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A NOTE ON THE INDEPENDENT DOMINATION NUMBER VERSUS THE DOMINATION NUMBER IN BIPARTITE GRAPHS

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Abstract. Let $\gamma(G)$ and $i(G)$ be the domination number and the independent domination number of G, respectively. Rad and Volkmann posted a conjecture that $i(G)/\gamma(G) \leqslant \Delta(G)/2$ for any graph G, where $\Delta(G)$ is its maximum degree (see N. J. Rad, L. Volkmann (2013)). In this work, we verify the conjecture for bipartite graphs. Several graph classes attaining the extremal bound and graphs containing odd cycles with the ratio larger than $\Delta(G)/2$ are provided as well.

Keywords: domination; independent domination

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Let $G = (V(G), E(G))$ be a simple undirected graph on n vertices. For $v \in V(G)$, $N_G(v) = \{w \in V(G)\colon vw \in E(G)\}$ is an open neighborhood of v in G. If $N_G(v) = \varphi$, v is called an isolated vertex. For $S \subseteq V(G)$, $N_G(S)$ is an open neighborhood of S and $G-S$ is a subgraph induced by $V(G)-S$. A vertex set $D \subseteq V(G)$ is a dominating set if every vertex of $V(G) - D$ is adjacent to some vertices of D. The minimum cardinality of a dominating set of G is called the domination number, denoted by $\gamma(G)$. Furthermore, a vertex set $I \subseteq V(G)$ is an independent dominating set if I is both an independent set and a dominating set in G, where an independent set is a set of vertices in a graph such that no two of them are adjacent. The minimum cardinality of an independent dominating set of G is called the independent domination number, denoted by $i(G)$. We refer to [8], [9] for notation and terminologies used but not defined here. Currently, lots of work related to domination number and independent domination number have been done, see [1], [2], [4].

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In general, it is very difficult to find the domination and independent domination numbers. Note that $i(G) \geqslant \gamma(G)$, which implies $i(G)/\gamma(G) \geqslant 1$. A natural question is to determine an upper bound for $i(G)/\gamma(G)$. Beyer, Proskurowski, Hedetniemi and Mitchell in [2] in 1977 showed that if L is a line graph of a tree, then $i(L)/\gamma(L) = 1$, where the line graph $L(G)$ of a connected graph G is a graph such that each vertex of $L(G)$ represents an edge of G and two vertices of $L(G)$ are adjacent if and only if their corresponding edges share a common endpoint in G. Since a line graph does not have an induced subgraph isomorphic to $K_{1,3}$, Allan and Laskar in [1] extended the previous result and obtained that if a graph G is a $K_{1,3}$-free graph, then $i(G)/\gamma(G) = 1$. In 2012, Goddard et al. in [5] continued the similar approach and proved that $i(G)/\gamma(G) \leqslant 3/2$ if G is a cubic graph. In 2013, Southey and Henning in [7] improved the previous bound to $i(G)/\gamma(G) \leqslant 4/3$ for a connected cubic graph G other than $K_{3,3}$. Additionally, Rad and Volkmann in [6] obtained an upper bound of $i(G)/\gamma(G)$ related to the maximum degree $\Delta(G)$ for any graph G and proposed a conjecture below.

Theorem 1 (Rad and Volkmann [6]). *If G is a graph, then*

$$\frac{i(G)}{\gamma(G)} \leqslant \begin{cases} \dfrac{\Delta(G)}{2} & \text{if } 3 \leqslant \Delta(G) \leqslant 5, \\[2ex] \Delta(G) - 3 + \dfrac{2}{\Delta(G) - 1} & \text{if } \Delta(G) \geqslant 6. \end{cases}$$

Conjecture 2 ([6]). *If G is a graph with $\Delta(G) \geqslant 3$, then $i(G)/\gamma(G) \leqslant \Delta(G)/2$.*

In 2014, Furuya et al. in [3] proved that $i(G)/\gamma(G) \leqslant \Delta(G) - 2\sqrt{\Delta(G)} + 2$ and gave a class of graphs which achieve the new upper bound. However, when $\Delta(G) \neq 4$, then $\Delta(G) - 2\sqrt{\Delta(G)} + 2 > \Delta(G)/2$. On the other hand, it is still very interesting to determine other classes of graphs for which Conjecture 2 holds.

Motivated by Conjecture 2 and previous results, we will show:

Theorem 3. *If G is a bipartite graph with maximum degree $\Delta(G) \geqslant 2$, then*

$$\frac{i(G)}{\gamma(G)} \leqslant \frac{\Delta(G)}{2}.$$

We now provide graphs containing some odd cycles, for which Conjecture 2 does not hold. For any large integer n, the graph G' consists of an odd cycle C_{2k+1} and $(2k+1)s$ vertices of degree 1 such that each vertex on C_{2k+1} is adjacent to exactly s vertices of degree 1, for any positive integers k, s. Then $\Delta(G') = s+2, \gamma(G') = 2k+1$ and $i(G') = k + (k+1)s$. By simple calculations, we can get $i(G')/\gamma(G') > \Delta(G')/2$ if $s > 2k + 2$.

P r o o f of Theorem 3. Let A and B be the partite sets of the bipartite graph G, and let D be a minimum dominating set of G. For conveniences, $\Delta(G) = \Delta$. Assume that I_0 is the set of isolated vertices in $G[D]$. Set $A_0 = A \cap I_0$, $A_1 = (D \setminus A_0) \cap A$, $B_0 = B \cap I_0$ and $B_1 = (D \setminus B_0) \cap B$. Then $|D| = |A_0| + |A_1| + |B_0| + |B_1|$. Renaming the sets A and B, if necessary, we may assume that $|A_1| \geqslant |B_1|$, which implies that $|B_1| \leqslant (|D| - |A_0| - |B_0|)/2 \leqslant |D|/2$. Let $A_2 = A \setminus (A_0 \cup A_1 \cup N_G(B_0))$. Since D is a dominating set of G, we have that $A_2 \subset N_G(B_1)$. Furthermore, by the choice of I_0, any vertex of A_1 (or B_1) is adjacent to at least one vertex of B_1 (or A_1, respectively) in $G[D]$. So, $|N_G(B_1) - A_1 - N_G(B_0)| \leqslant (\Delta - 1)|B_1|$, which implies that $|A_2| \leqslant (\Delta - 1)|B_1|$. We now consider the set $I = A_0 \cup A_1 \cup A_2 \cup B_0$. By the construction, we see that I is an independent set of G. We now show that I is a dominating set. In fact, if $v \in A \setminus I$, then v is dominated by B_0, and if $v \in B \setminus I$, then either $v \in B_1$, in which case v is dominated by $A_1 \cup A_2$, or $v \in B \setminus (B_0 \cup B_1)$, in which case v is dominated by $A_0 \cup A_1$. Thus, I is an independent dominating set of G, and so

$$\begin{aligned} i(G) &\leqslant |I| = |A_0| + |A_1| + |A_2| + |B_0| = |D| - |B_1| + |A_2| \\ &\leqslant |D| - |B_1| + (\Delta - 1)|B_1| = |D| + |B_1|(\Delta - 2) \\ &\leqslant |D| + \frac{1}{2}|D|(\Delta - 2) = \frac{1}{2}|D|\Delta = \frac{1}{2}\gamma(G)\Delta, \end{aligned}$$

or, equivalently, $i(G)/\gamma(G) \leqslant \Delta/2$. □

Remark. We see that Conjecture 2 holds for the bipartite graph G. The upper bound $\Delta/2$ can be achieved, and a balanced double star and a complete balanced bipartite graph are examples of bipartite graphs attaining the upper bound.

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