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EXTENDING GENERALIZED WHITNEY MAPS

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Abstract. For metrizable continua, there exists the well-known notion of a Whitney map. If X is a nonempty, compact, and metric space, then any Whitney map for any closed subset of 2^X can be extended to a Whitney map for 2^X [3, 16.10 Theorem].

The main purpose of this paper is to prove some generalizations of this theorem.

1. Introduction

All spaces in this paper are compact Hausdorff and all mappings are continuous. The weight of a space X is denoted by $\mathrm{w}(X)$. The cardinality of a set A is denoted by $\mathrm{card}(A)$.

Let X be a space. We define its hyperspaces as the following sets:

$$\begin{aligned} 2^X &= \{F \subseteq X : F \text{ is closed and non-empty}\}\,, \\ C(X) &= \{F \in 2^X : F \text{ is connected}\}\,, \\ X(n) &= \{F \in 2^X : F \text{ has at most } n \text{ points}\},\ n \in \mathbb{N}\,. \end{aligned}$$

For any finitely many subsets $S_1, \dots, S_n$, let

$$\langle S_1, \dots, S_n \rangle = \Big\{ F \in 2^X : F \subset \bigcup_{i=1}^{n} S_i, \text{ and } F \cap S_i \neq \emptyset, \text{ for each } i \Big\}\,.$$

The topology on 2^X is the Vietoris topology, i.e., the topology with a base $\{\langle U_1, \dots, U_n \rangle : U_i$ is an open subset of X for each i and each $n < \infty\}$, and $C(X)$ is a subspace of 2^X.

Let X and Y be the spaces and let $f\colon X \to Y$ be a mapping. Define $2^f : 2^X \to 2^Y$ by $2^f(F) = \{f(x) : x \in F\}$ for $F \in 2^X$. By [7, Theorem 5.10, p. 170] 2^f is continuous and $2^f(C(X)) \subset C(Y)$. The restriction $2^f \mid C(X)$ is denoted by $C(f)$.

The concept of Whitney maps is a very powerful tool in hyperspace theory of metric compact spaces. In the 1930's, Hassler Whitney constructed special types of functions on spaces of sets for the purpose of studying families of curves ([13] and [14]). In 1942, J. L. Kelley made significant use of Whitney's functions in studying

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hyperspaces of metric continua [5]. In 1978, Whitney's functions are called Whitney maps (see Chapter XIV of [8]).

Definition 1.1. Let Λ be a subspace of 2^X. By a *Whitney map* for Λ [8, p. 24, (0.50)] we will mean any mapping $g\colon \Lambda \to [0, +\infty)$ satisfying

a) if A, $B \in \Lambda$ such that $A \subset B$ and $A \neq B$, then $g(A) < g(B)$ and

b) $g(\{x\}) = 0$ for each $x \in X$ such that $\{x\} \in \Lambda$.

If X is a metric continuum, then there exists a Whitney map for 2^X and $C(X)$ [8, pp. 24–26], [3, p. 106]. If X is a metric continuum, then so is $C(X)$.

Whitney maps and Whitney levels are widely used in the theory of metrizable continua, for details, see the book [3].

In [1] first examples are presented of non-metrizable continua X which admit and ones which do not admit a Whitney map $\mu\colon C(X) \to [0, 1]$.

In the sequel we shall use the notion of inverse system as in [2, pp. 135-142]. An inverse system is denoted by $\mathbf{X} = \{X_a, p_{ab}, A\}$.

An element $\{x_a\}$ of the Cartesian product $\prod\{X_a : a \in A\}$ is called a *thread* of $\mathbf{X}$ if $p_{ab}(x_b) = x_a$ for any a, $b \in A$ satisfying $a \leq b$. The subspace of $\prod\{X_a : a \in A\}$ consisting of all threads of $\mathbf{X}$ is called the limit of the inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ and is denoted by $\lim \mathbf{X}$ or by $\lim\{X_a, p_{ab}, A\}$ [2, p. 135].

Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of compact spaces with the natural projections $p_a\colon \lim \mathbf{X} \to X_a$, for $a \in A$. Then $2^{\mathbf{X}} = \{2^{X_a}, 2^{p_{ab}}, A\}$, $\mathbf{X}(2) = \{X_a(2), p_{ab}(2), A\}$ and $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ form inverse systems.

Lemma 1.1. *Let $X = \lim \mathbf{X}$. Then $2^X = \lim 2^{\mathbf{X}}$, $X(2) = \lim \mathbf{X}(2)$ and $C(X) = \lim C(\mathbf{X})$.*

Remark 1.2. Let us observe that the inverse limit can be defined abstractly in an arbitrary category by means of a universal property. Let $\{X_a, p_{ab}, A\}$ be an inverse system of objects and morphisms in a category C. The inverse limit of this system is an object X in C together with morphisms $p_a\colon X \to X_a$ (called projections) satisfying $p_a = p_{ab}p_b$ for all $a \leq b$. The pair (X, p_a) must be universal in the sense that for any other such pair (Y, q_a) (i.e. $q_a\colon Y \to X_a$ with $q_a = p_{ab}q_b$ for all $a \leq b$) there exists a unique morphism $u\colon Y \to X$ such that $q_a = p_a u$ for all a.

For a cardinal τ we say that $\mathbf{X} = \{X_a, p_{ab}, A\}$ is τ-*directed* if for each $B \subseteq A$ with $\mathrm{card}(B) \leq \tau$ there is an $a \in A$ such that $a \geq b$ for each $b \in B$. Inverse system $\mathbf{X}$ is σ-*directed* if $\mathbf{X}$ is $\aleph_0$-directed. We say that an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is λ-*system* if it is λ-directed and $w(X_a) \leq \lambda$.

Theorem 1.3. *For each Tychonoff cube I^τ, $\tau \geq \aleph_1$, there exists $\lambda < \tau$ and an inverse λ-system $\mathbf{I} = \{I^a, P_{ab}, A\}$ of the cubes I^a, $\mathrm{card}(a) = \lambda$, such that I^τ is homeomorphic to $\lim \mathbf{I}$.*

Proof. **a)** Let us recall that the *Tychonoff cube* I^τ is the Cartesian product $\prod\{I_s : s \in S\}$, $\mathrm{card}(S) = \tau$, $I_s = [0, 1]$ [2, p. 114]. If $\mathrm{card}(S) = \aleph_0$, the Tychonoff cube I^τ is called the *Hilbert cube*. Let A be the set of all subsets of S of the cardinality λ ordered by inclusion. If $a \subseteq b$, then we write $a \leq b$. It is clear that A

is λ-directed. For each $a \in A$ there exists the cube I^a. If $a, b \in A$ and $a \leq b$, then there exists the projection $P_{ab}\colon I^b \to I^a$. Finally, we have system $\mathbf{I} = \{I^a, P_{ab}, A\}$.

b) Let us prove that I^τ is homeomorphic to $\lim \mathbf{I}$. Let $x \in I^\tau$. It is clear that $P_a(x) = x_a$ is a point of I^a and that $P_{ab}(x_b) = x_a$ if $a \leq b$. This means that (x_a) is a thread in $\mathbf{I} = \{I^a, P_{ab}, A\}$. Set $H(x) = (x_a)$. We have the mapping $H\colon I^m \to \lim \mathbf{I}$. It is clear that H is continuous, 1-1 and onto. Hence, H is a homeomorphism. □

Theorem 1.4. *Let X be a compact Hausdorff space such that* $\mathrm{w}(X) \geq \aleph_1$ *and let* $\aleph_0 \leq \lambda < \mathrm{w}(X)$. *Then there exists an inverse λ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that* $\mathrm{w}(X_a) \leq \lambda$ *and X is homeomorphic to* $\lim \mathbf{X}$.

Proof. By [2, Theorem 2.3.23] the space X is embeddable in $I^{w(X)}$. From Theorem 1.3 it follows that $I^{w(X)}$ is a limit of $\mathbf{I} = \{I^a, P_{ab}, A\}$ where A is the set from the proof of a) of Theorem 1.3. Now, X is homeomorphic to a closed subspace of $\lim \mathbf{I}$. For each $a \in A$ let $X_a = P_m(X)$, where $P_m\colon I^m \to I^a$ is a projection of the Tychonoff cube I^m onto the cube I^a. Let p_{ab} be the restriction of P_{ab} onto X_b. We have the inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that $w(X_a) \leq \lambda$. By virtue of [2, Corollary 2.5.7] X is homeomorphic to $\lim \mathbf{X}$. Moreover, $\mathbf{X}$ is an inverse λ-system since $\mathbf{I} = \{I^a, P_{ab}, A\}$ is an inverse λ-system. □

If $\lambda = \aleph_0$, then we say that $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that $w(X_a) \leq \aleph_0$ is σ-system.

A *cover* of a set X is a family $\{A_s : s \in S\}$ of subsets of X such that $X = \cup\{A_s : s \in S\}$. $\mathrm{Cov}(X)$ is the set of all coverings of topological space X. We say that a cover $\mathcal{B}$ of space X is *refinement* of a cover $\mathcal{A}$ of the same space if for every $B \in \mathcal{B}$ there exists $A \in \mathcal{A}$ such that $B \subset A$. If $\mathcal{U}, \mathcal{V} \in \mathrm{Cov}(X)$ and $\mathcal{V}$ refines $\mathcal{U}$, we write $\mathcal{V} \prec \mathcal{U}$.

Lemma 1.5. *Let $\mathbf{X} = \{X_a, p_{ab},\ A\}$ be an inverse system of compact spaces with surjective bonding mappings and limit X. For every finite cover $\mathcal{U} = \{U_1, U_2, \dots, U_n\}$ there exists an $a(\mathcal{U}) \in A$ such that for each $b \geq a(\mathcal{U})$ there is a finite cover $\mathcal{U}_b = \{U_{b1}, U_{b2}, \dots, U_{bm}\}$ of X_b such that $p_b^{-1}(\mathcal{U}_b) = \{p_b^{-1}(U_{b1}), p_b^{-1}(U_{b2}), \dots, p_b^{-1}(U_{bm})\}$ is refinement of $\mathcal{U} = \{U_1, U_2, \dots, U_n\}$, i.e., $p_b^{-1}(\mathcal{U}_b\} \prec \mathcal{U}$.*

Proof. By virtue of the definition of a base in X, for each $U_i \in \mathcal{U}$ we have $U_i = \cup\{p_a^{-1}(U_a) : a \in K_i \subset A\}$. Now, $\cup\{p_a^{-1}(U_a) : a \in K_i \cup K_2 \cup \dots \cup K_n\}$ is a cover oh X since $\mathcal{U} = \{U_1, U_2, \dots, U_n\}$ is cover of X. There is finite subfamily $\{p_{a1}^{-1}(U_{a1}), \dots, p_{am}^{-1}(U_{am})\}$ of $\cup\{p_a^{-1}(U_a) : a \in K_i \cup K_2 \cup \dots \cup K_n\}$ which covers X. We infer that there exists $a(\mathcal{U}) \in A$ such that $a(\mathcal{U}) \geq a1, \dots, am$. For each $b \geq a(\mathcal{U})$ we have a finite cover $\{U_{b1} = p_{a1b}^{-1}(U_{a1}), \dots, U_{bm} = p_{amb}^{-1}(U_{am})\}$, i.e., finite cover $\mathcal{U}_b = \{U_{b1}, U_{b2}, \dots, U_{bm}\}$ of X_b such that $p_b^{-1}(\mathcal{U}_b\} = \{p_b^{-1}(U_{b1}, \dots, p_b^{-1}(U_{bm})\}$ is refinement of $\mathcal{U} = \{U_1, \dots, U_n\}$. □

Theorem 1.6. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be a λ-directed inverse system of compact spaces with surjective bonding mappings and limit X. Let Y be a compact space of weight λ. For each surjective mapping $f\colon X \to Y$ there exists an $a \in A$ such that for each $b \geq a$ there exists a mapping $g_b\colon X_b \to Y$ such that $f = g_b p_b$.*

Proof. Let B be a basis of Y with $\mathrm{card}(B) = \lambda$ and let $\mathcal{V}$ be a collection of all finite subfamilies of B which cover Y. Clearly, $\mathrm{card}\,(\mathcal{V}) = \lambda$. Consider an enumeration

$\mathcal{V} = \{\mathcal{V}_\upsilon : \upsilon < \lambda\}$. For each $\mathcal{V}_\upsilon$ the family $f^{-1}(\mathcal{V}_\upsilon) = \{f^{-1}(U) : U \in \mathcal{V}_\upsilon\}$ is a covering of X. By virtue of Lemma 1.5 there exists an $a(\upsilon) \in A$ such that for each $b \geq a(\upsilon)$ there is a cover $\mathcal{V}_{\upsilon b}$ of X_b with $p_b^{-1}(\mathcal{V}_{\upsilon b}) \prec f^{-1}(\mathcal{V}_\upsilon)$. From the λ-directedness of A it follows that there is an $a \in A$ such that $a \geq a(\upsilon)$, for all $\upsilon < \lambda$. Let $b \geq a$. We claim that $f(p_b^{-1}(x_b))$ is degenerate. Suppose that there exists a pair u, v of distinct points of Y such that $u, v \in f(p_b^{-1}(x_b))$. Then there exists a pair x, y of distinct points of $p_b^{-1}(x_b)$ such that $f(x) = u$ and $f(y) = v$. Let U, V be a pair of disjoint open sets of Y such that $u \in U$ and $v \in V$. Consider the covering $\{U, V, Y\backslash\{u, v\}\}$. There exists a covering $\mathcal{V}_\upsilon \in \mathcal{V}$ such that $\mathcal{V}_\nu \prec \{U, V, X\backslash\{u, v\}\}$. We infer that there is a covering $\mathcal{V}_{\upsilon b}$ of X_b such that $p_b^{-1}(\mathcal{V}_{\upsilon b}) \prec f^{-1}(\mathcal{V}_\upsilon)$. It follows that $p_b(x) \neq p_b(y)$ since x and y lie in the disjoint members of the covering $f^{-1}(\mathcal{V}_\upsilon)$. This is impossible since $x, y \in p_b^{-1}(x_b)$. Thus, $f(p_b^{-1}(x_b))$ is degenerate. Now we define $g_b \colon X_b \to Y$ by $g_b(x_b) = f(p_b^{-1}(x_b))$. It is clear that $g_b p_b = f$. Let us prove that g_b is continuous. Let U be open in Y. Then $g_b^{-1}(U)$ is open since $p_b^{-1}(g_b^{-1}(U)) = f^{-1}(U)$ is open and p_b is quotient (as a closed mapping). □

2. Extending generalized Whitney maps

Main result of [12, Theorem 3.1] is the following theorem.

Theorem 2.1. *Let P be a compact metric partially ordered space such that* Min$P$ *and* Max$P$ *are disjoint closed sets and let Q be a closed subset of P such that* MinQ $\subset$ Min$P$ *and* Max$Q \subset$ Max$P$. *Then a Whitney map for Q can be extended to a Whitney map for P.*

Then the following corollaries are proved:

(1) If X is a continuum then any Whitney map for $C(X)$, the space of subcontinua of X, can be extended to a Whitney map for 2^X, the space of nonempty closed subsets of X.

(2) If Y is a continuum and X is a subcontinuum of Y, then any Whitney map for $C(X)$ (resp., 2^X) can be extended to a Whitney map for $C(Y)$ (resp., 2^Y).

In the sequel we shall use a version of Theorem 2.1 as it is in [3, 16.10 Theorem].

Theorem 2.2. *If X is a compact metric space, then any Whitney map for any closed subset of 2^X can be extended to a Whitney map for 2^X.*

Now we shall study the generalized Whitney maps.

Definition 2.1. A generalized arc is a continuum J with its topology given by a strict linear order $\rhd$. It is denoted by $\langle J, \rhd\rangle$.

Definition 2.2. If X is a continuum, a generalized Whitney map for $C(X)$ is a map $\mu\colon C(X) \to \langle J, \rhd\rangle$ where $\langle J, \rhd\rangle$ is a generalized arc and the following conditions hold:

a) $\mu(\{x\}) = \min J$ for each $x \in X$,

b) $\mu(A) \rhd \mu(B)$ whenever A, $B \in C(X)$ and $A \subset B$, and

c) $\mu(X) = \max J$.

Let X be a compact space, $\Lambda \in \{\ 2^X, X(2), C(X)\}$ and let F be any closed subset of Λ. We say that a generalized Whitney map $\sigma\colon F \to \langle J, \rhd\rangle$ is Λ-*extendable* if it can be extended to a Whitney map $\mu\colon \Lambda \to \langle J, \rhd\rangle$ for Λ.

2.1. **Case $\Lambda = 2^X$.** Now we shall prove the following result.

Theorem 2.3. *Let X be a compact space and let F be a closed subspace of 2^X. If a generalized Whitney map $\sigma\colon F \to \langle J, \rhd\rangle$ is 2^X-extendable then* $\mathrm{w}(X) = \mathrm{w}(\langle J, \rhd\rangle)$.

Proof. Set $\lambda = \mathrm{w}(\langle J, \rhd\rangle)$. By Theorem 1.4 there exists an inverse λ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that $\mathrm{w}(X_a) \leq \lambda$ and X is homeomorphic to $\lim \mathbf{X}$. Now $2^{\mathbf{X}}$ $= \{2^{X_a}, 2^{p_{ab}}, A\}$ and $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ form inverse systems suche X that $2^X = \lim 2^{\mathbf{X}}$and $C(X) = \lim C(\mathbf{X})$ (Lemma 1.1). If there exists an extension $\mu\colon 2^X \to \langle J, \rhd\rangle$, then by Theorem 1.6 there exists an $a \in A$ such that for each $b \geq a$ there exists a mapping $\mu_b\colon 2^{X_b} \to \langle J, \rhd\rangle$ such that $\mu = \mu_b P_b$, where P_b is the natural projection $P_b\colon \lim 2^{\mathbf{X}} \to 2^{X_b}$. Now we shall prove that every natural projection $p_b\colon \lim X \to X_b$ is a homeomorphism. It suffices to prove that p_b is 1-1. Suppose that there exists a point $x_b \in X_b$ such that $p_{ab}^{-1}(x_b)$ contains two different points x and y. Then $\{x\} \subset p_{ab}^{-1}(x_b)$ and $\{x\} \neq \{p_{ab}^{-1}(x_b)\}$. This means that $\mu_b(\{x\}) < \mu_b(\{p_{ab}^{-1}(x_b)\})$, i.e., $0 < \mu_b(\{p_{ab}^{-1}(x_b)\})$. On the other hand, $\mu(\{p_{ab}^{-1}(x_b)\}) = \mu_b P_b\{p_{ab}^{-1}(x_b)\} = \mu_b(\{x_b\})$ since $P_b\{p_{ab}^{-1}(x_b)\} = \{x_b\}$. This means that $\mu(\{p_{ab}^{-1}(x_b)\}) = 0$. This is impossible since $0 < \mu(\{p_{ab}^{-1}(x_b)\})$. We infer that p_b is a homeomorphism. Hence $\mathrm{w}(X) = \mathrm{w}(\langle J, \rhd\rangle)$. □

Corollary 2.4. *Let X be a compact space and let F be a closed subspace of 2^X. Then a Whitney map $\sigma\colon F \to [0,1]$ is 2^X-extendable if and only if X is metrizable.*

Proof. Apply Theorems 2.3 and 2.2. □

2.2. **Case $\Lambda = X(2)$.**

Theorem 2.5. *Let X be a compact space and let F be a closed subspace of $X(2)$. If a generalized Whitney map $\sigma\colon F \to \langle J, \rhd\rangle$ is $X(2)$-extendable then* $\mathrm{w}(X) = \mathrm{w}(\langle J, \rhd\rangle)$.

Proof. Set $\lambda = \mathrm{w}(\langle J, \rhd\rangle)$. By Theorem 1.4 there exists an inverse λ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that $\mathrm{w}(X_a) \leq \lambda$ and X is homeomorphic to $\lim \mathbf{X}$. Now $\mathbf{X}(2) = \{X_a(2), p_{ab}(2), A\}$ is an inverse system such that $X(2) = \lim \mathbf{X}(2)$ (Lemma 1.1). If there exists an extension $\mu\colon X(2) \to \langle J, \rhd\rangle$, then by Theorem 1.6 there exists an $a \in A$ such that for each $b \geq a$ there exists a mapping $\mu_b\colon X_b(2) \to \langle J, \rhd\rangle$ such that $\mu = \mu_b P_b$, where P_b is the natural projection $P_b\colon \lim 2^{\mathbf{X}} \to 2^{X_b}$. Now we shall prove that every natural projection $p_b\colon \lim X \to X_b$ is a homeomorphism. It suffices to prove that p_b is 1-1. Suppose that there exists a point $x_b \in X_b$ such that $p_{ab}^{-1}(x_b)$ contains two different points x and y. Then $\{x\} \subset p_{ab}^{-1}(x_b)$ and $\{x\} \neq \{p_{ab}^{-1}(x_b)\}$. This means that $\mu_b(\{x\}) < \mu_b(\{p_{ab}^{-1}(x_b)\})$, i.e., $0 < \mu_b(\{p_{ab}^{-1}(x_b)\})$. On the other hand, $\mu\ (\{p_{ab}^{-1}(x_b)\}) = \mu_b P_b\{p_{ab}^{-1}(x_b)\} = \mu_b(\{x_b\})$ since $P_b\{p_{ab}^{-1}(x_b)\} = \{x_b\}$. This means that $\mu(\{p_{ab}^{-1}(x_b)\}) = 0$. This is impossible since $0 < \mu(\{p_{ab}^{-1}(x_b)\})$. We infer that p_b is a homeomorphism. Hence $\mathrm{w}(X) = \mathrm{w}(\langle J, \rhd\rangle)$. □

Corollary 2.6. *Let X be a compact space and let F be a closed subspace of $X(2)$. Then a Whitney map $\sigma\colon F \to [0,1]$ is $X(2)$-extendable if and only if X is metrizable.*

Proof. Apply Theorems 2.5 and 2.2. $\square$

2.3. **Case $\Lambda = C(X)$.** In the remaining part of this paper we study the generalized Whitney maps on $C(X)$.

A mapping $f\colon X \to Y$ is said to be *hereditarily irreducible* [8, p. 204, (1.212.3)] provided that for any given subcontinuum Z of X, no proper subcontinuum of Z maps onto $f(Z)$.

Proposition 1 ([8, (1.212.3), p. 204]). *A mapping $f\colon X \to Y$ is hereditarily irreducible if and only if a mapping $C(f)\colon C(X) \to C(Y)$ is light.*

Theorem 2.7. *Let X be a non-metric continuum and let F be a subcontinuum of $C(X)$. If a generalized Whitney map $\sigma\colon F \to \langle J, \rhd\rangle$ is $C(X)$-extendable then, for each λ-directed inverse system $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ with $X = \lim \mathbf{X}$ there exists a cofinal subset $B \subset A$ such that for every $b \in B$ the projection $p_b\colon \lim \mathbf{X} \to X_b$ is hereditarily irreducible and $C(p_b)\colon C(\lim \mathbf{X}) \to C(X_b)$ is light.*

Proof. By Theorem 1.4 there exists an inverse λ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that $\mathrm{w}(X_a) \leq \lambda$ such that X is homeomorphic to $\lim \mathbf{X}$.
Now $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ (Lemma 1.1). If there exists an extension $\mu : C(X) \to \langle J, \rhd\rangle$, then by Theorem 1.6 there exists an $a \in A$ such that for each $b \geq a$ there exists a mapping $\mu_b\colon C(X_b) \to \langle J, \rhd\rangle$ such that $\mu = \mu_b P_b$, where P_b is the natural projection $P_b\colon \lim C(X) \to C(X_b)$. Now we shall prove that the projections $p_b\colon \lim X \to X_b$ are hereditarily irreducible. Suppose that p_b is not hereditarily irreducible. Then there exists a pair F, G of subcontinua of X with $F \subseteq G$, $F \neq G$, (i.e., F is a proper subcontinuum of G) such that $p_b(F) = p_b(G)$. It is clear that $C(p_b)(\{F\}) = C(p_b)(\{G\})$. This means that $\mu_b C(p_b)(\{F\}) = \mu_b C(p_b)(\{G\})$. From $\mu = \mu_b C(p_b)$ it follows that $\mu(\{F\}) = \mu(\{G\})$. This is impossible since μ is a Whitney map for $C(X)$ and from $F \subseteq G$, $F \neq G$ it follows $\mu(\{F\}) < \mu(\{G\})$. Hence, $p_b\colon \lim \mathbf{X} \to X_b$ is hereditarily irreducible and, by Proposition 1, $C(p_b)\colon C(\lim \mathbf{X}) \to C(X_b)$ is light. $\square$

Lemma 2.8. *If $f\colon X \to Y$ is monotone and hereditarily irreducible, then f is 1-1.*

Proposition 2. *Let $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ be σ-directed inverse system of compact spaces with $X = \lim \mathbf{X}$. If the projections $p_a\colon \lim \mathbf{X} \to X_a$ are hereditarily irreducible and monotone, then they are 1-1 and homeomorphisms.*

Proof. Use Lemma 2.8. $\square$

Theorem 2.9 ([6, Theorem 3.7]). *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of compact spaces and surjective bonding mappings p_{ab}. Then:*

1) *There exists an inverse system $M(\mathbf{X}) = \{M_a, m_{ab}, A\}$ of compact spaces such that m_{ab} are monotone surjections and $\lim \mathbf{X} = \lim M(\mathbf{X})$,*
2) *If $\mathbf{X}$ is σ-directed, then $M(\mathbf{X})$ is σ-directed.*

3) *If every* X_a *is a metric space and* $\lim \mathbf{X}$ *is locally connected* (*a rim-metrizable continuum*), *then every* M_a *is metrizable.*

Now we shall prove the following results.

Theorem 2.10. *Let* X *be a locally connected* (*or rim-metrizable*) *continuum and let* F *be a subcontinuum of* $C(X)$. *If a generalized Whitney map* $\sigma\colon F \to \langle J, \rhd\rangle$ *is* $C(X)$*-extendable, then* $\mathrm{w}(X) = \mathrm{w}(\langle J, \rhd\rangle)$.

Proof. Suppose that X is non-metric. By Theorem 1.4 there exists an inverse σ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that $\mathrm{w}(X_a) \leq \aleph_0$ (i.e. each X_a is a metric compact space) and X is homeomorphic to $\lim \mathbf{X}$. By Theorem 2.9 we have an inverse system $M(\mathbf{X}) = \{M_a, m_{ab}, A\}$ of compact spaces such that m_{ab} are monotone surjections and $\lim \mathbf{X} = \lim M(\mathbf{X})$. Now the projetions $m_b\colon\ \lim \mathbf{X} \to \mathbf{X}_b$ are monotone and hereditarily irreducible for each b in some cofinal subset B of A. From Proposition 2 it follows that $m_b\colon\ \lim \mathbf{X} \to \mathbf{X}_b, b \in B$, are homeomorphisms. Hence, $\mathrm{w}(X) = \mathrm{w}(\langle J, \rhd\rangle)$. □

Corollary 2.11. *Let* X *be a locally connected* (*or rim-metrizable*) *continuum and let* F *be a subcontinuum of* $C(X)$. *Then a Whitney map* $\sigma\colon F \to [0, 1]$ *is* $C(X)$*-extendable if and only if* X *is metrizable.*

Proof. By Theorem 2.10 we have $\mathrm{w}(X) = \mathrm{w}(\langle J, \rhd\rangle) = \mathrm{w}([0, 1]) = \aleph_0$. □

In the sequel we shall use the following result [9, Exercise 11.52, p. 226].

Lemma 2.12. *If* X *is a continuum and if* A *and* B *are mutually disjoint subcontinua of* X, *then there is a component* K *of* $X\setminus(A \cup B)$ *such that* $\mathrm{Cl}\, K \cap A \neq \emptyset$ *and* $\mathrm{Cl}\, K \cap B \neq \emptyset$.

Definition 2.3. A continuum X is called a *D-continuum* if for every pair C, D of its disjoint non-degenerate subcontinua there exists a subcontinuum $E \subset X$ such that $C \cap E \neq \emptyset \neq D \cap E$ and $(C \cup D)\setminus E \neq \emptyset$.

A family $\mathcal{N} = \{M_s : s \in S\}$ of a subsets of a topological space X is a *network* for X if for every point $x \in X$ and any neighbourhood U of x there exists an $s \in S$ such that $x \in M_s \subset U$ [2, p. 170]. The *network weight* of a space X is defined as the smallest cardinal number of the form $\mathrm{card}\,(\mathcal{N})$, where $\mathcal{N}$ is a network for X; this cardinal number is denoted by $nw(X)$.

Theorem 2.13 ([2, p. 171, Theorem 3.1.19])**.** *For every compact space* X *we have* $nw(X) = w(X)$.

Theorem 2.14. *Let* X *be a D-continuum and let* F *be a subcontinuum of* $C(X)$. *If a generalized Whitney map* $\sigma\colon F \to \langle J, \rhd\rangle$ *is* $C(X)$*-extendable, then* $\mathrm{w}(X) = \mathrm{w}(\langle J, \rhd\rangle)$.

Proof.
Step 1. *Let us prove that* $\mathrm{w}(C(X)\setminus X(1)) \leq \mathrm{w}(\langle J, \rhd\rangle)$. From Theorem 1.4 it follows that there exists a λ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of continua with $\mathrm{w}(X) = \mathrm{w}(\langle J, \rhd\rangle) = \lambda$ and surjective bonding mappings such that X is homeomorphic to $\lim \mathbf{X}$. Consider inverse system $C(\mathbf{X}) = \{C(X_a),\ C(p_{ab}),\ A\}$

whose limit is $C(X)$. From Theorem 2.7 it follows that the projections p_a are hereditarily irreducible and $C(p_a)$ are light. If $C(p_a)$ are one-to-one, then we have a homeomorphism $C(p_a)$ of $C(X)$ onto $C(p_a)(X)$. It follows that $\mathrm{w}(X) = \mathrm{w}(\langle J, \rhd \rangle$. Suppose that $C(p_a)$ is not one-to-one. Then there exists a continuum C_a in X_a and two continua C, D in X such that $p_a(C) = p_a(D) = C_a$. It is impossible that $C \subset D$ or $D \subset C$ since p_a is hereditarily irreducible. Otherwise, If $C \cap D \neq \emptyset$, then for a continuum $Y = C \cup D$ we have that C and D are subcontinua of Y and $p_a(Y) = p_a(C) = p_a(D) = C_a$ which is impossible since p_a is hereditarily irreducible. We infer that $C \cap D = \emptyset$. By Definition 2.3 there exists a subcontinuum E such that $C \subset E, D \neq D \cap E \neq \emptyset$ since X is a D-continuum. Now $p_a(E \cup D) = p_a(E)$ which is impossible since p_a is hereditarily irreducible. Furthermore, $C(p_a)^{-1}(X_a(1)) = X(1)$ since from the hereditarily irreducibility of p_a it follows that no non-degenerate subcontinuum of X maps under p_a onto a point. Let $Y_a = C(p_a)(C(X))$. We infer that $C(p_a)^{-1}[Y_a \diagdown X_a(1)] = C(X) \diagdown X(1)$. It follows that the restriction $P_a = C(p_a)|(C(X) \diagdown X(1))$ is one-to-one and closed [2, Proposition 2.1.4]. Hence, P_a is a homeomorphism. It follows that $\mathrm{w}(C(X) \diagdown X(1)) \leq \lambda = \mathrm{w}(\langle J, \rhd \rangle)$.

Step 2. *Let us prove that* $\mathrm{w}(X) = \mathrm{w}(\langle J, \rhd \rangle)$. By Step 1 we have that $\mathrm{w}(C(X) \diagdown X(1))$ $\leq \mathrm{w}(\langle J, \rhd \rangle)$. This means that there exists a base $\mathcal{B} = \{B_i : i \in \lambda\}$ of $C(X) \diagdown X(1)$. For each B_i let $C_i = \cup\{x \in X : x \in B,\ B \in B_i\}$, i.e., the union of all continua B contained in B_i.

Claim 1. *The family* $\{C_i : i \in N\}$ *is a network of* X. Let x be a point of X and let U be an open subsets of X such that $x \in U$. There exists and open set V such that $x \in V \subset \mathrm{Cl}V \subset U$. Let K be a component of $\mathrm{Cl}V$ containing x. By Boundary Bumping Theorem [9, p. 73, Theorem 5.4] K is non-degenerate and, consequently, $K \in C(X) \diagdown X(1)$. Now, $\langle U \rangle \cap (C(X) \diagdown X(1))$ is a neighbourhood of K in $C(X) \diagdown X(1)$. It follows that there exists a $B_i \in \mathcal{B}$ such that $K \in B_i \subset \langle U \rangle \cap (C(X) \diagdown X(1))$. It is clear that $C_i \subset U$ and $x \in C_i$ since $x \in K$. Hence, the family $\{C_i : i \in \lambda\}$ is a network of X.

Claim 2. $nw(X) = \mathrm{w}(\langle J, \rhd \rangle)$. Apply Claim 1.

Claim 3. $w(X) = \mathrm{w}(\langle J, \rhd \rangle)$. By Claim 2 we have $nw(X) = \mathrm{w}(\langle J, \rhd \rangle)$. Moreover, by Theorem 2.13 $w(X) = \mathrm{w}(\langle J, \rhd \rangle)$. □

Corollary 2.15. *Let X be a D-continuum and let F be a subcontinuum of $C(X)$. Then a Whitney map $\sigma \colon F \to [0,1]$ is $C(X)$-extendable if and only if X is metrizable.*

Proof. By Theorem 2.14 we have $\mathrm{w}(X) = \mathrm{w}(\langle J, \rhd \rangle) = \mathrm{w}([0,1]) = \aleph_0$. □

3. Examples of D-continua

The following continua are D-continua:

A *generalized arc* is a Hausdorff continuum with exactly two non-separating points (end points) x, y. Each separable arc is homeomorphic to the closed interval $I = [0,1]$.

We say that a space X is *arcwise connected* if for every pair x, y of points of X there exists a generalized arc L with end points x, y.

Lemma 3.1. *If X is an arcwise connected continuum, then X is a D-continuum.*

Proof. Let C, D be a pair of disjoint subcontiua of X. Take the points $c \in C$ and $d \in D$. There exists an arc L with the endpoints c and d. We have two cases. First, D is not a proper subset of L. Now, $E = C \cup L$ is a subcontinuum which contains C and $D \cap E$ is a non-empty proper subset of D. Secondly, let D be a proper subset of L. Then D is an arc with end points d and e. It is clear that e is not in C. Let $E = C \cup [c, e]$, where $[c, e]$ is a subarc of L with end points c and e. The continuum E contains C and $E \cap D = \{e\} \subset D$. Finally, we infer that X is a D-continuum. $\square$

Lemma 3.2. *If X is a locally connected continuum, then X is a D-continuum.*

Proof. Let C, D be a pair of disjoint subcontinua of X. Let d be a point of D. Let U_d be a connected neoghborhood of d such that $d \in U_d$, $\mathrm{Cl}\, U_d \cap C = \emptyset$ and $D \setminus \mathrm{Cl}\, U_d \neq \emptyset$. Set $U = X \setminus (\mathrm{Cl}\, U_d \cup D)$. Because of the Boundary Bumping Theorem, [9, Theorem 5.4, p. 73] there exists a component K of ClU such that $C \subset K$ and $K \cap Bd(U) \neq \emptyset$. If $K \cap C = \emptyset$, then $K \cap \mathrm{Cl}\, U_d \neq \emptyset$. Set $E = C \cup K$. From $d \in U_d$, $ClU_d \cap C = \emptyset$ and $D \setminus ClU_d \neq \emptyset$ it follows that $C \subset E$, $E \cap D \neq \emptyset$ and $D \setminus E \neq \emptyset$. The required continuum E is constructed. If $K \cap ClU_d = \emptyset$, then $K \cap (D \setminus \mathrm{Cl}\, U_d) \neq \emptyset$. Set $E = K$. It follows that $C \subset E$, $E \cap D \neq \emptyset$ and $D \setminus E \neq \emptyset$ since $\mathrm{Cl}\, U_d \subset D \setminus E$. Hence, X is a D-continuum. $\square$

A continuum is said to be *semi-aposyndetic*, [3, p. 238, Definition 29.1], if for every $p \neq q$ in X, there exists a subcontinuum M of X such that $\mathrm{Int}_X(M)$ contains one of the points p, q and $X \backslash M$ contains the other one. Each locally connected continuum is semi-aposyndetic.

Lemma 3.3. *If X is a semi-aposyndetic continuum, then X is a D-continuum.*

Proof. If X *is semi-aposyndetic, then for every pair* C, D *of disjoint non-degenerate subcontinua of* X *there exists a non-degenerate subcontinuum* $E \subset X$ *such that* $C \cap E \neq \emptyset \neq D \cap E$ *and* $(C \cup D) \setminus E \neq \emptyset$. We shall consider two cases.

a) If either $\mathrm{Int}_X(C) \neq \emptyset$ or $\mathrm{Int}_X(D) \neq \emptyset$, then it suffices to apply Lemma 2.12 to the union $C \cup D$ and obtain a component K of $X \setminus (C \cup D)$ such that $\mathrm{Cl}K \cap C \neq \emptyset$ and $\mathrm{Cl}\, K \cap D \neq \emptyset$. Then $E = \mathrm{Cl}K$ is a continuum with properties $C \cap E \neq \emptyset \neq D \cap E$ and $(C \cup D) \setminus E \neq \emptyset$ since $\mathrm{Int}_X(C) \cap E = \emptyset$ or $\mathrm{Int}_X(D) \cap E = \emptyset$.

b) Assume that $\mathrm{Int}_X(C) = \emptyset$ and $\mathrm{Int}_X(D) = \emptyset$. There exist $x, y \in C$ such that $x \neq y$. Moreover, there exists a subcontinuum M of X such that $\mathrm{Int}_X(M)$ contains one of the points x, y and $X \setminus M$ contains the other one since X is semi-aposyndetic. Suppose that $x \in \mathrm{Int}_X(M)$ and $y \in X \setminus M$. If $M \cap D \neq \emptyset$, then we set $E = M$ and we have the continuum E such that $C \cap E \neq \emptyset \neq D \cap E$ and $(C \cup D) \setminus E \neq \emptyset$ since $y \in X \setminus M$. Suppose that $M \cap D = \emptyset$. Applying Lemma 2.12 to the union $C \cup D \cup M$ we obtain a component K of $X \setminus (C \cup D \cup M)$ such that $\mathrm{Cl}\, K \cap (C \cup M) \neq \emptyset$ and $\mathrm{Cl}\, K \cap D \neq \emptyset$. It is clear that $x \notin \mathrm{Cl}\, K$. If $\mathrm{Cl}\, K \cap C \neq \emptyset$, then we set $E = \mathrm{Cl}\, K$ and obtain a continuum E such that $C \cap E \neq \emptyset \neq D \cap E$ and $(C \cup D) \setminus E \neq \emptyset$ since $x \notin \mathrm{Cl}\, K$. If $\mathrm{Cl}\, K \cap C = \emptyset$, then $\mathrm{Cl}\, K \cap M \neq \emptyset$ and we set $E = \mathrm{Cl}\, K \cup M$. Now $y \notin E$, $C \cap E \neq \emptyset \neq D \cap E$ and $(C \cup D) \setminus E \neq \emptyset$. $\square$

A continuum X is said to be a *C-continuum* provided for each triple x, y, z of points of X, there exists a subcontinuum C of X which contains x and exactly one of the points y and z, [15, p. 326].

A *generalized arc* is a Hausdorff continuum with exactly two non-separating points (end points) x, y. Each separable arc is homeomorphic to the closed interval $I = [0, 1]$.

We say that a space X is *arcwise connected* if for every pair x, y of points of X there exists a generalized arc L with end points x, y.

Lemma 3.4. *Each arcwise connected continuum is a C-continuum.*

Proof. Let x, y, z be a triple of points of an arcwise connected continuum X. There exists an arc $[x, y]$ with endpoints x and y. If $z \notin [x, y]$, then the proof is completed. If $z \in [x, y]$, then subarc $[x, z]$ contains x and z, but not y. The proof is completed. □

Lemma 3.5. *The cartesian product $X \times Y$ of two non-degenerate continua X and Y is a C-continuum.*

Proof. Let $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ be a triple of points of the product $X \times Y$. Now we have $x_2 \neq x_3$ or $y_2 \neq y_3$. We will give the proof in the case $x_2 \neq x_3$ since the proof in the case $y_2 \neq y_3$ is similar. Now we have two disjoint continua $Y_2 = \{(x_2, y) : y \in Y\}$ and $Y_3 = \{(x_3, y) : y \in Y\}$. If $(x_1, y_1) \in Y_2$ or $(x_1, y_1) \in Y_3$, the proof is completed. Let $(x_1, y_1) \notin Y_2$ and $(x_1, y_1) \notin Y_3$. Consider the continua $X_2 = \{(x, y_2) : x \in X\}$ and $X_3 = \{(x, y_3) : x \in X\}$. The continuum $Y_1 = \{(x_1, y) : y \in Y\}$ contains a point (x_1, p) such that $(x_1, p) \notin X_2 \cup X_3$. Let $X_p = \{(x, p) : x \in X\}$. It is clear that a continuum $Y_1 \cup X_p \cup Y_2$ contains the points (x_1, y_1) and (x_2, y_2) but not (x_3, y_3). Similarly, a continuum $Y_1 \cup X_p \cup Y_3$ contains the points (x_1, y_1) and (x_3, y_3) but not (x_2, y_2). The proof is completed. □

The concept of aposyndesis was introduced by Jones in [4]. A continuum X is *aposyndetic* provided it is true that if x and y are any two points of X, then some closed connected neighborhood of x misses y.

Proposition 3 ([15, Theorem 1, p. 326])**.** *If the continuum X is aposyndetic, the X is the C-continuum.*

Remark 1. There exists a C-continuum continuum which is not aposyndetic, [15, p. 327]. (See countable harmonic fan.)

Remark 2. There exists a C-continuum continuum which is not arcwise connected, [15, p. 328].

A continuum X is said to be *colocally connected* provided that for each point $x \in X$ and each open se $U \ni x$ there exists an open set V containing x such that $V \subset U$ and $X \setminus U$ is connected.

Lemma 3.6. *Each colocally connected continuum X is a C-continuum.*

Proof. Let x, y, z be a triple of points of X. Now, $U = X \setminus \{x, y\}$ is an open set U such that $z \in U$. From the colocal connectedness of X it follows that there exists an open set V such that $z \in V \subset U$ and $X \setminus V$ is connected.Hence, X is a C-continuum since the continuum $X \setminus V$ contains the points x and y. □

Lemma 3.7. *The cartesian product $X \times Y$ of two non-degenerate continua is a colocally connected continuum and, consequently, a C-continuum.*

Proof. Let (x, y) be a point of $X \times Y$. We have to prove that there exists a neighbourhood $U = U_x \times U_y$ of (x, y) such that $E = X \times Y \setminus U$ is connected. We may assume that $U_x \neq X$ and $U_y \neq Y$. Let (x_1, y_1), (x_2, y_2) be a pair of different points in E. For each point $(z, w) \in X \times Y$ we consider a continuum

$$E_{zw} = \{(z, y) : y \in Y\} \cup \{(x, w) : x \in X\}.$$

Claim 1. *For each point $(x', y') \in E$ there exists a point $(z, w) \in E$ such that $(x', y') \in E_{zw}$ and $E_{zw} \cap U = \emptyset$.* If $E_{x'y'} \cap U = \emptyset$ the proof is completed. In the opposite case we have either $\{(x', y) : y \in Y\} \cap U \neq \emptyset$ or $\{(x, y') : x \in X\} \cap U \neq \emptyset$. Suppose that $\{(x', y) : y \in Y\} \cap U \neq \emptyset$. Then $\{(x, y') : x \in X\} \cap U = \emptyset$. There exists a point $z \in X$ such that $z \notin U$. Setting $y' = w$, we obtain a point $(z, w) \in E$ such that $(x', y') \in E_{zw}$ and $E_{zw} \cap U = \emptyset$. The proof in the case $\{(x, y') : x \in X\} \cap U \neq \emptyset$ is similar.

Now, by Claim 1, for (x_1, y_1) there exists a continuum E_{z_1,w_1} such that $E_{z_1 w_1} \cap U = \emptyset$ and $(x_1, y_1) \in E_{z_1,w_1}$. Similarly, there exist a continuum E_{z_2,w_2} such that $E_{z_2 w_2} \cap U = \emptyset$ and $(x_2, y_2) \in E_{z_2,w_2}$.

Claim 2. *The union $E_{z_1,w_1} \cup E_{z_2,w_2}$ is a continuum which contains the points (x_1, y_1), (x_2, y_2) and is contained in $E = X \times Y \setminus U$.* Obvious.

Finally, we infer that $E = X \times Y \setminus U$ is connected. Hence, $X \times Y$ is colocally connected. From Lemma 3.6 it follows that $X \times Y$ is a C-continuum. The proof is completed. □

Now we shall prove the following result.

Theorem 3.8. *A C-continuum X is a D-continuum.*

Proof. *If X is a C-continuum, then for every pair C, D of disjoint non-degenerate subcontinua of X there exists a non-degenerate subcontinuum $E \subset X$ such that $C \cap E \neq \emptyset \neq D \cap E$ and $(C \cup D) \setminus E \neq \emptyset$.* Let $x \in C$ and y, $z \in D$. There exists a continuum E such that either x, $y \in E$, $z \in \lim \mathbf{X} \setminus E$ or x, $z \in E$, $y \in \lim \mathbf{X} \setminus E$, respectively since X is a C-continuum. We assume that x, $y \in E$, and $z \in X \setminus E$. It is clear that $C \cap E \neq \emptyset \neq D \cap E$ *and* $(C \cup D) \setminus E \neq \emptyset$ since $x \in C \cap E$, $y \in D \cap E$ and $z \in (C \cup D) \setminus E$. □

Theorem 3.9. *Let X be an arcwise connected (locally connected, semi-aposyndetic, product of two continua, colocally connected, C-continuum). If a generalized Whitney map for any closed subset of $C(X)$ can be extended to a Whitney map for $C(X)$, then $w(X) = \mathrm{w}(\langle J, \rhd \rangle)$.*

Theorem 3.10. *Let X be an arcwise connected (locally connected, semi-aposyndetic, product of two continua, colocally connected, C-continuum). If any Whitney map for any closed subset of $C(X)$ can be extended to a Whitney map for $C(X)$, then X is metrizable.*

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