Peng-Jie Wong

On critical values of twisted Artin L-functions

Czechoslovak Mathematical Journal, Vol. 67 (2017), No. 2, 551–555

Czechoslovak Mathematical Journal, 67 (142) (2017), 551–555

ON CRITICAL VALUES OF TWISTED ARTIN L-FUNCTIONS

Peng-Jie Wong, Kingston

Received March 18, 2016. First published March 28, 2017.

Dedicated to the memory of my teacher, Chao-Liang Shen

Abstract. We give a simple proof that critical values of any Artin L-function attached to a representation ϱ with character χ_ϱ are stable under twisting by a totally even character χ, up to the $\dim\varrho$-th power of the Gauss sum related to χ and an element in the field generated by the values of χ_ϱ and χ over $\mathbb{Q}$. This extends a result of Coates and Lichtenbaum as well as the previous work of Ward.

Keywords: Artin L-function; character; Galois Gauss sum; special value

MSC 2010: 11F67, 11F80, 11L05, 11M06

1. Introduction

Let K/k denote a Galois extension of number fields with Galois group G. Let ϱ be a representation of G with underlying vector space V, χ_ϱ its character, and $L(s,\chi_\varrho,K/k)$ the Artin L-function attached to χ_ϱ. For G abelian and ϱ (absolutely) irreducible, Siegel in [5] and Klingen in [2] utilized the theory of (Hilbert) modular forms to derive the algebricity of $L(m,\chi_\varrho,K/k)$ whenever m is a negative integer. Via the work of Siegel-Klingen and the Brauer induction theorem, Coates and Lichtenbaum in [1] showed that $s=m$ is a critical value of $L(s,\chi_\varrho,K/k)$ if

R: m is an odd negative integer and the fixed field $K^{\operatorname{Ker}\varrho}$ of the kernel $\operatorname{Ker}\varrho$ of ϱ is totally real; or

C: m is even and negative, the fixed field $K^{\operatorname{Ker}\varrho}$ is totally imaginary, conjugation is central in $\operatorname{Gal}(K^{\operatorname{Ker}\varrho}/k)$, and $\chi_\varrho(c)=-\dim\varrho$, where $c\in G$ is the complex conjugation;

or, equivalently, for m negative, $s = m$ is a critical point of $L(s, \chi_\varrho, K/k)$ if and only if $V = V^{(-1)^{1-m}}$, where $V^{\pm 1}$ is the $\pm$-eigenspace of $\chi_\varrho(c)$. In each of these cases, $L(m, \chi_\varrho, K/k)$ lies in $\mathbb{Q}(\chi_\varrho)$.

Let k be the field of rational numbers $\mathbb{Q}$. Ward in [6] recently extended the above-mentioned theorem of Coates and Lichtenbaum by showing that critical values of the Artin L-function attached to χ_ϱ are stable under *twisting* by an even Dirichlet character χ, up to the $\dim \varrho$-th power of the Gauss sum of χ and an element in $\mathbb{Q}(\chi_\varrho, \chi)$.

Naturally, one may wonder if such a stability result holds for any Galois extension K/k of number fields. In this note, we will answer this question by showing the following.

Theorem 1.1. *Suppose that χ is a totally even character of G, i.e., it is trivial on any complex conjugation. If $s = m \in \mathbb{Z}^-$ is a critical point of $L(s, \chi_\varrho, K/k)$, then*

$$L(1-m, \chi \otimes \chi_\varrho, K/k) \sim_{\mathbb{Q}(\chi_\varrho, \chi)} \tau(\overline{\chi})^{\dim \varrho} L(1-m, \chi(1)\chi_\varrho, K/k), \tag{1.1}$$

where $\tau(\overline{\chi})$ is the Gauss sum of $\overline{\chi}$ and $\mathbb{Q}(\chi_\varrho, \chi)$ denotes the field generated by the values of χ_ϱ and χ over $\mathbb{Q}$.

2. Proof of Theorem 1.1

Unlike Ward, who utilized the Brauer induction theorem and the theory of Gauss sums of Dirichlet characters, we invoke the general theory of (Galois) Gauss sums of characters due to Fröhlich. Now we begin by reviewing some aspects of this theory.

As before, let K/k be a Galois extension of number fields with Galois group G, and σ be a character of G. We make some use of notation as in [3] and [4]. The (Artin) root number of σ is denoted as $W(\sigma)$, $W_\infty(\sigma)$ denotes the infinite part of σ, and $\mathfrak{f}(\sigma)$ denotes the (global) Artin conductor of σ. The (Galois) Gauss sum of σ is defined by

$$\tau(\sigma) = W(\overline{\sigma}) W_\infty(\sigma)^{-1} \sqrt{N_{k/\mathbb{Q}} \mathfrak{f}(\sigma)},$$

where $N_{k/\mathbb{Q}}$ denotes the usual ideal norm. Since the determinant of a matrix only depends on its trace, we denote the determinant of a representation whose character is σ by $\det_\sigma$. These Gauss sums of characters share the following properties similar to the Gauss sums of Dirichlet characters.

Proposition 2.1 ([3], Propositions 4.1 and 7.1)**.**

I: *For any characters σ_1 and σ_2 of G, $\tau(\sigma_1+\sigma_2)=\tau(\sigma_1)\tau(\sigma_2)$.*

II: *For any character σ of G, $\tau(\sigma)\tau(\overline{\sigma})=N_{k/\mathbb{Q}}\mathfrak{f}(\sigma)\det_\sigma(-1)$.*

On the other hand, as it can be shown that

$$\mathfrak{f}(\chi)=\mathfrak{f}(\overline{\chi}),\quad W_\infty(\overline{\chi})=W_\infty(\chi),$$

the following equality holds:

$$W(\chi)=\frac{\tau(\overline{\chi})W_\infty(\chi)}{\sqrt{N_{k/\mathbb{Q}}\mathfrak{f}(\chi)}}.$$

For a topological group G, the group G^{ab} denotes the quotient of G by the closure of its commutator subgroup. For any subgroup H of G, we denote by Ver the transfer from G^{ab} to H^{ab}, which was introduced by Schur and rediscovered by Artin. Thanks to their work, we have the following Artin maps:

L: Ver: $\mathrm{Gal}(\overline{\mathbb{Q}_p}/F)^{ab}\to\mathrm{Gal}(\overline{\mathbb{Q}_p}/E)^{ab}$ where E/F is a finite extension of the field of p-adic numbers $\mathbb{Q}_p$; and

G: Ver: $\mathrm{Gal}(\overline{F}/F)^{ab}\to\mathrm{Gal}(\overline{F}/E)^{ab}$ where E/F is an extension of number fields.

Following [3], we write $\mathrm{Ver}_{E/F}$ for the transfers involved in these maps (for the complete details see [3], II. 3). After Artin, Fröhlich proved the following theorem, which gives a formula for the Galois action on the (Galois) Gauss sums.

Theorem 2.2 (Fröhlich)**.** *For every $\omega\in\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$,*

$$\omega(\tau(\omega^{-1}(\chi)))=\tau(\chi)\det_\chi(\mathrm{Ver}_{k/\mathbb{Q}}(\omega)).$$

P r o o f of Theorem 1.1. As in [6], using the classical theory of Artin L-functions, for every character σ of G one can write the functional equation of $L(s,\sigma,K/k)$ (see for example [4], VII. 12):

$$L(1-s,\sigma,K/k)=W(\sigma)\{d_k^{\dim\sigma}N_{k/\mathbb{Q}}\mathfrak{f}(\sigma)\}^{s-1/2}\cos^{n_\sigma^+}\left(\frac{\pi s}{2}\right)\times\sin^{n_\sigma^-}\left(\frac{\pi s}{2}\right)\{2(2\pi)^{-s}\Gamma(s)\}^{[k:\mathbb{Q}]\dim\sigma}L(s,\overline{\sigma},K/k), \tag{2.1}$$

where d_k is the absolute discriminant of k,

$$n_\sigma^+=\frac{[k:\mathbb{Q}]\sigma(1)}{2}+\frac{1}{2}\sum_{\mathfrak{p}|\infty,\ \mathfrak{p}\text{ real}}\sigma(\sigma_{\mathfrak{P}}),\quad n_\sigma^-=\frac{[k:\mathbb{Q}]\sigma(1)}{2}-\frac{1}{2}\sum_{\mathfrak{p}|\infty,\ \mathfrak{p}\text{ real}}\sigma(\sigma_{\mathfrak{P}}).$$

(For each prime $\mathfrak{p}$ appearing in the sums, $\mathfrak{P}$ denotes a prime of K above $\mathfrak{p}$, and $\sigma_{\mathfrak{P}}$ is a generator of the decomposition group $D_{\mathfrak{P}}(K/k)$.)

Under the same notation as above, since χ is totally even, one has

$$\frac{\chi(1)+\chi(\sigma_{\mathfrak{P}})}{2} = \dim V_\chi^+ = \chi(1),$$

where V_χ is the underlying vector space of χ and V_χ^+ is the $+$-eigenspace of $\chi(\sigma_{\mathfrak{P}})$. Therefore, $n^{\pm}_{\chi\otimes\chi_\varrho} = n^{\pm}_{\chi(1)\chi_\varrho}$, and the functional equation of $L(s,\chi\otimes\chi_\varrho,K/k)$ has the same form as the functional equation of $L(s,\chi(1)\chi_\varrho,K/k)$. In particular, one has

$$\frac{L(1-m,\chi\otimes\chi_\varrho,K/k)}{L(1-m,\chi(1)\chi_\varrho,K/k)} = \alpha\Big(\frac{\tau(\overline{\chi\otimes\chi_\varrho})}{\tau(\overline{\chi_\varrho})^{\chi(1)}}\Big)\frac{L(m,\overline{\chi\otimes\chi_\varrho},K/k)}{L(m,\overline{\chi(1)\chi_\varrho},K/k)} \tag{2.2}$$

for some rational number α. On the other hand, for any $\omega\in\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\chi_\varrho,\chi))$, one has

$$\frac{\tau(\overline{\chi\otimes\chi_\varrho})}{\tau(\overline{\chi})^{\dim\varrho}\tau(\overline{\chi_\varrho})^{\chi(1)}} = \frac{\tau(\omega(\overline{\chi\otimes\chi_\varrho}))}{\tau(\omega(\overline{\chi}))^{\dim\varrho}\tau(\omega(\overline{\chi_\varrho}))^{\chi(1)}}. \tag{2.3}$$

Applying the formula given by Fröhlich's theorem, for every $\omega\in\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\chi_\varrho,\chi))$ we have

$$\begin{aligned}
&\omega\Big(\frac{\tau(\overline{\chi\otimes\chi_\varrho})}{\tau(\overline{\chi})^{\dim\varrho}\tau(\overline{\chi_\varrho})^{\chi(1)}}\Big)\\
&\quad= \frac{\tau(\omega(\overline{\chi\otimes\chi_\varrho}))\det_{\overline{\chi\otimes\chi_\varrho}}(\mathrm{Ver}_{k/\mathbb{Q}}(\omega))}{\{\tau(\omega(\overline{\chi}))\det_{\overline{\chi}}(\mathrm{Ver}_{k/\mathbb{Q}}(\omega))\}^{\dim\varrho}\{\tau(\omega(\overline{\chi_\varrho}))\det_{\overline{\chi_\varrho}}(\mathrm{Ver}_{k/\mathbb{Q}}(\omega))\}^{\chi(1)}}\\
&\quad= \frac{\tau(\overline{\chi\otimes\chi_\varrho})}{\tau(\overline{\chi})^{\dim\varrho}\tau(\overline{\chi_\varrho})^{\chi(1)}},
\end{aligned}$$

where the last equality holds thanks to equalities (2.3) and

$$\det_{\overline{\chi\otimes\chi_\varrho}}(\mathrm{Ver}_{k/\mathbb{Q}}(\omega)) = \det_{\overline{\chi}}(\mathrm{Ver}_{k/\mathbb{Q}}(\omega))^{\dim\varrho}\det_{\overline{\chi_\varrho}}(\mathrm{Ver}_{k/\mathbb{Q}}(\omega))^{\chi(1)}.$$

Therefore, as every $\omega\in\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\chi_\varrho,\chi))$ fixes this quotient, one has

$$\frac{\tau(\overline{\chi\otimes\chi_\varrho})}{\tau(\overline{\chi})^{\dim\varrho}\tau(\overline{\chi_\varrho})^{\chi(1)}} \in \mathbb{Q}(\chi_\varrho,\chi). \tag{2.4}$$

Furthermore, [1], Theorem 1.2, asserts

$$\frac{L(m,\chi\otimes\chi_\varrho,K/k)}{L(m,\chi(1)\chi_\varrho,K/k)} \in \mathbb{Q}(\chi_\varrho,\chi), \tag{2.5}$$

which together with (2.2) and (2.4) completes the proof of Theorem 1.1. $\square$

Acknowledgment. The author would like to thank Professor Ram Murty for his encouragement and valuable suggestions, and the reviewers for their helpful comments.

References

[1] *J. Coates, S. Lichtenbaum*: On l-adic zeta functions. Ann. Math. (2) *98* (1973), 498–550.
[2] *H. Klingen*: Über die Werte der Dedekindschen Zetafunktion. Math. Ann. *145* (1962), 265–272. (In German.)
[3] *J. Martinet*: Character theory and Artin L-functions. Algebraic Number Fields. Proc. Symp. London math. Soc., Univ. Durham 1975, Academic Press, London, 1977, pp. 1–87.
[4] *J. Neukirch*: Algebraic Number Theory. Grundlehren der Mathematischen Wissenschaften 322, Springer, Berlin, 1999.
[5] *C. L. Siegel*: Über die Fourierschen Koeffizienten von Modulformen. Nachr. Akad. Wiss. Göttingen, II. Math.-Phys. Kl. *3* (1970), 15–56. (In German.)
[6] *K. Ward*: Values of twisted Artin L-functions. Arch. Math. *103* (2014), 285–290.

Authors' addresses: Peng-Jie Wong, Department of Mathematics and Statistics, Queen's University, Jeffery Hall, 48 University Ave., Kingston, K7L 3N6, Ontario, Canada, e-mail: pjwong@mast.queensu.ca.