \mapsto (0, 0, (1-q)z_0, z_1, 0, 1-q),$

where

$$\rho_{0R_0}(h(x),\xi_0) = \rho_0(h(x(t)), \xi_0(t), \alpha_{R_0}(.), \varphi_{R_0}(.)),$$

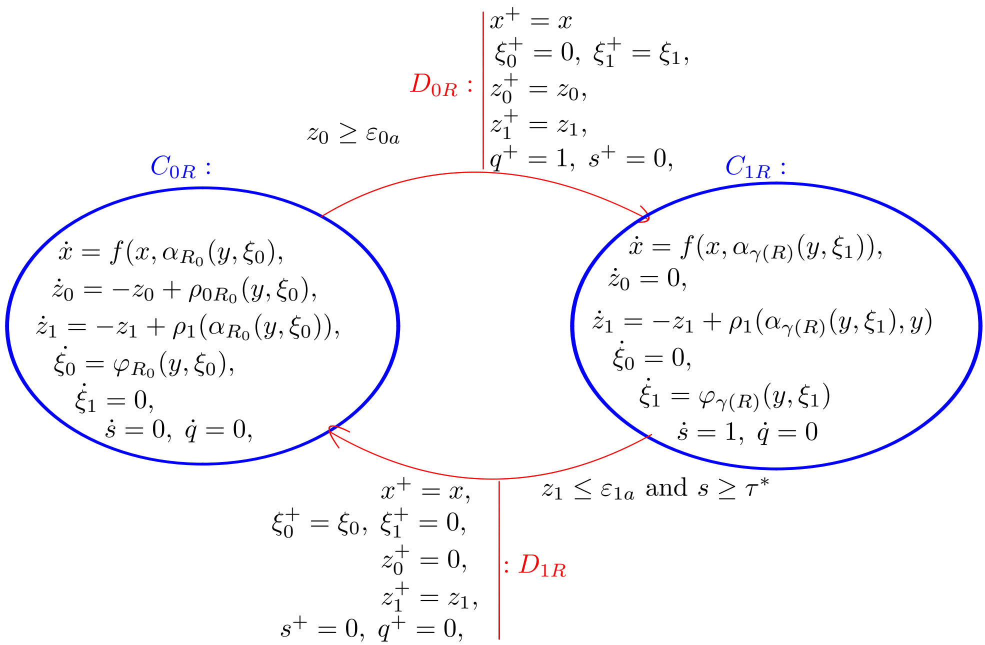
with

- $C_{0R} = \{\xi : 0 \le z_0 \le \varepsilon_{0a},\ 0 \le z_1,\ \xi_1 = 0,\ s = 0, q = 0\}$,
- $C_{1R} = \{\xi : 0 \le z_0,\ 0 \le z_1,\ \xi_0 = 0,\ s \ge 0,\ q = 1\}$,
- $D_{0R} = \{\xi : \xi_1 = 0,\ z_0 \ge \varepsilon_{0a},\ z_1 \ge 0,\ s = 0, q = 0\}$,
- $D_{1R} = \{\xi : 0 \le z_0 \le \varepsilon_{0a},\ 0 \le z_1 \le \varepsilon_{1a},\ \xi_0 = 0,\ s \ge \tau^*,\ q = 1\}$,

and $\gamma(R)$ is a positive real to be selected later. System (5) in closed loop with the hybrid output controller $\mathfrak{U}_R$ is defined as the hybrid system:

$$\begin{aligned} &\text{if}\ \ \xi \in C_R, && \begin{cases} \dot{x} = f(x, u_R(h(x),\xi)), \\ \dot{\xi} = v_R(h(x),\xi), \end{cases} \\ &\text{if}\ \ \xi \in D_R, && \begin{cases} x^+ = x, \\ \xi^+ = w_R(h(x),\xi). \end{cases} \end{aligned} \tag{19}$$

The closed loop system (19) can be represented by the following automate



2.6. The main Theorem

The main result of this paper is summarized in the following Theorem.

Theorem 2.3. Consider the system (5) with assumptions 1, 2 and 3. Let τ^* be an arbitrarily strictly positive real time. Then, there exists $\gamma(R) > 0$ such that the family of hybrid output controller $\left(\mathfrak{U}_R\right)_{R>R_0}$ defined above stabilizes semiglobally the origin of system (5). Furthermore, the set $B_{n_p}(0,R) \times B_{l_1}(0,R) \times B_{l_1}(0,R) \times \mathbb{R}^4 \cap \left(\mathbb{R}^{n_p} \times (C_R \cup D_R)\right)$ is included in the region of attraction of the closed loop system (19).

Let us give the main ideas of the proof of Theorem 2.3 Intuitively, due to the expression of C_R , for large initial conditions, the trajectories of (5) in closed loop with the controller $\mathfrak{U}_R$ are trajectories of (6) as long as the state variable z_1 of the second norm estimator does not attain the value ε_{1a}. Due to item $(\mathcal{I}_2)$ and item $(\mathcal{I}_4)$ of Assumption 2, for sufficiently large time, the state variable z_1 becomes smaller than ε_{1a}. Then the trajectory enters D_{1R} and C_{0R} successively. It is possible that, as the first time when we enter C_{0R}, we are not in the region of attraction of (7). However using (11), (13), (15), (16) , and item $(\mathcal{I}_3)$ of Assumption 2, we may prove that, for sufficiently large time, $V_0(x, z_0)$ is smaller than ε_{0bR}, and thus we eventually are in C_{0R} and also in the region of attraction of the fast local controller. Due to the expression of C_{0R}, we continue to follow the trajectories of (7), and, with item $(\mathcal{I}_1)$ of Assumption 2, we converge to the origin.

Some remarks about the new hybrid output controller are given.

Observation 2.4. The main difficulty in our problem of uniting two local output controllers $\mathcal{U}_{R_0}$ and $\mathcal{U}_{\gamma(R)}$ ($\gamma(R)$ is defined in the proof of Theorem 2.3 in (30)) with regions of attraction Ω_0 and Ω_1 with $\Omega_0 \subset \Omega_1$, is the following. When we switch between the two controllers, we must not leave the region of attraction Ω_1. This is completely different from the problem of uniting local and global output controllers which has been solved in [21]. This difficulty has been surmounted in [24] by adding a *sufficiently large* trigger time τ^* when the "global" controller is applied to guarantee the entry of the trajectories to the region of attraction of the local controller. From a performance point of view, the strategy proposed in the cited paper is not good since we are forced to use the slow controller frequently.

To solve our problem, for the given compact set $B_{n_p}(0,R)$ and positive trigger time τ^*, we consider two controllers $\mathcal{U}_{R_0}$ and $\mathcal{U}_{\gamma(R)}$ where $\gamma(R)$ is chosen sufficiently large (see (30)). The new hybrid controller $\mathfrak{U}_R$ is equal to the fast controller $\mathcal{U}_{R_0}$ near the origin and equal to the slow controller $\mathcal{U}_{\gamma(R)}$ away from the origin. And we impose that any switch from $\mathcal{U}_{\gamma(R)}$ to $\mathcal{U}_{R_0}$ must occur after the trigger time τ^*. As we will see in the proof of Theorem 2.3, this strategy of switch is designed such that the trajectories of system (5) with hybrid controller $\mathfrak{U}_R$ do not leave the region of attraction of system (5) with continuous controller $\mathcal{U}_{\gamma(R)}$.

To guarantee a high performance with the new hybrid controller $\mathfrak{U}_R$ i. e. to minimize the use of the slow continuous controller $\mathcal{U}_{\gamma(R)}$, the trigger time τ^* can be chosen *arbitrarily small.* Noting that, if τ^* tends to 0, then $\gamma(R)$ converges to $+\infty$. In addition, if we select τ^* large then in the hybrid controller $\mathfrak{U}_R$ we will use frequently the slow continuous controller $\mathcal{U}_{\gamma(R)}$. And this is not good for the performance. Thus, to obtain a high performance we chose τ^* "not small" and "not large". An "optimal" value of τ^* (if there exists) depends on the system itself and on the semiglobal continuous controller. We believe that a computation of the optimal value of τ^* is hard even for linear systems subject to saturation actuators (3) – (4).

Finally, we note that the trigger time τ^* should not to be used when we apply the fast continuous controller $\mathcal{U}_{R_0}$, otherwise, the trajectory can leave the two regions of attraction of the two controllers.

Observation 2.5. The main drawback in the proposed hybrid controller $(\mathfrak{U}_R)_{R\geq R_0}$ is the following. According to Proposition 3.1 and for $R > R_0$, we use a continuous controller $\mathcal{U}_{\gamma(R)}$ to steer the system trajectories in the region of attraction of the fast controller $\mathcal{U}_{R_0}$, but from a practical point of view the convergence becomes slow if $\gamma(R)$ is large. This is not good from a performance point of view. It seems possible to improve the performance of the new hybrid controller $(\mathfrak{U}_R)_{R>R_0}$ by using the controller $\mathcal{U}_{R+\varepsilon}$ instead of $\mathcal{U}_{\gamma(R)}$ to steer the system trajectories in the region of attraction of the fast controller, for small positive real ε. This scenario will be feasible if we solve the problem of the unification of two local output controllers $\mathcal{U}_{R_0}$ and $\mathcal{U}_R$ with region of attractions Ω_0 and Ω_1 such that $\Omega_0 \subset \Omega_1$.

Observation 2.6. For linear systems, the detectability implies IOSS. Note that for detectable linear time unvarying system, we can construct a convergent Luenberger observer that estimates not only the norm of the state but also the state itself. We believe that when we change the two norm estimators in the hybrid controller $(\mathfrak{U}_R)_{R>R_0}$ by the state of the Luenberger observer, this leads to better improve the performance for linear systems.

3. PROOF OF THEOREM 2.3

In this section, we give a constructive proof of the main result of this paper which is summarized in Theorem 2.3. To make the proof easy to follow, we break it up into four steps. In the first step, we prove the existence of the solutions of the closed loop system (19). The second step is devoted to the local stability of the origin of the closed loop system (19). In the third step, we prove that there exists $\gamma(R) > 0$, such that all solutions of the closed loop system (19) starting from an initial condition in the set $B_{n_p}(0,R) \times B_{l_1}(0,R) \times B_{l_1}(0,R) \times \mathbb{R}^4 \cap \big(\mathbb{R}^{n_p} \times (C_R \cup D_R)\big)$ do not leave the set $B_{n_p}(0,\gamma(R)) \times B_{l_1}(0,\gamma(R)) \times B_{l_1}(0,\gamma(R)) \times \mathbb{R}^4 \cap \big(\mathbb{R}^{n_p} \times (C_R \cup D_R)\big)$. Finally, the fourth step is devoted to prove that all solutions started from $B_{n_p}(0,R) \times B_{l_1}(0,R) \times B_{l_1}(0,R) \times \mathbb{R}^4 \cap \big(\mathbb{R}^{n_p} \times (C_R \cup D_R)\big)$ converge to the origin.

3.1. Existence of hybrid solutions of the closed loop system (19)

In the following, we prove the existence of maximal solutions of the closed loop system (19) using the viability conditions

- $v_R(y,\xi) \cap T_{C_R}(\xi) \neq \emptyset$, for all $(y,\xi) \in \mathbb{R}^p \times \partial C_R \backslash D_R$,
- $w_R(y,\xi) \in C_R \cup D_R$, for all $(y,\xi) \in \mathbb{R}^p \times D_R$,

where ∂C_R denotes the boundary of C_R and $T_{C_R}(\xi)$ the tangent cone of C_R in ξ (See [9] or the recent textbook [8] for more details).

Let us prove the first item. Observe that $z_q = 0$, for all $\xi \in (\partial C_R \backslash D_R)$. From (11) and (16), we get $\dot{z}_q \geq 0$, in $(\partial C_R \backslash D_R)$, for all $q \in \{0,1\}$. Then, when flowing from the boundary of C_R, we enter C_R, which gives $v_R(y,\xi) \cap T_{C_R}(\xi) \neq \emptyset$, for all $(y,\xi) \in \mathbb{R}^p \times \partial C_R \backslash D_R$.

Concerning the second item, in view of the expression of w_R, we have $w_R(y,\xi) = (0,0,0,z_1,0,1) \in C_{1R}$ if $\xi \in D_{0R}$, and $w_R(y,\xi) = (0,0,0,z_1,0,0) \in C_{0R}$ if $\xi \in D_{1R}$. Thus, for all $(y,\xi) \in \mathbb{R}^p \times D_R$, $w_R(y,\xi) \in C_R \cup D_R$. This completes the proof of the first step.

3.2. Stability of the origin of the closed loop system (19): Proof of item (I2)

Let $R > R_0$. The proof of this step can be found in [21]. Precisely, we can choose δ small enough and independent from R such that for all the initial conditions (x^0, ξ^0) satisfying $|(x^0, \xi^0)| \leq \delta$, the trajectory of (19) do not leave C_{0R}. Thus, we obtain the local stability and moreover Item (I2) of our problem is satisfied with $M = (I_{l_1}\ 0_{l_1 \times (l_1+4)})$, where I_{l_1} is the $l_1 \times l_1$ identity matrix and $0_{l_1 \times (l_1+4)}$ is the $l_1 \times (l_1 + 4)$ null matrix.

3.3. Region of attraction of the closed loop system (19): Proof of item (I1)

This step is the main contribution of this work. In the following proposition, we prove that all the solutions of the closed loop system (19) do not leave some compact set as shown in Figure 2.

Proposition 3.1. For all $R > R_0$, there exists a positive constant $\gamma(R) > R$, such that for all initial conditions $(x^0, \xi^0) \in B_{n_p}(0,R) \times B_{l_1}(0,R) \times B_{l_1}(0,R) \times \mathbb{R}^4 \cap \big(\mathbb{R}^{n_p} \times (C_R \cup D_R)\big)$, all trajectories $(x(t,j), \xi(t,j))$ of the closed loop system (19), does not leave the set $B_{n_p}(0,\gamma(R)) \times B_{l_1}(0,\gamma(R)) \times B_{l_1}(0,\gamma(R)) \times \mathbb{R}^4 \cap \big(\mathbb{R}^{n_p} \times (C_R \cup D_R)\big)$.

P r o o f. Let $(x^0, \xi^0) \in B_{n_p}(0,R) \times B_{l_1}(0,R) \times B_{l_1}(0,R) \times \mathbb{R}^4 \cap \big(\mathbb{R}^{n_p} \times (C_R \cup D_R)\big)$ and $(x(t,j), \xi(t,j))$ a trajectory of hybrid system (19) starting from the initial condition (x^0, ξ^0) and a hybrid time domain $dom(x,\xi)$.

Let $((t_n, j_n))_{n \in I_N}$ be a sequence of hybrid time of $dom(x,\xi)$ such that $t_0 = j_0 = 0$, $(x(0,0), \xi(0,0)) = (x^0, \xi^0)$ and $I_N = \{0, 1, \ldots, N\}$, where $N \in \mathbb{N} \cup \{+\infty\}$. We assume that we have no jump between two points of the previous sequence.

In the following, we prove that there exists $\gamma(R) > R$, such that

$$(x(t,j_n), \xi_0(t,j_n), \xi_1(t,j_n)) \in B_{n_p}(0,\gamma(R)) \times B_{l_1}(0,\gamma(R)) \times B_{l_1}(0,\gamma(R)), \tag{20}$$

for all $t \in [t_n, t_{n+1}[$ and all $n \in I_N$.

Consider the case where $\xi^0 \in C_{0R}$. Using (9), from (17) and the definition of C_{0R}, we get

$$\begin{aligned} V_0(x(t,0), \xi_0(t,0)) &\leq z_0(t,0) + (V_0(x^0, \xi_0^0) + |z_0^0|)e^{-t}, \\ &\leq \varepsilon_{0a} + (\kappa_2(R) + \frac{1}{2}R^2)e^{-t} \\ &\leq \varepsilon_{0a} + \kappa_2(R) + \frac{1}{2}R^2. \end{aligned} \tag{21}$$

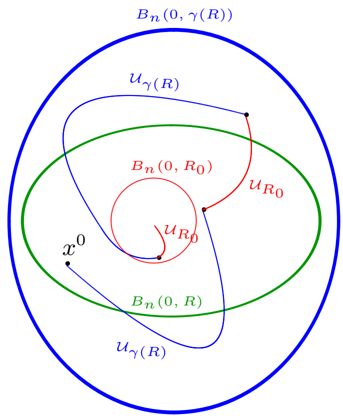


Fig. 2. Time evolution of a solution $x(t, j)$ of system (19) with the hybrid controller $\mathfrak{U}_R$.

Thus, it follows that for all $(t, 0) \in dom(x, \xi)$,

$$|x(t,0)| \leq \kappa_1^{-1}\left(\varepsilon_{0a} + \kappa_2(R) + \frac{1}{2}R^2\right) := \gamma_0(R), \tag{22}$$

and

$$|\xi_0(t,0)| \leq \sqrt{2\left(\varepsilon_{0a} + \kappa_2(R) + \frac{1}{2}R^2\right)} := \gamma_1(R). \tag{23}$$

Define

$$\gamma_2(R) = \max\{R, \gamma_0(R), \gamma_1(R)\}.$$

Since in C_{0R}, $\xi_1(t, 0) = 0$, we conclude that

$$(x(t,0), \xi_0(t,0), \xi_1(t,0)) \in B_{n_p}(0, \gamma_2(R)) \times B_{l_1}(0, \gamma_2(R)) \times B_{l_1}(0, \gamma_2(R)),$$

for all $t \in [t_0, t_1[$.

Now, we treat the case when the initial condition $\xi^0 \in D_R \backslash C_R$. In this case, we switch to C_R, with

$$(x(0,1), \xi_0(0,1), \xi_1(0,1)) = (x(0,1), 0, 0) \in B_{n_p}(0, R) \times B_{l_1}(0, R) \times B_{l_1}(0, R).$$

In view of the above discussion about the initial condition ξ^0 and neglecting the case in the subsection 3.2 where the solutions do not leave C_{0R}, without loss of generality we may assume that the trajectory $(x(t, j), \xi(t, j))$ of system (19) satisfies

- $\xi(t, j_{2n}) \in C_{1R}$, for all $t \in [t_{2n}, t_{2n+1}[$, and
- $\xi(t, j_{2n+1}) \in C_{0R}$, for all $t \in [t_{2n+1}, t_{2n+2}[$,

for all $n \in I_N$, where

$$(x(0,0), \xi_0(0,0), \xi_1(0,0)) \in B_{n_p}(0, \gamma_2(R)) \times B_{l_1}(0, \gamma_2(R)) \times B_{l_1}(0, \gamma_2(R)).$$

In the sequel, we denote

$$\tilde{R}_n := V_1(x(t_n, j_n)).$$

Using (17), and since $\xi_1(t_{2n}, j_{2n}) = 0, \xi_0(t, j_{2n}) = 0$ in C_{1R}, we have

$$V_0(x(t, j_{2n}), 0) \quad \leq \quad z_0(t, j_{2n}) + (\tilde{R}_{2n} + z_0(t_{2n}, j_{2n}))e^{-(t-t_{2n})}.$$

for all $t \in [t_{2n}, t_{2n+1}[$. From the definition of C_{0R} and D_{0R}, we deduce that $z_0(t_{2n}, j_{2n}) = \varepsilon_{0a}$. Note that since in C_{1R}, we have $\dot{z}_0(t, j_{2n}) = 0$, it yields that $z_0(t, j_{2n}) = \varepsilon_{0a}$, for all $t \in [t_{2n}, t_{2n+1}[$. Since $V_1(x) \leq V_0(x, \xi_0)$, when t tends to t_{2n+1}, it yields

$$\tilde{R}_{2n+1} \quad \leq \quad \varepsilon_{0a} + (\tilde{R}_{2n} + \varepsilon_{0a})e^{-\tau^*}. \tag{24}$$

Again, using (17) and since $\xi_0(t_{2n+1}, j_{2n+1}) = 0, z_0(t_{2n+1}, j_{2n+1}) = 0$, and $\xi_1(t, j_{2n+1}) = 0$, in C_{0R}, we get

$$V_0(x(t, j_{2n+1}), \xi_0(t, j_{2n+1})) \quad \leq \quad z_0(t, j_{2n+1}) + \tilde{R}_{2n+1}e^{-(t-t_{2n+1})},$$

for all $t \in [t_{2n+1}, t_{2n+2}[$. Since $V_1(x) \leq V_0(x, \xi_0)$ and in C_{0R}, $z_0(t, j_{2n+1}) \leq \varepsilon_{0a}$, when t tends to t_{2n+2}, it yields

$$\begin{aligned} \tilde{R}_{2n+2} \quad &\leq \quad \varepsilon_{0a} + \tilde{R}_{2n+1}, \\ &\leq \quad 2\varepsilon_{0a} + (\tilde{R}_{2n} + \varepsilon_{0a})e^{-\tau^*}. \end{aligned}$$

Thus, we get

$$\tilde{R}_{2n+2} \leq 2\varepsilon_{0a} + \varepsilon_{0a}e^{-\tau^*} + \tilde{R}_{2n}e^{-\tau^*}. \tag{25}$$

From estimation (25), it follows

$$\begin{aligned} \tilde{R}_{2n} \quad &\leq \quad \tilde{R}_0 e^{-n\tau^*} + (2\varepsilon_{0a} + \varepsilon_{0a}e^{-\tau^*}) \sum_{i=0}^{n-1} \left(e^{-\tau^*}\right)^i && (26) \\ &\leq \quad \kappa_2(\gamma_2(R)) + \left(2\varepsilon_{0a} + \varepsilon_{0a}e^{-\tau^*}\right) \frac{1}{1 - e^{-\tau^*}} := \tilde{\gamma}_2(R), && (27) \end{aligned}$$

for all $n \in I_N$. From (9), we get

$$|(x(t_{2n}, j_{2n})| \leq \kappa_1^{-1}(\tilde{\gamma}_2(R)). \tag{28}$$

Using (24), it yields

$$\begin{aligned} \tilde{R}_{2n+1} \quad &\leq \quad \varepsilon_{0a} + (\tilde{R}_{2n} + \varepsilon_{0a})e^{-\tau^*}, \\ &\leq \quad \varepsilon_{0a} + \varepsilon_{0a}e^{-\tau^*} + \tilde{\gamma}_2(R)e^{-\tau^*} := \tilde{\gamma}_3(R). \end{aligned}$$

In view of (9), we get

$$|x(t_{2n+1}, j_{2n+1})| \leq \kappa_1^{-1}(\tilde{\gamma}_3(R)). \tag{29}$$

Now, define $\gamma(R)$ as follows

$$\gamma(R) \quad = \quad \max\left\{\gamma_2(R), \kappa_1^{-1}(\tilde{\gamma}_2(R)), \sqrt{2(\varepsilon_{0a}+\tilde{\gamma}_3(R))}, \kappa_1^{-1}(\varepsilon_{0a}+\tilde{\gamma}_3(R))\right\}. \quad (30)$$

Since $\xi_1(t_n, j_n) = \xi_0(t_n, j_n) = 0$, from (28) and (29) we obtain

$$(x(t_n, j_n), \xi_0(t_n, j_n), \xi_1(t_n, j_n)) \in B_{n_p}(0, \gamma(R)) \times B_{l_1}(0, \gamma(R)) \times B_{l_1}(0, \gamma(R)).$$

Now, let prove (20). First, in $[t_{2n+1}, t_{2n+2}[$, from (17) it yields

$$\begin{aligned} V_0(x(t, j_{2n+1}), \xi_0(t, j_{2n+1})) &\leq z_0(t, j_{2n+1}) + \tilde{R}_{2n+1}e^{-(t-t_{2n+1})}, \\ &\leq \varepsilon_{0a} + \tilde{R}_{2n+1}, \\ &\leq \varepsilon_{0a} + \tilde{\gamma}_3(R). \end{aligned}$$

Then,

$$V_1(x(t, j_{2n+1})) \leq \varepsilon_{0a} + \tilde{\gamma}_3(R),$$

and

$$\frac{1}{2}|\xi_0(t, j_{2n+1})|^2 \leq \varepsilon_{0a} + \tilde{\gamma}_3(R),$$

which implies that

$$|x(t, j_{2n+1})| \leq \kappa_1^{-1}(\varepsilon_{0a} + \tilde{\gamma}_3(R)) \leq \gamma(R),$$

and

$$|\xi_0(t, j_{2n+1})| \leq \sqrt{2(\varepsilon_{0a} + \tilde{\gamma}_3(R))} \leq \gamma(R).$$

Since $\xi_1(t, j_{2n+1}) = 0$, for all $t \in [t_{2n+1}, t_{2n+2}[$, it follows that

$$(x(t, j_{2n+1}), \xi_0(t, j_{2n+1}), \xi_1(t, j_{2n+1})) \in B_{n_p}(0, \gamma(R)) \times B_{l_1}(0, \gamma(R)) \times B_{l_1}(0, \gamma(R)),$$

for all $t \in [t_{2n+1}, t_{2n+2}[$.

Finally, in $[t_{2n}, t_{2n+1}[$, $(x(t, j_{2n}), \xi_1(t, j_{2n}))$ is a trajectory of system $(\mathcal{S}_{\gamma(R)})$

$$\left\{ \begin{array}{l} \dot{x} = f(x, \alpha_{\gamma(R)}(\xi_1, y)) \\ \dot{\xi}_1 = \varphi_{\gamma(R)}(\xi_1, y) \end{array} \right. \quad (31)$$

with initial condition $(x(t_{2n}, j_{2n}), \xi_1(t_{2n}, j_{2n})) \in B_{n_p}(0, \gamma(R)) \times B_{l_1}(0, \gamma(R))$ which is included in the region of attraction of (31). Since $\xi_0(t, j_{2n}) = 0$, for all $t \in [t_{2n}, t_{2n+1}[$, it follows that $(x(t, j_{2n}), \xi_0(t, j_{2n}), \xi_1(t, j_{2n}))$ do not leave $B_{n_p}(0, \gamma(R)) \times B_{l_1}(0, \gamma(R)) \times B_{l_1}(0, \gamma(R))$, for all $t \in [t_{2n}, t_{2n+1}[$.

So, for all $n \in I_N$, we have (20) and this achieves the proof of Proposition 3.1. □

3.4. Convergence of hybrid trajectories

Let $(x^0, \xi^0) \in B_{n_p}(0, R) \times B_{l_1}(0, R) \times B_{l_1}(0, R) \times \mathbb{R}^4 \cap \left(\mathbb{R}^{n_p} \times (C_R \cup D_R)\right)$. Using three Lemmas, we prove the convergence to the origin of any hybrid trajectory of system (19) with initial condition (x^0, ξ^0).

Lemma 3.2. There exists a hybrid time $(t, j) \in dom(x, \xi)$, such that

$$q(t, j) = 0.$$

P r o o f. By contradiction, assume that

$$q(t,j) = 1, \forall (t,j) \in dom(x,\xi), \tag{32}$$

then, $dom(x,\xi) = [0,T) \times \{0\}$. If $T < +\infty$, then the hybrid trajectory $(x(t,0), \xi(t,0))$ eventually leaves any compact subset of $\mathbb{R}^{n_p} \times \mathbb{R}^l$. Since in C_{1R}, $\xi_0 = 0$, and using (11), $z_1(t,0)$ and $z_0(t,0)$, cannot grow unbounded if $(x(t,0), \xi_1(t,0))$ is bounded. Then, $(x(t,0), \xi_1(t,0))$ is a trajectory of (31) which grows unbounded. This is impossible since the initial condition (x^0, ξ_1^0) is in the set $B_{n_p}(0,\gamma(R)) \times B_{l_1}(0,\gamma(R))$, which is included in the region of attraction of system (31). Therefore, $T = +\infty$ and $(x(t,0), \xi_1(t,0))$ is trajectory of (31) starting from (x^0, ξ_1^0) in $B_{n_p}(0,R) \times B_{l_1}(0,R) \subset B_{n_p}(0,\gamma(R)) \times B_{l_1}(0,\gamma(R))$ which is included in the region of attraction of system (31), then the trajectory $(x(t,0), \xi_1(t,0))$ converges to the origin. Therefore, from (11), there exists a time $\bar{t} \geq \tau^*$, such that $z_1(\bar{t},0) \leq \varepsilon_{1a}$ and $z_0(\bar{t},0) \leq \varepsilon_{0a}$. Then, we enter successively, in D_{1R} and in C_{0R} and this contradicts (32). □

Lemma 3.3. Suppose that there exists a hybrid time $(\bar{t}, \bar{j}) \in dom(x,\xi)$, such that,

$$q(t,j) = 0, \forall (t,j) \in dom(x,\xi), \quad (t,j) \geq (\bar{t}, \bar{j}). \tag{33}$$

Then, any hybrid trajectory $(x(t,j), \xi(t,j))$ of system (19) is complete and converges to the origin.

P r o o f. Assume (33). Then, there exists $j_0 \in \mathbb{N}$, such that $\xi(t,j_0) \in C_{0R}$, for all $t \in [\bar{t}, T)$, where

$$T = \sup\{t, \exists j,\ (t,j) \in dom(x,\xi)\}.$$

By contradiction, suppose that T is finite. Then, one of the components of the state $(x(t,j), \xi(t,j))$ is unbounded. Note that in C_{0R}, $z_0(t,j_0)$ is bounded, $\xi_1(t,j_0) = 0$ and $s(t,j_0) = 0$, for all $t \in [\bar{t}, T[$. Therefore, $(x(t,j_0), \xi_0(t,j_0), z_1(t,j_0))$ is unbounded. Note that if $(x(t,j_0), \xi_0(t,j_0))$ is bounded, then $z_1(t,j_0)$ is bounded. So, $(x(t,j_0), \xi_0(t,j_0))$ is unbounded. Thus, by (16) the component $z_0(t,j_0)$ is unbounded and this contradicts with the fact that in C_{0R}, $0 \leq z_0(t,j_0) \leq \varepsilon_{0a}$. Therefore, $T = +\infty$ and the hybrid trajectory $(x(t,j), \xi(t,j))$ is complete.

Due to (17) and in $C_{0R}, z_0(t,j_0) \leq \varepsilon_{0a}$ and $\xi_1(t,j_0) = 0$, there exists a time $\tilde{t} \geq \bar{t}$, such that

$$V_0(x(\tilde{t},j_0), \xi_0(\tilde{t},j_0), 0) \leq \varepsilon_{0b},$$

and then $(x(\tilde{t},j_0), \xi_0(\tilde{t},j_0))$ belongs to $B_{n_p}(0,R_0) \times B_{l_1}(0,R_0)$ which is included in the region of attraction of system $(\mathcal{S}_{R_0})$. Hence, $(x(t,j_0), \xi_0(t,j_0))$ converges to the origin. Since systems (16) and (11) are ISS, $z_0(t,j_0)$ and $z_1(t,j_0)$ tends also to 0. Moreover, $\xi_1(t,j_0) = s(t,j_0) = q(t,j_0) = 0$, for all $t \geq \bar{t}$. Then, the hybrid trajectory $(x(t,j), \xi(t,j))$ converges to the origin. □

Lemma 3.4. There does not exist a non decreasing infinite sequence of hybrid times $((t_n, j_n))_{n \in \mathbb{N}}$ in $dom(x,\xi)$, such that,

$$q(t_{2n}, j_{2n}) = 1,\ q(t_{2n+1}, j_{2n+1}) = 0,\ \forall\, n \in \mathbb{N}. \tag{34}$$

P r o o f . By contradiction, assume that there exists a non decreasing sequence of hybrid times $(t_n, j_n) \in dom(x,\xi)$ satisfying (34), for all $n \in \mathbb{N}$. Without loss of generality, we may assume that we have no jump between two points of this sequence. From (34), the trajectory $\xi(t, j_{2n})$ is in C_{1R} for all $t \in [t_{2n}, t_{2n+1}[$, and $\xi(t, j_{2n+1})$ is in C_{0R} for all $t \in [t_{2n+1}, t_{2n+2}[$, for all $n \in \mathbb{N}$. Then,

$$t_{2n} \geq n\tau^*, \ \forall n \in \mathbb{N},$$

which implies that $T = +\infty$. Using inequality (13) and the continuity of $x(t,j)$ and $z_1(t,j)$ with respect to t, it yields

$$V_1(x(t,j)) \leq z_1(t,j) + (V_1(x^0) + |z_1^0|)e^{-t}, \tag{35}$$

for all $(t,j) \in dom(x,\xi)$. Since $z_1(t_{2N+1}, j_{2N+1}) \leq \varepsilon_{1a}$, from (35) we obtain

$$V_1(x(t_{2N+1}, j_{2N+1})) \quad \leq \quad \varepsilon_{1a} + (V_1(x^0) + |z_1^0|)e^{-N\tau^*}.$$

Then, we can find a positive integer N, such that

$$V_1(x(t_{2N+1}, j_{2N+1})) \quad \leq \quad \varepsilon_{1b}.$$

Thus, $(x(t, j_{2N+1}), \xi_0(t, j_{2N+1}))$ is a trajectory of $(\mathcal{S}_{R_0})$ for $t \in [t_{2N+1}, t_{2N+2}[$ with the initial condition $(x(t_{2N+1}, j_{2N+1}), \xi_0(t_{2N+1}, j_{2N+1}))$ which belongs in the set

$$\{(x, \xi_0), V_1(x) \leq \varepsilon_{1b}, \xi_0 = 0\}.$$

Using $(\mathcal{I}_3)$, we get

$$\rho_0(h(x(t, j_{2N+1})), \xi_0(t, j_{2N+1}), 0, \alpha_{R_0}(.), \varphi_{R_0}(.), 0) < \varepsilon_{0a}, \ \forall t \in [t_{2N+1}, t_{2N+2}[.$$

Thus, from (16) we obtain

$$z_0(t, j_{2N+1}) < \varepsilon_{0a}, \ \forall t \in [t_{2N+1}, t_{2N+2}[. \tag{36}$$

Then, we do not enter C_{1R}, abut this contradicts with (34). This achieves the proof of this lemma. □

4. ILLUSTRATIVE EXAMPLES

Example 4.1. To compare the performance of our hybrid controller with the continuous semiglobal output controller and hybrid output controller introduced in [24], we consider the following linear saturated system

$$\begin{cases} \dot{x} &= Ax + Bsat(u), \\ y &= Cx, \end{cases} \tag{37}$$

where $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$, $A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $C = (1\ 0)$ and $sat(.)$ is defined by (2) with $\bar{u} = 100$. An output controller of system (37) is given by

$$u = K\hat{x}, \tag{38}$$

where $K = (k_1, k_2) \in \mathbb{R}^2$, and $\hat{x} = \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix}$ is the solution of the linear Luenberger observer

$$\dot{\hat{x}} = A\hat{x} + Bsat(u) + L(Cx - C\hat{x}), \tag{39}$$

where $L = \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} \in \mathbb{R}^2$. The closed loop system (37) – (38) – (39) is written in a compact form as

$$\begin{pmatrix} \dot{x} \\ \dot{\hat{x}} \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ \hat{x} \end{pmatrix} + \mathbf{B}\, sat(K\hat{x}), \tag{40}$$

where, $\mathbf{A} = \begin{pmatrix} A & 0 \\ LC & A - LC \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} B \\ B \end{pmatrix}$. Gains matrices K and L are computed as follows, $K = YP^{-1}$, $L = Q^{-1}T$, where Q and P are symmetric positive definite 2×2 matrices, T is 2×1 matrix and Y is 1×2 matrix satisfying the following linear matrix inequalities (LMI):

$$PA^T + AP + BY + Y^T B^T \leq -2\lambda P, \tag{41}$$

$$QA + A^T Q - C^T T^T - TC \leq -2\lambda Q. \tag{42}$$

where λ is a positive constant. To compute a set included in the region of attraction of system (40), we can use for example [29]. Such set has the form

$$\Omega_\lambda = \left\{ \mathcal{X},\ \mathcal{X} W_\lambda^{-1} \mathcal{X}^T \leq 1 \right\},$$

where $\mathcal{X} = (x_1, x_2, \hat{x}_1, \hat{x}_2)$ and W_λ is 4×4 symmetric positive definite matrix satisfying the following LMIs:

$$\begin{pmatrix} W_\lambda \mathbf{A}^T + \mathbf{A} W_\lambda & \mathbf{B}S - Z^T \\ * & -2S \end{pmatrix} < 0, \tag{43}$$

and,

$$\begin{pmatrix} W_\lambda & W_\lambda K^T - Z^T \\ * & \bar{u}^2 \end{pmatrix} \geq 0. \tag{44}$$

It is worthy to note that the origin of the closed loop system (40) is exponentially stable with a decay rate λ and a region of attraction containing Ω_λ. Note that if λ is small, then the region of attraction Ω_λ will be large and if λ is large, then the region of attraction Ω_λ will be small. To improve the performance of the semiglobal controller (38) – (39) when Ω_λ is large, i. e. λ is small, let us construct a hybrid controller by uniting two local controllers. It is simple to see that Assumptions 1 and 3 are satisfied. For Assumption 2, we can choose the gain matrices K and L bounded for small values of λ (i. e. large values of R in Assumption 2.) by choosing in LMIs (41) and (42), $P \geq I_2$ and $YY^T \leq 1, T^T T \leq 1$.

The fast controller is chosen with $\lambda = \lambda_0 = 20$. Computations give

$$K_0 = \begin{pmatrix} -800.5492 & -41 \end{pmatrix},\ L_0 = \begin{pmatrix} 41.0008 \\ 882.5390 \end{pmatrix}$$

and

$$W_{\lambda_0} = 10^3 \begin{pmatrix} 0.0037 & -0.0281 & 0.0038 & -0.0615 \\ -0.0281 & 0.3055 & -0.0269 & 0.5488 \\ 0.0038 & -0.0269 & 0.0042 & -0.0598 \\ -0.0615 & 0.5488 & -0.0598 & 1.1615 \end{pmatrix}.$$

The maximum value of R_0 such that $B_2(0,R_0) \times B_2(0,R_0) \subset \Omega_{\lambda_0}$ is

$$R_0 = \sqrt{\frac{1}{\lambda_{\max}(W_{\lambda_0}^{-1})}} = 0.2686.$$

We use the Algorithm 4.3 in [21] to compute ε_{0b} and ε_{1b}. Calculations give $\varepsilon_{0b} = \varepsilon_{1b} = 0.3196$. Let $\varepsilon_{1a} = \varepsilon_{0a} = 0.2876$. The IOSS-Lyapunov function of system (37) is selected as follows

$$V(x_1,x_2) = (x_1,x_2)P_1(x_1,x_2)^T,$$

where P_1 is 2×2 symmetric positive definite matrix satisfying the following LMI

$$P_1A + A^TP_1 + \mathcal{M}_1C + C^T\mathcal{M}_1^T < -2P_1.$$

where $\mathcal{M}_1$ is 2×1 unknown matrix. Denote $\mathcal{M} = P_1^{-1}\mathcal{M}_1$. A solution to the previous LMI is the following

$$P_1 = \begin{pmatrix} 17.0058 & -4.0741 \\ -4.0741 & 2.0370 \end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix} -3.0001 \\ -8.3486 \end{pmatrix}.$$

Note that $\kappa_1(r) = r^2$ and $\kappa_2(r) = 18.0429r^2$. Then, the dynamics of the norm estimators z_0 and z_1 are given by (16) and (11) with

$$\rho_1(y,u) = 2(u,y)\begin{pmatrix} B^TP_1B & 0 \\ 0 & \mathcal{M}^TP_1\mathcal{M} \end{pmatrix}(u,y)^T = 2.0370u^2 + 90.9569y^2,$$

and

$$\begin{aligned} \rho_{0R_0}(h(x),\xi_0) &= 2.0370(K_0\xi_0)^2 + 90.9569x_1^2 + |\xi_0|^2 \\ &\quad + \frac{1}{2}\left|A\xi_0 + Bsat(K_0\xi_0) + L_0(Cx - C\xi_0)\right|^2, \end{aligned}$$

where $\xi_0 = (\hat{x}_{01}, \hat{x}_{02})$ and $\xi_1 = (\hat{x}_{11}, \hat{x}_{12})$ are governed by the dynamics

$$\dot{\xi}_0 = A\xi_0 + Bsat(K_0\xi_0) + L_0(Cx - C\xi_0),$$

and

$$\dot{\xi}_1 = A\xi_1 + Bsat(K_2\xi_1) + L_2(Cx - C\xi_1).$$

The slow controller is chosen with $\lambda = \lambda_1 = 7$. Solutions K_1 and L_1 to (41) and (42) with $\lambda = 7$ are given by

$$K_1 = (-98.6110 \quad -15.0008), \quad L_1 = (15.0003 \ \ 128.4948)^T.$$

To compute W_{λ_1} we solve LMIs (43) and (44) and such that $\Omega_{\lambda_0} \subset \Omega_{\lambda_1}$ which is equivalent to the following LMI

$$\begin{pmatrix} W_{\lambda_1} & I_4 \\ I_4 & W_{\lambda_0}^{-1} \end{pmatrix} \geq 0.$$

Calculations give

$$W_{\lambda_1} = 10^3 \begin{pmatrix} 0.1512 & -0.3883 & 0.1558 & -0.8331 \\ -0.3883 & 1.5713 & -0.3745 & 2.6878 \\ 0.1558 & -0.3745 & 0.1719 & -0.8057 \\ -0.8331 & 2.6878 & -0.8057 & 5.5413 \end{pmatrix},$$

and the maximum value of R such that $B_2(0,R) \times B_2(0,R) \subset \Omega_{\lambda_1}$ is

$$R = \sqrt{\frac{1}{\lambda_{\max}(W_{\lambda_1}^{-1})}} = 1.6697.$$

Now, we choose $\tau^* = 0.1$. Then, we obtain successively

- $\gamma_0(R) = \kappa_1^{-1}\left(\varepsilon_{0a} + \kappa_2(R) + \frac{1}{2}R^2\right) = 5.5409$,
- $\gamma_1(R) = \kappa_1^{-1}\big(2\varepsilon_{1a} + \tilde{\gamma}_1(R)\big) = 5.3275$,
- $\gamma_2(R) = 6.3062$,
- $\tilde{\gamma}_2(R) = 38.9062$,
- $\tilde{\gamma}_3(R) = 39.4815$.

Thus,

$$\gamma(R) = 6.3062.$$

In order to obtain $\Omega_R \subset \Omega_{\gamma(R)}$ and $B_2(0,\gamma(R)) \times B_2(0,\gamma(R)) \subset \Omega_{\gamma(R)}$, we take $\lambda = \lambda_2 = 2$, and we obtain from (41) and (42)

$$K_2 = (-8.6970 \quad -5), \quad L_2 = (5.0001 \quad 18.4350)^T.$$

Now, let us compare the performance of the continuous controller with $\lambda = \lambda_1 = 7$ with region of attraction Ω_{λ_1} that contains $B_2(0,R) \times B_2(0,R)$ to the hybrid controller which is composed of the two continuous controllers with $\lambda_0 = 20$ and $\lambda_2 = 2$, respectively, and with regions of attraction Ω_{λ_0} and Ω_{λ_2} that contain respectively $B_2(0,R_0) \times B_2(0,R_0)$ and $B_2(0,\gamma(R)) \times B_2(0,\gamma(R))$. To compare the two controllers, let $x_1^0 = 0$, $x_2^0 = 0.075$, $\hat{x}_{01}^0 = \hat{x}_{02}^0 = 0$, $\hat{x}_{11}^0 = \hat{x}_{12}^0 = 0$, $z_0^0 = 0, z_1^0 = 0, s^0 = 0, q^0 = 0$. Figure 4 shows a trajectory of system (37) in closed loop with hybrid controller when $\varepsilon_{0a} = 0.3196$, $\varepsilon_{1a} = 0.2876$ and $\tau^* = 0.1$. The trajectory starts from $(x_1, x_2) = (0, 0.075)$ with the fast controller $\mathcal{U}_{R_0}$ $(q = 0)$ until the time $t \approx 0.1$, where $z_0(0.1) \approx 0.285 > \varepsilon_{0a}$, triggering a jump to $q = 1$, thus the slow controller is used. At about $t \approx 0.2 \geq \tau^*$, z_1 reaches ε_{1a} and s is above τ^*. Then a jump to the fast controller occurs. In that mode, the trajectory converges to the origin at about $t \approx 0.55$. Figure 3 shows a trajectory to the system (37) starting from the same initial condition $(x_1, x_2) = (0, 0.075)$ with the continuous controller $\mathcal{U}_R$. We see that the trajectory converges to the origin at about $t \approx 1.2$. We conclude that the trajectory with the hybrid controller converges more rapidly than with the continuous one, which indicates that the performance of the hybrid controller is better than the continuous one.

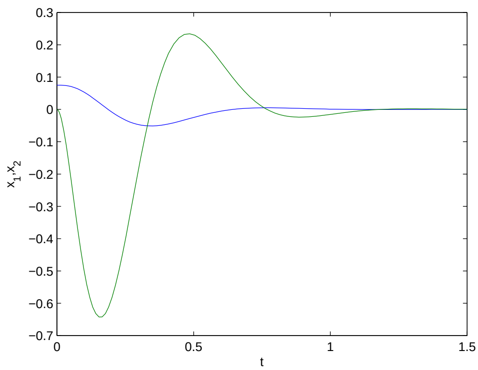


Fig. 3. Time evolution of $(x_1(t), x_2(t))$ of the system (37) in closed loop with the continuous controller with a decay rate $\lambda = \lambda_1 = 7$ and initial condition $x_1^0 = 0$, $x_2^0 = 0.075$, $\hat{x}_{01}^0 = \hat{x}_{02}^0 = 0$.

Now, we compute the hybrid controller using the design procedure in [24]. To obtain an OSS-Lyapunov functions, we rewrite system (37) in closed loop with output controller $\mathcal{U}_i, i = 0, 1$ as

$$\begin{pmatrix} \dot{x} \\ \dot{\hat{x}} \end{pmatrix} = \mathbf{A}_i \begin{pmatrix} x \\ \hat{x} \end{pmatrix} + \mathbf{B}\ sat(\mathcal{K}_i(x\ \hat{x})^T) + \begin{pmatrix} L_i \\ L_i \end{pmatrix} y, \tag{45}$$

where, $\mathbf{A}_i = diag(A - L_iC, A - L_iC)$ and $\mathcal{K}_i = (0\ 0\ K_i)$. An OSS-Lyapunov function $V_i(x, \hat{x}) = (x, \hat{x})^T W_i(x, \hat{x})$ of system (45) with output $y = Cx$ is obtained by solving the following LMI

$$\begin{pmatrix} W_i\mathbf{A}_i^T + \mathbf{A}_iW_i + 2W_i & \mathbf{B}S_i - Z_i^T \\ * & -2S \end{pmatrix} < 0, \tag{46}$$

and,

$$\begin{pmatrix} W_i & W_i\mathcal{K}_i^T - Z^T \\ * & \bar{u}^2 \end{pmatrix} \geq 0. \tag{47}$$

It is not difficult to prove that the derivative of V_i along the solutions of (45) satisfies

$$\dot{V}_i(x, \hat{x}) \leq -V_i(x, \hat{x}) + \gamma_i(|y|), \tag{48}$$

where $\gamma_i(r) = a_ir^2$ and $a_i = \begin{pmatrix} L_i \\ L_i \end{pmatrix}^T W_i \begin{pmatrix} L_i \\ L_i \end{pmatrix}$. Furthermore, $\alpha_{i,1}(r) = b_{i,1}r^2$ and $\alpha_{i,2}(r) = b_{i,2}r^2$ where $b_{i,1} = \lambda_{\min}(W_i)$ and $b_{i,2} = \lambda_{\max}(W_i)$. Let,

$$\varepsilon_{0b} = \lambda_{\max}(W_0)/\lambda_{\min}(W_{\lambda_0}),$$

such that $\Gamma_0 = \{(x, \hat{x}) \in \mathbb{R}^2 \times \mathbb{R}^2,\ (x, \hat{x})^T W_0(x, \hat{x}) \leq \varepsilon_{0b}\}$ is a subset of the region of attraction $\Omega_{\lambda_0} = \{(x, \hat{x}) \in \mathbb{R}^2 \times \mathbb{R}^2,\ (x, \hat{x})^T W_{\lambda_0}(x, \hat{x}) \leq 1\}$ for the asymptotic stabilization of the origin of system (37) with local controller $\mathcal{U}_0$ defined by (38)

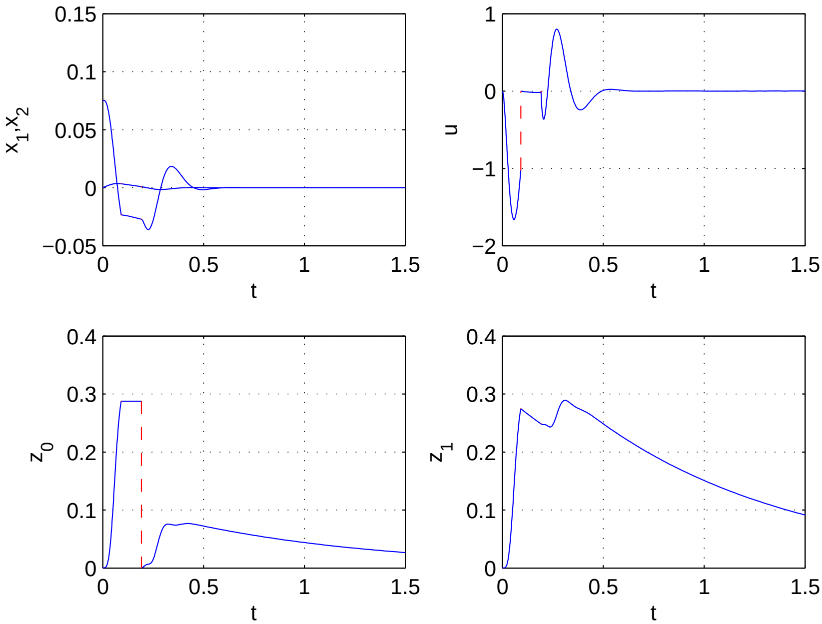


Fig. 4. Time evolution of $(x_1(t,j), x_2(t,j), z_0(t,j), z_1(t,j))$ of the system (37) in closed loop with the hybrid controller with $\lambda_2 = 2, \lambda_0 = 20$ and initial condition $x_1^0 = 0$, $x_2^0 = 0.075$, $\hat{x}_{01}^0 = \hat{x}_{02}^0 = \hat{x}_{11}^0 = \hat{x}_{12}^0 = z_0^0 = z_1^0 = s^0 = q^0 = 0$.

and (39) with $L = L_0$ and $K = K_0$. Furthermore, $\varepsilon_{1b} = \varepsilon_{0b}\lambda_{\max}(\tilde{W}_1)/\lambda_{\min}(\tilde{W}_0)$ is such that $\Gamma_1 = \{x \in \mathbb{R}^2,\ (x,0)^T W_1(x,0) \le \varepsilon_{1b}\} \times \{0\}$ is a subset of Γ_0, where $\tilde{W}_i = (I_2\ 0_{2\times 2})W_i(I_2\ 0_{2\times 2})^T$, I_2 is 2×2 identity matrix and $0_{2\times 2}$ is 2×2 null matrix. Now, we compute ε_{0a} such that every solution $\big(x(t), \hat{x}(t)\big)$ to (37) with local controller $\mathcal{U}_0$ from Γ_1 satisfies $\gamma_0\big(h_0(x(t))\big) = \gamma_0\big(y(t)\big) = a_0 y^2(t) \le \varepsilon_{0a}$. Note that Γ_1 is a subset of

$$\Omega_{\lambda_0,c} = \{(x,\hat{x}) \in \mathbb{R}^2 \times \mathbb{R}^2,\ (x,\hat{x})^T W_{\lambda_0}(x,\hat{x}) \le c\},$$

where $c = \lambda_{\max}(\tilde{W}_{\lambda_0})\varepsilon_{1b}/\lambda_{\max}(W_1)$. Since $\Omega_{\lambda_0,c}$ is an invariant set, then

$$(x(t),\hat{x}(t))^T W_{\lambda_0}(x(t),\hat{x}(t)) \le c, \quad \forall t \ge 0.$$

Thus,

$$\begin{aligned}\gamma_0\big(h_0(x(t))\big) &\le \frac{a_0}{\lambda_{\min}(W_{\lambda_0})}(x(t),\hat{x}(t))^T W_{\lambda_0}(x(t),\hat{x}(t)),\\ &\le \frac{a_0 c}{\lambda_{\min}(W_{\lambda_0})} := \varepsilon_{0a}.\end{aligned}$$

Note that $\Delta = 0, \Delta_1 = \varepsilon_{0a}$ and $\Delta_2 = \sqrt{\frac{\varepsilon_{0a}+2b_{02}(R+\varepsilon_{0a})^2}{b_{01}}}$, where Δ, Δ_1 and Δ_2 are defined in [24]. Furthermore, the following condition

$$\alpha_{1,2}\Big(\alpha_{1,1}^{-1}\big(\varepsilon_{1,a} + \bar{\beta}_1\big(\Delta + \alpha_{0,1}^{-1}\big(\Delta_1 + \bar{\beta}_0(\Delta_2, \tau^*)\big), \tau^*\big)\big)\Big) < \varepsilon_{1b}, \tag{49}$$

introduced in [24] is equivalent to

$$ae^{-2\tau^*} + be^{-\tau^*} + d < 0, \tag{50}$$

where

$$a = \frac{4b_{12}^2 b_{02}\Delta_2^2}{b_{01}b_{11}}, \quad b = \frac{2b_{12}^2\varepsilon_{0a}}{b_{01}b_{11}}, \quad d = \frac{b_{12}}{b_{11}\varepsilon_{1a}} - \varepsilon_{1b}.$$

For $\varepsilon_{1a} = 0,00001$, from (50) we obtain $\tau^* = 52.6579$ which is very large and thus, the hybrid controller of [24] frequently uses the slow controller which is not good from performance point of view. While, in this example and for our hybrid controller we have picked $\tau^* = 0, 1$.

Example 4.2. Consider the following nonlinear control system :

$$\begin{cases} \dot{x}_1 &= -x_1 + (u - x_2)x_1^2, \\ \dot{x}_2 &= -x_2 + x_1^2, \\ y &= x_1, \end{cases} \tag{51}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$ is the plant's state, $y \in \mathbb{R}$ stands for the output and $u \in \mathbb{R}$ stands for the input. A global output stabilizer of (51) is given in [3] by

$$\begin{cases} \dot{w} &= -w + y^2 - 2y^5, \\ u &= w. \end{cases} \tag{52}$$

Note that the derivative of the Lyapunov function

$$U(x_1, x_2, w) = x_1^4 + x_2^2 + (w - x_2)^2,$$

along the solutions of the closed loop system (51)−(52) satisfies

$$\dot{U}(x_1, x_2, w) \leq -3x_1^4 - x_2^2 - 2(w - x_2)^2.$$

To accelerate the convergence, we consider the controller

$$\begin{cases} \dot{w} &= -w + y^2 - 2y^5, \\ u &= w - ky, \end{cases} \tag{53}$$

where k is a positive constant. Indeed, the derivative of U along the solutions of the closed loop system (51)−(53) is such that

$$\begin{aligned} \dot{U}(x_1, x_2, w) &\leq -3x_1^4 - x_2^2 - 2(w - x_2)^2 - 4kx_1^6, \\ &\leq -U(x_1, x_2, w) - 4kx_1^6. \end{aligned}$$

We point out that for large values of k, the solutions of the closed loop system (51) – (53) converge rapidly due to the presence of the negative term $-4kx_1^6$.

From practical point of view, the controller u must be bounded. Under bounded control, the following output controller

$$(\mathcal{U}_{k,l}): \begin{cases} \dot{w} &= -w + y^2 - 2y^5, \\ u &= sat(w - ky), \end{cases} \tag{54}$$

stabilizes semiglobally the origin of system (51) by tuning the parameter k and the level of saturation l. Due to the presence of the saturation function in the controller, the region of attraction of system (51) in closed loop with the bounded controller $(\mathcal{U}_{k,l})$ is inversely proportional to k. The system (51) in closed loop with a fast controller $(\mathcal{U}_{k_0,l_0})$ can be rewritten as (7) with $\xi_0 = w$, $\alpha_{k_0}(w, y) = sat(w - k_0 y)$ and $\varphi_{k_0}(w, y) = -w + y^2 - 2y^5$, where k_0 is a positive constant will be chosen later. Moreover, $V_1(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$ is an IOSS-Lyapunov function of system (51) . In fact, simple computations give, for all (x, u) in $\mathbb{R}^2 \times \mathbb{R}$,

$$\begin{aligned} \dot{V}_1(x_1, x_2) &= -x_1^2 - x_1^3 x_2 - x_2^2 + x_2 x_1^2 + u x_1^3, \\ &\leq -x_1^2 - \frac{1}{2}x_2^2 + \frac{3}{2}x_1^6 + x_1^4 + \frac{1}{2}u^2. \end{aligned}$$

Then,

$$\dot{V}_1(x_1, x_2) \leq -V_1(x_1, x_2) + \sigma_1(|u|) + \sigma_2(|y|),$$

where $\sigma_1(|u|) = \frac{1}{2}u^2$ and $\sigma_2(|y|) = \frac{3}{2}y^6 + y^4$. Thus, the two norm estimators are given by $\dot{z}_1 = -z_1 + \rho_1(u, y)$ where $\rho_1(u, y) = \sigma_1(|u|) + \sigma_2(|y|)$ and $\dot{z}_0 = -z_0 + \rho_{0k_0}(y, w)$ where $\rho_{0k_0}(y, w) = \sigma_1(|\alpha_{k_0}(w, y)|) + \sigma_2(y) + \frac{1}{2}\varphi_{k_0}^2(w, y) + w^2$.

For a given positive constant k, an invariant region of attraction for the closed loop system (51) – (54) is given by

$$\Omega_{k,c} = \{(x_1, x_2, w) \in \mathbb{R}^3,\ U(x_1, x_2, w) \leq c\},$$

where c is chosen such that

$$\Omega_{k,c} \subset \{(x_1, x_2, w) \in \mathbb{R}^3,\ -l \leq w - kx_1 \leq l\}. \tag{55}$$

Let $(x_1, x_2, w) \in \Omega_{k,c}$. By using $-\frac{w^2}{2} - 2x_2^2 \leq -2wx_2$, we obtain

$$x_1^4 + \frac{w^2}{2} \leq U(x_1, x_2, w) \leq c.$$

Thus, $w \leq \sqrt{2(c - x_1^4)}$ and $x_1^4 \leq c$, it follows

$$w - kx_1 \leq \sqrt{2(c - x_1^4)} - kx_1, \quad \text{for } x_1 \in [-\sqrt[4]{c}, \sqrt[4]{c}].$$

Then, by picking

$$l = \max_{x_1 \in [-\sqrt[4]{c},\, \sqrt[4]{c}]} \sqrt{2(c - x_1^4)} - kx_1, \tag{56}$$

(55) holds.

To put ourself in the setting of the assumption, we give a sufficient condition relating R and c such that

$$B_2(0,R)\times]-R,R[\subset\Omega_{k,c}. \tag{57}$$

Let $(x_1,x_2,w)\in B_2(0,R)\times]-R,R[$, we have $x_2^2\le R^2-x_1^2$ and $w^2\le R^2$. Thus,

$$\begin{aligned} U(x_1,x_2,w)=x_1^4+x_2^2+(w-x_2)^2 &\le x_1^4+3x_2^2+2w^2,\\ &\le x_1^4+3(R^2-x_1^2)+2R^2,\\ &\le x_1^4-3x_1^2+5R^2. \end{aligned}$$

By maximizing the function $x_1^4-3x_1^2+5R^2$ on the interval $]-R,R[$, we conclude that

$$U(x_1,x_2,w)\le\begin{cases} 5R^2, & \text{if } R\le\sqrt{\frac{3}{2}},\\ \max(5R^2,R^4+2R^2), & \text{if } R\ge\sqrt{\frac{3}{2}}. \end{cases}$$

So, $U(x_1,x_2,w)\le\max(5R^2,R^4+2R^2)$. Therefore by letting

$$\max(5R^2,R^4+2R^2)=c,$$

(57) holds. Simple computations give

$$R=\max(\sqrt{c/5},\sqrt{-1+\sqrt{1+c}}). \tag{58}$$

Now, we find a positive constant ε_{0b} that satisfies item $(\mathcal{I}_1)$, i.e.

$$\{(x_1,x_2,w),\ U(x_1,x_2,w)\le\varepsilon_{0b}\}\subset B_2(0,R)\times]-R,R[. \tag{59}$$

Let $(x_1,x_2,w)\in\{(x_1,x_2,w),\ U(x_1,x_2,w)\le\varepsilon_{0b}\}$. Using the estimation $-\frac{w^2}{2}-2x_2^2\le -2x_2w$, we get

$$\frac{w^2}{2}\le x_1^4+\frac{w^2}{2}\le x_1^4+x_2^2+(w-x_2)^2\le\varepsilon_{0b}. \tag{60}$$

Then, if we pick $\varepsilon_{0b}\le\frac{R^2}{2}$, it follows that $w\in]-R,R[$. Furthermore, from (60) we obtain $x_2^2\le\varepsilon_{0b}-x_1^4$, where $x_1\in[-\sqrt[4]{\varepsilon_{0b}},\ \sqrt[4]{\varepsilon_{0b}}]$. It yields $x_1^2+x_2^2\le x_1^2+\varepsilon_{0b}-x_1^4$. Maximizing the function $x_1^2+\varepsilon_{0b}-x_1^4$ with respect to x_1 over the interval $[-\sqrt[4]{\varepsilon_{0b}}\sqrt[4]{\varepsilon_{0b}}]$, we get

$$\begin{cases} \sqrt{\varepsilon_{0b}}<R^2 & \text{if } \sqrt{\varepsilon_{0b}}\le\frac{1}{2},\\ \varepsilon_{0b}+\frac{1}{4}<R^2 & \text{if } \sqrt{\varepsilon_{0b}}\ge\frac{1}{2}. \end{cases}$$

Then, if we pick ε_{0b} satisfying

$$\max(\sqrt{\varepsilon_{0b}},\varepsilon_{0b}+\frac{1}{4})<R^2, \tag{61}$$

(59) is verified and then $(\mathcal{I}_1)$ holds.

Now, we move to design a positive constant ε_{1b} that satisfies $(\mathcal{I}_3)$. Let $(x_1(0), x_2(0), w(0))$ an initial condition in the set $\{(x_1, x_2, w),\ V_1(x_1, x_2) \leq \varepsilon_{1b}, w = 0\}$, then x_1 is in $[-\sqrt{2\varepsilon_{1b}}, \sqrt{2\varepsilon_{1b}}]$. First, we find a sufficiently small positive constant $\tilde{c}$ such that

$$\{(x_1, x_2, w),\ V_1(x_1, x_2) \leq \varepsilon_{1b}, w = 0\} \subset \Omega_{k,\tilde{c}}. \tag{62}$$

Since $w = 0$ and $x_2^2 \leq 2\varepsilon_{1b} - x_1^2$, we have and

$$U(x_1, x_2, w) = x_1^4 + 2x_2^2 \leq x_1^4 + 4\varepsilon_{1b} - 2x_1^2.$$

By maximizing the function $x_1^4 + 4\varepsilon_{1b} - 2x_1^2$ with respect to x_1 over the interval $[-\sqrt{2\varepsilon_{1b}}, \sqrt{2\varepsilon_{1b}}]$, we obtain $U(x_1, x_2, w) \leq 4\varepsilon_{1b}$. Then by letting $\tilde{c} = \min(1, 4\varepsilon_{1b}) < c$, the initial condition $(x_1(0), x_2(0), w(0))$ is in the region $\Omega_{k,\tilde{c}}$ and then (62) holds. Moreover, for all $t \geq 0$, we have $x_1^4(t) \leq U(x_1(t), x_2(t), w(t)) \leq \tilde{c} \leq 1$.

Furthermore, we find a relation between ε_{0a} , ε_{1b} and k such that item $(\mathcal{I}_3)$ holds. In fact, since the initial condition $(x_1(0), x_2(0), w(0))$ is in the set $\{(x_1, x_2, w), V_1(x_1, x_2) \leq \varepsilon_{1b}, w = 0\}$ which is included in $\Omega_{k,\tilde{c}}$, then the trajectory $(x_1(t), x_2(t), w(t))$ of (51)–(54) don't leave the invariant set $\Omega_{k,\tilde{c}} \subset \Omega_{k,c} \subset \{(x_1, x_2, w),\ -l \leq w - kx_1 \leq l\}$. Thus, we have $sat(w(t) - kx_1(t)) = w(t) - kx_1(t)$ for all $t \geq 0$. Furthermore, for all $t \geq 0$, we have

$$\begin{aligned}
\rho_{0k}(y(t), w(t)) &= \frac{1}{2}(w - kx_1)^2 + \frac{3}{2}x_1^2 + x_1^4 + \frac{1}{2}(-w + x_1^2 - 2x_1^5)^2 + w^2, \\
&\leq 2w^2 + k^2x_1^2 + \frac{3}{2}x_1^2 + x_1^4 + \frac{3}{2}(w^2 + x_1^4 + 4x_1^{10}), \\
&\leq \frac{7}{2}w^2 + (k^2 + \frac{3}{2})x_1^2 + \frac{5}{2}x_1^4 + 6x_1^{10}, \\
&\leq \frac{7}{2}w^2 + (k^2 + \frac{3}{2})x_1^2 + \frac{17}{2}x_1^4, \\
&\leq (k^2 + \frac{3}{2})2\varepsilon_{1b} + \frac{17}{2}(w^2 + x_1^4),
\end{aligned} \tag{63}$$

where the estimations $x_1^{10} \leq x_1^4$ and $x_1^2 \leq 2\varepsilon_{1b}$ are used. We point out that we have omitted to indicate any time-dependence in the above estimations. Using $w^2 = (w - x_2)^2 + x_2^2 + 2x_2(w - x_2)$, from (63), it yields

$$\begin{aligned}
\rho_{0k}(y(t), w(t)) &\leq (2k^2 + 3)\varepsilon_{1b} + \frac{17}{2}(x_2^2 + (w - x_2)^2 + 2x_2(w - x_2) + x_1^4), \\
&\leq (2k^2 + 3)\varepsilon_{1b} + \frac{17}{2}(x_1^4 + 2x_2^2 + 2(w - x_2)^2), \\
&\leq (2k^2 + 3)\varepsilon_{1b} + 17U(x_1, x_2, w), \\
&\leq (2k^2 + 3)\varepsilon_{1b} + 17\tilde{c}.
\end{aligned}$$

Thus, picking ε_{1b} such that

$$17\min(1, \varepsilon_{1b}) + (2k^2 + 3)\varepsilon_{1b} \leq \varepsilon_{0a}, \tag{64}$$

it yields

$$\rho_{0k}(y(t), w(t)) \leq \varepsilon_{0a}, \quad \text{for all } t \geq 0.$$

Then item $(\mathcal{I}_3)$ holds. Since the origin of $\mathbb{R}^3$ is asymptotically stable for the system (51) – (54), item $(\mathcal{I}_4)$ holds for a sufficiently small positive value ε_2.

By picking $c = 10$ and $k = 1$. From (56) and (58), we get $l = 3.168$, $R = 1.52$. Furthermore $\varepsilon_{0b} = 5.3668$ satisfies the condition (61). Let $\varepsilon_{0a} = 4.8301 \leq \varepsilon_{0b}$, thus (64) holds by choosing $\varepsilon_{1b} = 0.2195$. Let $\varepsilon_{1a} = 0.1976 < \varepsilon_{1b}$. By using (30) in the procedure of our work, we compute the controller $(\mathcal{U}_{k_\gamma,l_\gamma})$, we find $\gamma(R) = 17.58$. Pick $k_\gamma = 0.5$, thus the conditions (56) and (58) give $l_\gamma = 13373$. For the fast controller $(\mathcal{U}_{k_0,l_0})$, pick $c_0 = 3.12$, $k_0 = 10$, a computations give $R_0 = 1.015$ and $l_0 = 7.81$.

Now, we compute the hybrid controller of [24], by breaking up the three steps of Corollary 3.8. An OSS-Lyapunov function of system (51) in closed loop with output controller (54) is given by

$$V(x_1, x_2, w) = V_1(x_1, x_2) + 3w^2 = \frac{1}{2}(x_1^2 + x_2^2 + 6w^2).$$

In fact, the derivative of V along the trajectory of the closed loop system (51) – (54) gives

$$\dot{V}(x_1, x_2, w) \leq -V(x_1, x_2, w) + \phi_k(y),$$

where $\phi_k(y) = k^2y^2 + 9y^4 + 36y^{10}$. Furthermore, we have

$$\alpha_{k,1}(|(x_1, x_2, w)|) \leq V(x_1, x_2, w) \leq \alpha_{k,2}(|(x_1, x_2, w)|),$$

where, $\alpha_{k,1}(r) = \frac{1}{2}r^2$, and $\alpha_{k,2}(r) = 3r^2$.

In the step 1 of the design procedure in Corollary 3.8, we find ε_{0b} to obtain the inclusion

$$\Gamma_{k_0} = \{(x_1, x_2, w) \in \mathbb{R}^3,\ V(x_1, x_2, w) \leq \varepsilon_{0b}\} \subset \Omega_{k_0,c}. \tag{65}$$

We have

$$V(x_1, x_2, w) \quad \geq \quad \frac{1}{2}(x_1^2 + x_2^2 + w^2),$$

and

$$\begin{aligned} U(x_1, x_2, w) &= x_1^4 + x_2^2 + (w - x_2)^2, \\ &\leq 3(x_1^4 + x_2^2 + w^2). \end{aligned}$$

Thus, if $(x_1, x_2, w) \in \Gamma_{k_0}$, $\frac{1}{2}(x_1^2 + x_2^2 + w^2) \leq \varepsilon_{0b}$, it follows $U(x_1, x_2, w) \leq 6\varepsilon_{0b}$. By picking $\varepsilon_{0b} = \frac{c}{6}$, (65) holds. We move to the step 2. Pick $\varepsilon_{1b} \leq \varepsilon_{0b}$, it follows that

$$\Gamma_k = \{(x_1, x_2, 0) \in \mathbb{R}^3,\ V(x_1, x_2, w) \leq \varepsilon_{1b}\} \subset \Gamma_{k_0}.$$

Indeed, if $(x_1, x_2, w) \in \Gamma_k$, we have $V(x_1, x_2, w) = V(x_1, x_2, 0) = \frac{1}{2}(x_1^2 + x_2^2) \leq \varepsilon_{1b}$. Then if $\varepsilon_{1b} \leq \varepsilon_{0b}$, $(x_1, x_2, w) \in \Gamma_{k_0}$. Moreover, we compute ε_{0a} such that every trajectory of (51) with the local controller $(\mathcal{U}_{k_0,l_0})$ starting from the Γ_1 satisfies $\phi_{k_0}(y) = k_0^2y^2 + 9y^4 + 36y^{10} \leq \varepsilon_{0a}$. Using the fact that Γ_1 is a subset of the invariant set $\{(x_1, x_2, w) \in$

$\mathbb{R}^3$, $U(x_1, x_2, w) \le c_1\}$, where $c_1 = 4\varepsilon_{1b} < 1$, it follows that $y(t) = x_1(t) < 1$ for all $t \ge 0$. We deduce that

$$\begin{aligned}\phi_{k_0}(y(t)) = k_0^2 x_1^2 + 9x_1^4 + 36x_1^{10} &\le k_0^2 x_1^2 + 45x_1^4,\\ &\le 2k_0^2\sqrt{\varepsilon_{1b}} + 45c_1,\\ &= 2k_0^2\sqrt{\varepsilon_{1b}} + 180\varepsilon_{1b} := \varepsilon_{0a}.\end{aligned}$$

Pick $\varepsilon_{1b} = 0.0075$, we have $\varepsilon_{0a} = 1.5 \le \varepsilon_{0b} = \frac{c}{6} = 1.6667$. To design ε_{1a} and τ^* in step 3, first, we obtain the following values after straightforward computations: $\Delta = 0$, $\Delta_1 = \varepsilon_{0a}$, $\Delta_2 = \alpha_{k_0,1}^{-1}(\varepsilon_{0a} + \alpha_{k_0,2}(R + \varepsilon_{0a}))$, $\alpha_{k_0,1}^{-1}(r) = \sqrt{2r}$, $\alpha_{k_0,2}(r) = 3r^2$ and $\alpha_{k,1}^{-1}(r) = \sqrt{2r}$. Using $\varepsilon_{0a} = 1.5$ then $\Delta_2 = 10.611$. Moreover, the condition in Step 3

$$\alpha_{k,2}\Big(\alpha_{k,1}^{-1}\big(\varepsilon_{1,a} + \bar\beta_k\big(\Delta + \alpha_{k_0,1}^{-1}\big(\Delta_1 + \bar\beta_{k_0}(\Delta_2, \tau^*)\big), \tau^*\big)\big)\Big) < \varepsilon_{1b},$$

where $\bar\beta_k(r, s) = \alpha_{k,2}(r)e^{-s}$ is satisfied with $\varepsilon_{1a} = 0.0012$ and $\tau^* = 12.11$.

Now, we compare the performance of our controller to the performance of controller proposed by [24]. Figure 5 and Figure 6 show the trajectories of system (51) in closed loop with the hybrid controller of our approach and the hybrid controller proposed in [24], respectively.

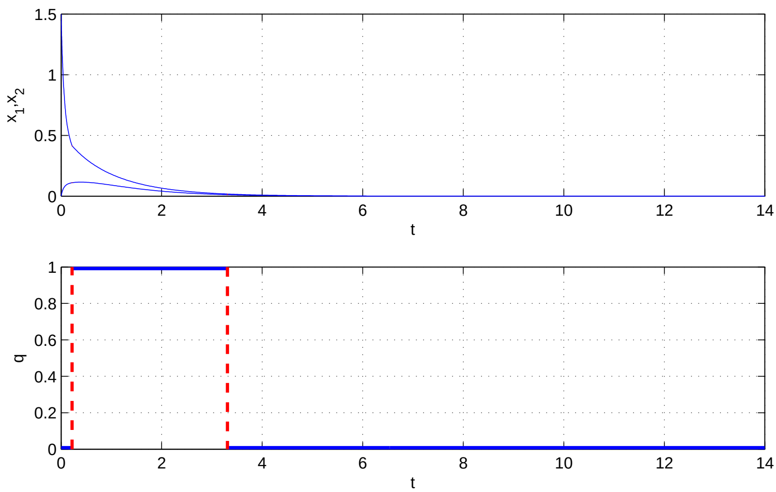


Fig. 5. Time evolution of $(x_1(t), x_2(t))$ and $q(t)$ of the system (51) in closed loop with the hybrid controller of our approach with $k = 1, l = 3.168$ and $k_0 = 10$, $l_0 = 7.81$ with initial condition $x_1^0 = 1.5$, $x_2^0 = 0$, $w^0 = z_0^0 = z_1^0 = s^0 = q^0 = 0$.

In Figure 5, the trajectory starts with fast controller and a switch occurs to the slow controller at about the time $t \approx 0.25$ since the local norm estimator z_0 becomes greater to ε_{0a}. At about $t \approx 3.3$, the norm estimator z_1 becomes less than ε_{1a} and then a switch to the fast controller occurs.

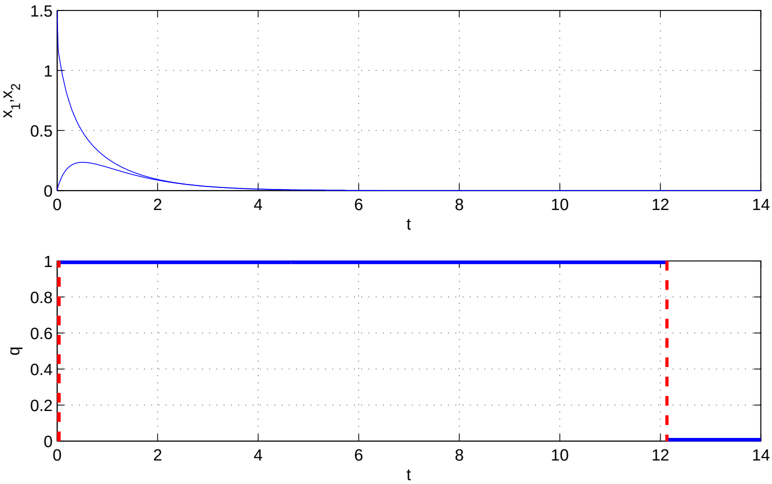


Fig. 6. Time evolution of $(x_1(t), x_2(t))$ and $q(t)$ of the system (51) in closed loop with the hybrid controller of [24] with $k = 1, l = 3.168$ and $k_0 = 10,\ l_0 = 7.81$ with initial condition $x_1^0 = 1.5,\ x_2^0 = 0,$ $w^0 = z_0^0 = z_1^0 = s^0 = q^0 = 0.$

To emphasize the difference between the two hybrid controllers, we plot $U(x_1(t), x_2(t), w(t))$. As showed by Figure 7, after the time $t \approx 0.5$, $U(x_1(t), x_2(t), w(t))$ becomes close to zero. While, in Figure 8, using the strategy of [24], $U(x_1(t), x_2(t), w(t))$ becomes close to zero after the time $t \approx 1.5$. Hence, the convergence to the origin of the trajectories using our strategy is rapid than the convergence of the trajectories using the strategy of [24]. The main drawback of the strategy introduced in [24] is that τ^* is selected sufficiently large to guaranteed that the solution with global controller enters the region of attraction of the system in closed loop with local controller.

5. CONCLUSION AND DISCUSSION

For a given family of dynamic output feedback controllers $(\mathcal{U}_R)_{R>0}$ that ensures semiglobal stability of the origin of an IOSS nonlinear control system, we propose a new hybrid output controller $(\mathfrak{U}_R)_{R>R_0}$ that preserves the semiglobal stability of the origin and locally improves the performance. The new hybrid output controller is based on two norm estimators that estimate the norm of the state and a timer to trigger τ^* the switch between a fast controller and a slow one. Note that the trigger time τ^* can be chosen arbitrarily small contrarily to [24] where it is chosen sufficiently large.

Unfortunately, there are some drawbacks in the proposed hybrid controller $(\mathfrak{U}_R)_{R>R_0}$ and it can be improved in the future in many directions. First, according to proposition 3.1 and for $R > R_0$, we use a controller $\mathcal{U}_{\gamma(R)}$ to steer the system trajectories in the region of attraction of the fast controller $\mathcal{U}_{R_0}$, but from a practical point of view the

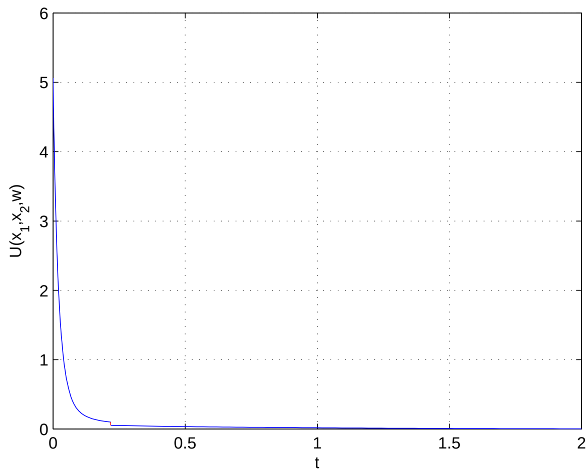


Fig. 7. Time evolution of $U(x_1(t), x_2(t), w(t))$ of the system (51) in closed loop with the hybrid controller of our approach with $k = 1, l = 3.168$ and $k_0 = 10,\ l_0 = 7.81$ with initial condition $x_1^0 = 1.5$, $x_2^0 = 0,\ w^0 = z_0^0 = z_1^0 = s^0 = q^0 = 0$.

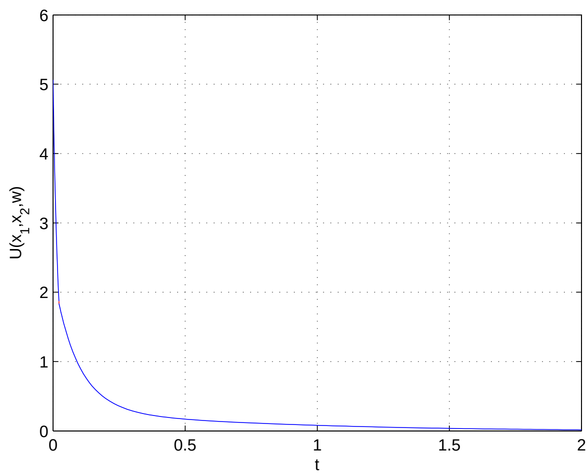


Fig. 8. Time evolution of $U(x_1(t), x_2(t), w(t))$ of the system (51) in closed loop with the hybrid controller of [24] with $k = 1, l = 3.168$ and $k_0 = 10,\ l_0 = 7.81$ with initial condition $x_1^0 = 1.5,\ x_2^0 = 0$, $w^0 = z_0^0 = z_1^0 = s^0 = q^0 = 0$.

convergence becomes slow if $\gamma(R)$ is large. It seems possible to improve the performance of the new hybrid controller $(\mathfrak{U}_R)_{R>R_0}$ by using the controller $\mathcal{U}_{R+\varepsilon}$ instead of $\mathcal{U}_{\gamma(R)}$ to steer the system trajectories in the region of attraction of the fast controller, where ε is small positive real number. This can be possible if we solve the problem of uniting two local output controllers.

Moreover, the number of switch between the fast controller $\mathcal{U}_{R_0}$ and the slow controller $\mathcal{U}_{\gamma(R)}$ can be great. We believe that it is possible to diminish the number of switch by accelerating the convergence of the norm estimators.

Finally, the IOSS assumption is restrictive. Recently, as shown in [24] it is possible to replace the IOSS assumption by output-to-state stability (OSS) of the two closed loop systems with two continuous output controllers.

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REFERENCES

[1] T. Ahmed-Ali, V. Van Assche, J. Massieu, and P. Dorleans: Continuous-discrete observer for state affine systems with sampled and delayed measurements. IEEE Trans. Automat. Control *58* (2013), 4, 1085–1091. DOI:10.1109/tac.2012.2225555

[2] T. Ahmed-Ali, I. Karafyllis, and F. Lamnabhi-Lagarrigue: Global exponential sampled-data observers for nonlinear systems with delayed measurements. System Control Lett. *62* (2013), 7, 539–549. DOI:10.1016/j.sysconle.2013.03.008

[3] V. Andrieu and L. Praly: A unifying point of view on output feedback designs for global asymptotic stabilization. Automatica *45* (2009), 8, 1789–1798. DOI:10.1016/j.automatica.2009.04.015

[4] D. V. Efimov: Uniting global and local controllers under acting disturbances. Automatica *42* (2006), 489–495. DOI:10.1016/j.automatica.2005.11.003

[5] D. V. Efimov, A. Loria, and E. Panteley: Robust output stabilization: improving performance via supervisory control. Int. J. Robust Nonlinear Control *21* (2011), 10, 1219–1236. DOI:10.1002/rnc.1660

[6] R. Freeman and P. Kokotovic: Robust Nonlinear Control Design: State-Space and Lyapunov Techniques. Birkhauser, Boston 1996. DOI:10.1007/978-0-8176-4759-9

[7] J. Gauthier, H. Hammouri, and S. Othman: A simple observer for nonlinear systems: Application to bioreactor. IEEE Trans. Automat. Control *37* (1992), 875–880. DOI:10.1109/9.256352

[8] R. Goebel, R. G. Sanfelice, and A. R. Teel: Hybrid Dynamical Systems: Modeling, Stability, and Robustness. Princeton University Press 2012.

[9] R. Geobel and A. R. Teel: Solutions to hybrid inclusions via set and graphical convergence with stability theory applications. Automatica *42* (2006), 573–587. DOI:10.1016/j.automatica.2005.12.019

[10] A. Isidori: Nonlinear Control Systems. (Third edition) Springer Verlag, London 1995. DOI:10.1007/978-1-84628-615-5

[11] Z. P. Jiang: Discussion on the paper "Global asymptotic output feedback stabilization of feedforward systems", by F. Mazenc and J. C. Vivalda. Europ. J. Control *8* (2002), 6, 531–534. DOI:10.3166/ejc.8.531-534

[12] P. Jouan and J. Gauthier: Finite singularities of nonlinear systems, output stabilization, observability and observers. J. Dynamical Control Systems *2* (1996), 2, 255-288. DOI:10.1007/bf02259528

[13] H. Khalil and F. Esfandiari: Semiglobal stabilization of a class of nonlinear systems using output feedback. IEEE Trans. Automat. Control *38* (1993), 2, 1412–1415. DOI:10.1109/9.237658

[14] M. Krichman, E. Sontag, and Y. Wang: Input-output-to-state stability. SIAM J. Control Optim. *39* (2001), 1874–1928. DOI:10.1137/s0363012999365352

[15] R. Marino and P. Tomei: A class of globally output feedback stabilizable nonlinear nonminimum phase systems. IEEE Trans. Automat. Control *50* (2005), 2097–2101. DOI:10.1109/tac.2005.858652

[16] F. Mazenc and J. C. Vivalda: Global asymptotic output feedback stabilization of feedforward systems. Europ. J. Control *8* (2002), 6, 519–530. DOI:10.3166/ejc.8.519-530

[17] Z. Pan, K. Ezal, A. J. Krener, and P. V. Kokotovic: Backstepping design with local optimality matching. IEEE Trans. Automat. Control *46* (2001), 7, 1014–1027. DOI:10.1109/9.935055

[18] L. Praly: Asymptotic stabilization via output feedback for lower triangular systems with output dependent incremental rate. IEEE Trans. Automat. Control *48* (2003), 12, 1103–1108. DOI:10.1109/tac.2003.812819

[19] L. Praly and A. R. Teel: Tools for semiglobal stabilization by partial state and output feedback. SIAM J. Control Optim. *33* (1995), 5, 1443–1488. DOI:10.1137/s0363012992241430

[20] C. Prieur, R. Goebel, and A. R. Teel: Hybrid feedback control and robust stabilization of nonlinear systems. IEEE Trans. Automat. Control *52* (2007), 11, 2103–2117. DOI:10.1109/tac.2007.908320

[21] C. Prieur and A. R. Teel: Uniting local and global output feedback controllers. IEEE Trans. Automat. Control *56* (2011), 1636–1649. DOI:10.1109/tac.2010.2091436

[22] C. Qian and W. Lin: Output feedback control of a class of nonlinear systems: a nonseparation principle paradigm. IEEE Trans. Automat. Control *47* (2002), 10, 1710–1715. DOI:10.1109/tac.2002.803542

[23] R. G. Sanfelice and R. Goebel: Generalized solutions to hybrid dynamical systems. ESAIM: Control, Optimisation and Calculus of Variations *14* (2008), 699–724. DOI:10.1051/cocv:2008008

[24] R. G. Sanfelice and C. Prieur: Robust supervisory control for uniting two output-feedback hybrid controllers with different objectives. Automatica *49* (2013), 1958–1969. DOI:10.1016/j.automatica.2013.03.009

[25] Y. Shen, D. Zhang, Y. Huang, and Y. Liu: Global $\mathcal{K}$-exponential stabilization of a class of nonlinear networked control systems. Int. J. Systems Sci. *47* (2016), 15, 3545–3553. DOI:10.1080/00207721.2015.1091899

[26] Y. Shen, D. Zhang, and X. Xia: Continuous output feedback stabilization for nonlinear systems based on sampled and delayed output measurements. Int. J. Robust. Nonlinear Control *26* (2016), 14, 3075–3087. DOI:10.1080/00207721.2015.1091899

[27] E. D. Sontag: Mathematical Control Theory: Deterministic Finite Dimensional Systems. (Second edition) Springer, New York 1998. DOI:10.1007/978-1-4612-0577-7

[28] E. D. Sontag and Y. Wang: Output-to-state stability and detectability of nonlinear systems. Systems Control Lett. *29* (1997), 5, 279–290. DOI:10.1016/s0167-6911(97)90013-x

[29] S. Tarbouriech, G. Garcia, J. M. G. da Silva Jr., and I. Queinnec: Stability and Stabilization of Linear Systems with Saturating Actuators. Springer-Verlag, London 2011. DOI:10.1007/978-0-85729-941-3

[30] A. R. Teel and N. Kapoor: Uniting global and local controllers. In: European Control Conference, Brussels 1997.

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