Czechoslovak Mathematical Journal

Dumitru Popa

Copies of l_p^n's uniformly in the spaces $\Pi_2(C[0,1],X)$ and $\Pi_1(C[0,1],X)$

Czechoslovak Mathematical Journal, Vol. 67 (2017), No. 2, 457–467

Persistent URL: http://dml.cz/dmlcz/146768

Czechoslovak Mathematical Journal, 67 (142) (2017), 457–467

COPIES OF l_p^n'S UNIFORMLY IN THE SPACES $\Pi_2(C[0,1],X)$ AND $\Pi_1(C[0,1],X)$

Dumitru Popa, Constanţa

Received January 8, 2016. First published May 4, 2017.

Abstract. We study the presence of copies of l_p^n's uniformly in the spaces $\Pi_2(C[0,1],X)$ and $\Pi_1(C[0,1],X)$. By using Dvoretzky's theorem we deduce that if X is an infinite-dimensional Banach space, then $\Pi_2(C[0,1],X)$ contains $\lambda\sqrt{2}$-uniformly copies of l_∞^n's and $\Pi_1(C[0,1],X)$ contains λ-uniformly copies of l_2^n's for all $\lambda > 1$. As an application, we show that if X is an infinite-dimensional Banach space then the spaces $\Pi_2(C[0,1],X)$ and $\Pi_1(C[0,1],X)$ are distinct, extending the well-known result that the spaces $\Pi_2(C[0,1],X)$ and $\mathcal{N}(C[0,1],X)$ are distinct.

Keywords: p-summing linear operators; copies of l_p^n's uniformly; local structure of a Banach space; multiplication operator; average

MSC 2010: 46B07, 47B10, 47L20, 46B28

1. Introduction and notation

The main purpose of this paper is to study the presence of copies of l_p^n's uniformly in the spaces $\Pi_2(C[0,1],X)$ and $\Pi_1(C[0,1],X)$. Let us fix some notation and concepts used below. The scalar field $\mathbb{R}$ (or $\mathbb{C}$) is denoted by $\mathbb{K}$ and if $n \in \mathbb{N}$, $1 \leqslant p \leqslant \infty$, then $l_p^n = (\mathbb{K}^n, \|\cdot\|_p)$, where $\|(\alpha_1,\ldots,\alpha_n)\|_p = \Bigl(\sum\limits_{i=1}^n |\alpha_i|^p\Bigr)^{1/p}$ if $p < \infty$ and $\|(\alpha_1,\ldots,\alpha_n)\|_\infty = \max\limits_{1\leqslant i\leqslant n} |\alpha_i|$. By $(e_i)_{1\leqslant i\leqslant n}$ we denote the standard unit vectors in $\mathbb{K}^n$, i.e. $e_i = (0,\ldots,0,1,0,\ldots,0)$. For $1 \leqslant p \leqslant \infty$ we write, as usual, p^* for the conjugate of p, i.e. $1/p + 1/p^* = 1$. If $\alpha = (\alpha_i)_{1\leqslant i\leqslant n} \in \mathbb{K}^n$, $1 \leqslant p,q \leqslant \infty$, $M_\alpha\colon l_p^n \to l_q^n$ is the multiplication operator, i.e. $M_\alpha((\xi_i)_{1\leqslant i\leqslant n}) := (\alpha_i\xi_i)_{1\leqslant i\leqslant n}$. By $r_n\colon [0,1] \to \mathbb{R}$, $r_n(t) = (-1)^{[2^n t]}$ we denote the Rademacher functions ($[\cdot]$ denotes the integer part) and $C[0,1]$ is the space of all scalar-valued continuous functions on $[0,1]$ under the uniform norm.

DOI: 10.21136/CMJ.2017.0009-16

Let $1 \leqslant p \leqslant \infty$ and $1 < \lambda < \infty$. We say that a Banach space X *contains* l_p^n*'s* λ*-uniformly* or that X *contains* λ*-uniformly copies of* l_p^n if for every $n \in \mathbb{N}$ there exists a linear operator $J_n\colon\ l_p^n \to X$ such that

$$\|\alpha\|_p \leqslant \|J_n(\alpha)\|_X \leqslant \lambda\|\alpha\|_p, \quad \alpha \in l_p^n$$

(see [3], page 260). Let X, Y be Banach spaces and $1 \leqslant p < \infty$. A linear operator $T\colon\ X \to Y$ is *p-summing* if there exists a constant $C \geqslant 0$ such that for every $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ the relation $\Big(\sum\limits_{i=1}^{n} \|T(x_i)\|^p\Big)^{1/p} \leqslant C \sup\limits_{\|x^*\|\leqslant 1} \Big(\sum\limits_{i=1}^{n} |x^*(x_i)|^p\Big)^{1/p}$ holds and the p-summing norm of T is defined by $\pi_p(T) := \min\{C\colon\ C \text{ as above}\}$. We denote by $\Pi_p(X,Y)$ the class of all p-summing operators from X into Y (see [2], [3], [4], [6]). Let X and Y be Banach spaces. If A is a set, the notation $(x_n)_{n\in\mathbb{N}} \subset A$ means that $x_n \in A$ for every $n \in \mathbb{N}$. A bounded linear operator $T\colon\ X \to Y$ is called *nuclear* if there exist $(x_n^*)_{n\in\mathbb{N}} \subset X^*$, $(y_n)_{n\in\mathbb{N}} \subset Y$such that $\sum\limits_{n=1}^{\infty} \|x_n^*\|\|y_n\| < \infty$ and $T(x) = \sum\limits_{n=1}^{\infty} x_n^*(x)y_n$ for $x \in X$; such a representation is called a *nuclear representation* of T and the nuclear norm of T is defined by $\|T\|_{\text{nuc}} := \inf\Big\{\sum\limits_{n=1}^{\infty} \|x_n^*\|\|y_n\|\Big\}$, where the infimum is taken over all the nuclear representations of T. We denote by $\mathcal{N}(X,Y)$ the space of all nuclear operators from X into Y (see [2], [3], [4], [6]). In [10], Theorem 4.2, it was shown that, if X is an infinite-dimensional Banach space, then $\mathcal{N}(C[0,1],X) \neq \Pi_2(C[0,1],X)$. As a natural consequence of our results, we recover the folklore result that if X is an infinite dimensional Banach space, then $\Pi_1(C[0,1],X) \neq \Pi_2(C[0,1],X)$, and hence $\mathcal{N}(C[0,1],X) \neq \Pi_2(C[0,1],X)$, see Corollary 1.

All notation and terminology, not otherwise explained, are as in [2], [3], [4], [6].

Preliminary results

The next Lemma is essentially well-known (see [8], Lemma 10).

Lemma 1. *Let* $1 \leqslant p \leqslant \infty$, $n \in \mathbb{N}$, $\alpha = (\alpha_i)_{1\leqslant i\leqslant n} \in \mathbb{K}^n$ *and let* $U_\alpha^n\colon\ C[0,1] \to l_p^n$ *be the operator defined by* $U_\alpha^n(f) = \big(\alpha_i \int_0^1 f(t)r_i(t)\,\mathrm{d}t\big)_{1\leqslant i\leqslant n}$. *Then:*

(i) $2^{-1/2}\|\alpha\|_r \leqslant \|U_\alpha^n\| \leqslant \pi_2(U_\alpha^n) \leqslant \|\alpha\|_r$ *if* $1 \leqslant p < 2$, *where* $1/p = 1/2 + 1/r$ *and* $2^{-1/2}\|\alpha\|_\infty \leqslant \|U_\alpha^n\| \leqslant \pi_2(U_\alpha^n) \leqslant \|\alpha\|_\infty$ *if* $2 \leqslant p \leqslant \infty$.

(ii) $\pi_1(U_\alpha^n) = \|\alpha\|_p$.

P r o o f. The representing measure of U_α^n is $G_\alpha^n\colon\ \Sigma \to l_p^n$ defined by $G_\alpha^n(E) := \big(\alpha_i \int_E r_i(t)\,\mathrm{d}t\big)_{1\leqslant i\leqslant n}$, where Σ is the σ-algebra of all borelian subsets of $[0,1]$, see [4], Theorem 1, page 152. Let $h^n_\alpha\colon [0,1] \to l^n_p$ be given by $h^n_\alpha(t) = (\alpha_i r_i(t))_{1\leqslant i\leqslant n}$ and observe that $G^n_\alpha(E) = \int_E h^n_\alpha(t)\,\mathrm{d}t$ for $E \in \Sigma$ (the Bochner integral).

(i) From [4], Theorem 1, page 152, and Proposition 11, page 4, we have

$$\|U^n_\alpha\| = \|G^n_\alpha\|([0,1]) = \sup_{\|y^*\|\leqslant 1} |y^* \circ G^n_\alpha|([0,1]) = \sup_{\|y^*\|\leqslant 1} \int_0^1 |\langle y^*, h^n_\alpha(t)\rangle|\,\mathrm{d}t$$

because $(y^* \circ G^n_\alpha)(E) = \int_E \langle y^*, h^n_\alpha(t)\rangle\,\mathrm{d}t$ and $|y^* \circ G^n_\alpha|([0,1]) = \int_0^1 |\langle y^*, h^n_\alpha(t)\rangle|\,\mathrm{d}t$. However, for any $y^* = (\xi_i)_{1\leqslant i\leqslant n} \in (l^n_p)^* = l^n_{p^*}$ we have $\langle y^*, h^n_\alpha(t)\rangle = \sum_{i=1}^n \xi_i\alpha_i r_i(t)$ and by Khinchin's inequality $2^{-1/2}\Bigl(\sum_{i=1}^n |\xi_i\alpha_i|^2\Bigr)^{1/2} \leqslant \int_0^1 |\langle y^*, h^n_\alpha(t)\rangle|\,\mathrm{d}t$, hence $2^{-1/2}\|M_\alpha\| \leqslant \|G^n_\alpha\|([0,1])$, where $M_\alpha\colon l^n_{p^*} \to l^n_2$ is the multiplication operator. Thus we have shown that $2^{-1/2}\|M_\alpha\colon l^n_{p^*} \to l^n_2\| \leqslant \|U^n_\alpha\|$. Let us note that always $\|U^n_\alpha\| \leqslant \pi_2(U^n_\alpha)$. Further, $U^n_\alpha\colon C[0,1] \overset{J}{\hookrightarrow} L_2[0,1] \overset{R}{\to} l^n_2 \overset{M_\alpha}{\to} l^n_p$ is a factorization of U^n_α, where J is the canonical inclusion and $R(f) = \bigl(\int_0^1 f(t)r_i(t)\,\mathrm{d}t\bigr)_{1\leqslant i\leqslant n}$. Since J is 2-summing with $\pi_2(J) = 1$ and $\|R\| = 1$, we deduce that $\pi_2(U^n_\alpha) \leqslant \|M_\alpha\colon l^n_2 \to l^n_p\|$. Now, as is well known, $\|M_\alpha\colon l^n_{p^*} \to l^n_2\| = \|M_\alpha\colon l^n_2 \to l^n_p\| = \|\alpha\|_r$ if $1 \leqslant p < 2$, where $1/p = 1/2 + 1/r$ and $\|M_\alpha\colon l^n_{p^*} \to l^n_2\| = \|M_\alpha\colon l^n_2 \to l^n_p\| = \max_{1\leqslant i\leqslant n} |\alpha_i| = \|\alpha\|_\infty$ if $2 \leqslant p \leqslant \infty$, see [1], page 218, and the proof of (i) is finished.

(ii) From [4], Theorem 3, page 162, $\pi_1(U^n_\alpha) = |G^n_\alpha|([0,1]) = \int_0^1 \|h^n_\alpha(t)\|_p\,\mathrm{d}t = \|\alpha\|_p$. □

In the sequel the technique named Average of a finite number of elements, introduced in [7], [9] is used to construct a useful kind of operators. Let us now fix some notation and recall this concept.

Let n be a natural number. For $(\lambda_1,\ldots,\lambda_n) \in \mathbb{K}^n$ we define the finite system denoted by $\mathrm{Average}(\lambda_i\colon 1\leqslant i\leqslant n)$ as being the system with 2^n elements obtained by arranging in the lexicographical order of $D_n := \{-1,1\}^n$ the elements $\varepsilon_1\lambda_1 + \ldots + \varepsilon_n\lambda_n$ for $(\varepsilon_1,\ldots,\varepsilon_n) \in D_n$. (On $\{-1,1\}$ we consider the natural order). Thus, as sets we have

$$\mathrm{Average}(\lambda_i\colon 1\leqslant i\leqslant n) = \{\varepsilon_1\lambda_1 + \ldots + \varepsilon_n\lambda_n\colon (\varepsilon_1,\ldots,\varepsilon_n) \in D_n\}.$$

Let us note that if $(\lambda_i)_{1\leqslant i\leqslant n} \in \mathbb{K}^n$ and $(e_{(\varepsilon_1,\ldots,\varepsilon_n)})_{(\varepsilon_1,\ldots,\varepsilon_n)\in D_n}$ are the standard unit vectors in $\mathbb{K}^{2^n}$ ordered in the lexicographical order of D_n, then the following equality in $\mathbb{K}^{2^n}$ holds:

$$(1)\qquad \mathrm{Average}(\lambda_i\colon 1\leqslant i\leqslant n) = \sum_{(\varepsilon_1,\ldots,\varepsilon_n)\in D_n} (\varepsilon_1\lambda_1 + \ldots + \varepsilon_n\lambda_n)e_{(\varepsilon_1,\ldots,\varepsilon_n)}.$$

If $1 \leqslant p < \infty$, by Khinchin's inequality we have

$$A_p\|(\lambda_1,\ldots,\lambda_n)\|_2 \leqslant \Big\|\operatorname{Average}\Big(\frac{1}{2^{n/p}}\lambda_i\colon\ 1\leqslant i\leqslant n\Big)\Big\|_p \tag{2}$$
$$= \bigg(\frac{1}{2^n}\sum_{(\varepsilon_1,\ldots,\varepsilon_n)\in D_n}|\varepsilon_1\lambda_1+\ldots+\varepsilon_n\lambda_n|^p\bigg)^{1/p}$$
$$\leqslant B_p\|(\lambda_1,\ldots,\lambda_n)\|_2.$$

Above and in the sequel A_p, B_p are Khinchin's constants (see [3]).

Lemma 2. *Let $1 \leqslant p < \infty$, $n \in \mathbb{N}$, $\alpha = (\alpha_i)_{1\leqslant i\leqslant n} \in \mathbb{K}^n$ and let $Av^n_\alpha\colon C[0,1] \to l_p^{2^n}$ be the operator defined by*

$$Av^n_\alpha(f) = \operatorname{Average}\bigg(\frac{\alpha_i}{2^{n/p}}\int_0^1 f(t)r_i(t)\,\mathrm{d}t\colon\ 1\leqslant i\leqslant n\bigg).$$

Then:

(i) $A_p 2^{-1/2}\|\alpha\|_\infty \leqslant \pi_2(Av^n_\alpha) \leqslant B_p\|\alpha\|_\infty$.

(ii) $A_p\|\alpha\|_2 \leqslant \pi_1(Av^n_\alpha) \leqslant B_p\|\alpha\|_2$.

P r o o f. Let $f \in C[0,1]$. From the relation (2) we have

$$A_p\|U^n_\alpha(f)\|_2 \leqslant \|Av^n_\alpha(f)\| \leqslant B_p\|U^n_\alpha(f)\|_2$$

where $U^n_\alpha\colon C[0,1] \to l_2^n$ is defined by $U^n_\alpha(f) = \big(\alpha_i\int_0^1 f(t)r_i(t)\,\mathrm{d}t\big)_{1\leqslant i\leqslant n}$. Thus

$$A_p\pi_2(U^n_\alpha) \leqslant \pi_2(Av^n_\alpha) \leqslant B_p\pi_2(U^n_\alpha) \quad\text{and}\quad A_p\pi_1(U^n_\alpha) \leqslant \pi_1(Av^n_\alpha) \leqslant B_p\pi_1(U^n_\alpha).$$

The conclusion follows, because in this case, by Lemma 1, $2^{-1/2}\|\alpha\|_\infty \leqslant \pi_2(U^n_\alpha) \leqslant \|\alpha\|_\infty$ and $\pi_1(U^n_\alpha) = \|\alpha\|_2$. □

We need also the second average which we describe next. Let n be a natural number. Let us note that if $(\lambda_1,\ldots,\lambda_n) \in \mathbb{K}^n$ then

$$c_{\mathbb{K}}\sum_{i=1}^n|\lambda_i| \leqslant \|\operatorname{Average}(\lambda_i\colon\ 1\leqslant i\leqslant n)\|_\infty \leqslant \sum_{i=1}^n|\lambda_i| \tag{3}$$

where $c_{\mathbb{K}} = 1$ if $\mathbb{K} := \mathbb{R}$; $c_{\mathbb{K}} = 1/2$ if $\mathbb{K} := \mathbb{C}$ (in this case consider the real and the imaginary part).

For $(\lambda_1,\ldots,\lambda_n) \in \mathbb{K}^n$ let us denote the 2^n elements of the set $\operatorname{Average}(\lambda_i\colon\ 1\leqslant i\leqslant n)$ by $\{\beta_1,\beta_2,\ldots,\beta_{2^n}\}$ and apply the same procedure; we define

$$\begin{aligned}\operatorname{Saverage}(\lambda_i\colon\ 1\leqslant i\leqslant n) &:= \operatorname{Average}(\beta_i\colon\ 1\leqslant i\leqslant 2^n)\\ &= \{\varepsilon_1\beta_1+\ldots+\varepsilon_{2^n}\beta_{2^n}\colon\ (\varepsilon_1,\ldots,\varepsilon_{2^n})\in D_{2^n}\} \subset \mathbb{K}^{2^{2^n}}.\end{aligned}$$

From the relation (3) we have

$$\frac{c_{\mathbb{K}}}{2^n}\|(\beta_1,\ldots,\beta_{2^n})\|_1 \leqslant \frac{1}{2^n}\|\operatorname{Saverage}(\lambda_i\colon\ 1\leqslant i\leqslant n)\|_\infty \leqslant \frac{1}{2^n}\|(\beta_1,\ldots,\beta_{2^n})\|_1$$

and since by Khinchin's inequality

$$\begin{aligned}\frac{1}{\sqrt{2}}\|(\lambda_1,\ldots,\lambda_n)\|_2 &\leqslant \frac{1}{2^n}\sum_{i=1}^{2^n}|\beta_i| = \frac{1}{2^n}\sum_{(\varepsilon_1,\ldots,\varepsilon_n)\in D_n}|\varepsilon_1\lambda_1+\ldots+\varepsilon_n\lambda_n|\\ &\leqslant \|(\lambda_1,\ldots,\lambda_n)\|_2\end{aligned}$$

we get

$$\frac{c_{\mathbb{K}}}{\sqrt{2}}\|(\lambda_1,\ldots,\lambda_n)\|_2 \leqslant \frac{1}{2^n}\|\operatorname{Saverage}(\lambda_i\colon\ 1\leqslant i\leqslant n)\|_\infty \leqslant \|(\lambda_1,\ldots,\lambda_n)\|_2. \tag{4}$$

Lemma 3. (a) *Let* $n\in\mathbb{N}$, $\alpha=(\alpha_i)_{1\leqslant i\leqslant n}\in\mathbb{K}^n$ *and let* $Av_\alpha^n\colon\ C[0,1]\to l_\infty^{2^n}$ *be the operator defined by*

$$Av_\alpha^n(f)=\operatorname{Average}\Big(\alpha_i\int_0^1 f(t)r_i(t)\,\mathrm{d}t\colon\ 1\leqslant i\leqslant n\Big).$$

Then:

(i) $c_{\mathbb{K}}2^{-1/2}\|\alpha\|_2\leqslant\pi_2(Av_\alpha^n)\leqslant\|\alpha\|_2$.

(ii) $c_{\mathbb{K}}\|\alpha\|_1\leqslant\pi_1(Av_\alpha^n)\leqslant\|\alpha\|_1$.

(b) *Let* $n\in\mathbb{N}$, $\alpha=(\alpha_i)_{1\leqslant i\leqslant n}\in\mathbb{K}^n$ *and let* $\operatorname{Sav}_\alpha^n\colon\ C[0,1]\to l_\infty^{2^{2^n}}$ *be the operator defined by*

$$\operatorname{Sav}_\alpha^n(f):=\operatorname{Saverage}\Big(\frac{1}{2^n}\alpha_i\int_0^1 f(t)r_i(t)\,\mathrm{d}t\colon\ 1\leqslant i\leqslant n\Big).$$

Then:

(i) $c_{\mathbb{K}}2^{-1}\|\alpha\|_\infty\leqslant\pi_2(\operatorname{Sav}_\alpha^n)\leqslant\|\alpha\|_\infty$.

(ii) $c_{\mathbb{K}}2^{-1/2}\|\alpha\|_2\leqslant\pi_1(\operatorname{Sav}_\alpha^n)\leqslant\|\alpha\|_2$.

Proof. (a) Let $f\in C[0,1]$. From the relation (3) we have

$$c_{\mathbb{K}}\|U_\alpha^n(f)\|_1\leqslant\|Av_\alpha^n(f)\|_\infty\leqslant\|U_\alpha^n(f)\|_1$$

where $U_\alpha^n\colon\ C[0,1]\to l_1^n$ is defined by $U_\alpha^n(f)=(\alpha_i\int_0^1 f(t)r_i(t)\,\mathrm{d}t)_{1\leqslant i\leqslant n}$. Thus, easily,

$$c_{\mathbb{K}}\pi_2(U_\alpha^n)\leqslant\pi_2(Av_\alpha^n)\leqslant\pi_2(U_\alpha^n)\text{ and }c_{\mathbb{K}}\pi_1(U_\alpha^n)\leqslant\pi_2(Av_\alpha^n)\leqslant\pi_1(U_\alpha^n).$$

The conclusion follows, because in this case, by Lemma 1, $2^{-1/2}\|\alpha\|_2 \leqslant \pi_2(U_\alpha^n) \leqslant \|\alpha\|_2$ and $\pi_1(U_\alpha^n) = \|\alpha\|_1$.

(b) Let $f \in C[0,1]$. From the relation (4) we have

$$\frac{c_{\mathbb{K}}}{\sqrt{2}}\|U_\alpha^n(f)\|_2 \leqslant \|\operatorname{Sav}_\alpha^n(f)\|_\infty \leqslant \|U_\alpha^n(f)\|_2$$

where $U_\alpha^n\colon C[0,1] \to l_2^n$ is defined by $U_\alpha^n(f) = \bigl(\alpha_i \int_0^1 f(t) r_i(t)\,\mathrm{d}t\bigr)_{1\leqslant i\leqslant n}$. Thus

$$\frac{c_{\mathbb{K}}}{\sqrt{2}}\pi_2(U_\alpha^n) \leqslant \pi_2(\operatorname{Sav}_\alpha^n) \leqslant \pi_2(U_\alpha^n); \quad \frac{c_{\mathbb{K}}}{\sqrt{2}}\pi_1(U_\alpha^n) \leqslant \pi_1(\operatorname{Sav}_\alpha^n) \leqslant \pi_1(U_\alpha^n).$$

The conclusion follows, because in this case, by Lemma 1, $2^{-1/2}\|\alpha\|_\infty \leqslant \pi_2(U_\alpha^n) \leqslant \|\alpha\|_\infty$ and $\pi_1(U_\alpha^n) = \|\alpha\|_2$. □

The results

In the next theorem, which is the main result of this paper, we show how the local structure of the spaces $\Pi_2(C[0,1],X)$ and $\Pi_1(C[0,1],X)$ depends on the local structure of X.

Theorem 4. *Let $1 \leqslant p \leqslant \infty$, $1 < \lambda < \infty$ and let X be a Banach space which contains l_p^n's λ-uniformly. Then:*

(i) *For $1 \leqslant p < 2$, $\Pi_2(C[0,1],X)$ contains $\lambda\sqrt{2}$-uniformly copies of l_r^n's where $1/p = 1/2 + 1/r$.*

(ii) *For $2 \leqslant p \leqslant \infty$, $\Pi_2(C[0,1],X)$ contains $\lambda\sqrt{2}$-uniformly copies of l_∞^n's.*

(iii) *For $1 \leqslant p < \infty$, $\Pi_2(C[0,1],X)$ contains $\lambda B_p\sqrt{2}/A_p$-uniformly copies of l_∞^n's.*

(iv) *$\Pi_1(C[0,1],X)$ contains λ-uniformly copies of l_p^n's.*

(v) *For $1 \leqslant p < \infty$, $\Pi_1(C[0,1],X)$ contains $\lambda B_p/A_p$-uniformly copies of l_2^n's.*

(vi) *For $1 \leqslant p < \infty$, the spaces $\Pi_2(C[0,1],X)$ and $\Pi_1(C[0,1],X)$ are distinct; in particular, $\Pi_2(C[0,1],X) \neq \mathcal{N}(C[0,1],X)$.*

P r o o f. (i), (ii) and (iv). Let $n \in \mathbb{N}$ be arbitrary. By hypothesis there exists a bounded linear operator $J_n\colon l_p^n \to X$ such that

$$\|\alpha\|_p \leqslant \|J_n(\alpha)\|_X \leqslant \lambda\|\alpha\|_p, \quad \alpha \in l_p^n. \tag{5}$$

Let us define $A_n\colon \mathbb{K}^n \to L(C[0,1],X)$ by $A_n(\alpha) = J_n \circ U_\alpha^n$, where $U_\alpha^n\colon C[0,1] \to l_p^n$ is the operator from Lemma 1. Though not needed in the sequel, let us note that if $\alpha = (\alpha_i)_{1\leqslant i\leqslant n} \in \mathbb{K}^n$ and $f \in C[0,1]$ then

$$A_n(\alpha)(f) = \sum_{i=1}^n \alpha_i \Bigl(\int_0^1 f(t) r_i(t)\,\mathrm{d}t\Bigr) J_n(e_i).$$

Let $\alpha \in \mathbb{K}^n$. For every $f \in C[0,1]$ by (5) we have

$$\|U_\alpha^n(f)\|_p \leqslant \|[A_n(\alpha)](f)\|_X = \|J_n(U_\alpha^n(f))\|_X \leqslant \lambda \|U_\alpha^n(f)\|_p$$

and by the definition of p-summing operators we deduce that

$$(6) \qquad \pi_2(U_\alpha^n) \leqslant \pi_2(A_n(\alpha)) \leqslant \lambda \pi_2(U_\alpha^n) \quad \text{and} \quad \pi_1(U_\alpha^n) \leqslant \pi_1(A_n(\alpha)) \leqslant \lambda \pi_1(U_\alpha^n).$$

From (6) and Lemma 1 we obtain

$$\begin{aligned} \|\alpha\|_r &\leqslant \pi_2(\sqrt{2}A_n(\alpha)) \leqslant \lambda\sqrt{2}\|\alpha\|_r \quad \text{if } 1 \leqslant p < 2, \text{ where } \frac{1}{p} = \frac{1}{2} + \frac{1}{r}, \\ \|\alpha\|_\infty &\leqslant \pi_2(\sqrt{2}A_n(\alpha)) \leqslant \lambda\sqrt{2}\|\alpha\|_\infty \quad \text{if } 2 \leqslant p < \infty, \\ \|\alpha\|_p &\leqslant \pi_1(A_n(\alpha)) \leqslant \lambda\|\alpha\|_p, \end{aligned}$$

which ends the proof of (i), (ii) and (iv).

(iii) and (v). Let $n \in \mathbb{N}$ be arbitrary. By hypothesis there exists a bounded linear operator $J_{2^n}\colon l_p^{2^n} \to X$ such that

$$(7) \qquad \|\xi\|_p \leqslant \|J_{2^n}(\xi)\|_X \leqslant \lambda\|\xi\|_p, \quad \xi \in l_p^{2^n}.$$

We define $Av_n\colon \mathbb{K}^n \to L(C[0,1], X)$ by $Av_n(\alpha) = J_{2^n} \circ Av_\alpha^n$, where $Av_\alpha^n\colon C[0,1] \to l_p^{2^n}$ is the operator from Lemma 2. Again, though not needed in the sequel, let us note that if $\alpha = (\alpha_i)_{1 \leqslant i \leqslant n} \in \mathbb{K}^n$ and $f \in C[0,1]$ we have

$$\begin{aligned} [Av_n(\alpha)](f) = \frac{1}{2^{n/p}} \sum_{(\varepsilon_1,\ldots,\varepsilon_n)\in D_n} \Bigl(\varepsilon_1\alpha_1 \int_0^1 f(t)r_1(t)\,\mathrm{d}t + \ldots \\ + \varepsilon_n\alpha_n \int_0^1 f(t)r_n(t)\,\mathrm{d}t\Bigr) J_{2^n}(e_{(\varepsilon_1,\ldots,\varepsilon_n)}). \end{aligned}$$

Let $\alpha \in \mathbb{K}^n$. For every $f \in C[0,1]$ by (7) we have

$$\|Av_\alpha^n(f)\|_p \leqslant \|[Av_n(\alpha)](f)\|_X = \|J_{2^n}(Av_\alpha^n(f))\|_X \leqslant \lambda\|Av_\alpha^n(f)\|_p$$

and by the definition of p-summing operators we deduce that

$$(8) \quad \pi_2(Av_\alpha^n) \leqslant \pi_2(Av_n(\alpha)) \leqslant \lambda\pi_2(Av_\alpha^n) \quad \text{and} \quad \pi_1(Av_\alpha^n) \leqslant \pi_1(Av_n(\alpha)) \leqslant \lambda\pi_1(Av_\alpha^n).$$

Since by Lemma 2

$$\frac{A_p}{\sqrt{2}}\|\alpha\|_\infty \leqslant \pi_2(Av_n(\alpha)) \leqslant B_p\|\alpha\|_\infty \quad \text{and} \quad A_p\|\alpha\|_2 \leqslant \pi_1(Av_n(\alpha)) \leqslant B_p\|\alpha\|_2,$$

from (8) we obtain

$$\|\alpha\|_\infty \leqslant \pi_2\Big(\frac{\sqrt{2}}{A_p}Av_n(\alpha)\Big) \leqslant \frac{\lambda B_p\sqrt{2}}{A_p}\|\alpha\|_\infty; \quad \|\alpha\|_2 \leqslant \pi_1\Big(\frac{Av_n(\alpha)}{A_p}\Big) \leqslant \frac{\lambda B_p}{A_p}\|\alpha\|_2,$$

which ends the proof of (iii) and (v).

(vi) If $\Pi_2(C[0,1],X) = \Pi_1(C[0,1],X)$, then by the open mapping theorem it follows that there exists $C>0$ such that $\pi_1(T) \leqslant C\pi_2(T)$ for all $T \in \Pi_1(C[0,1],X)$. In particular, $\pi_1(A_n(\alpha)) \leqslant C\pi_2(A_n(\alpha))$ for all natural numbers n and all $\alpha \in \mathbb{K}^n$. By (i), (ii) and (iv) for all natural numbers n and all $\alpha \in \mathbb{K}^n$ we have $\|\alpha\|_p \leqslant C\|\alpha\|_r$ if $1 \leqslant p < 2$, where $1/p = 1/2 + 1/r$, or $\|\alpha\|_p \leqslant C\|\alpha\|_\infty$ if $2 \leqslant p < \infty$. Taking

$$\alpha = (\underbrace{1,\ldots,1}_{n\text{-times}})$$

we get that for all natural numbers n we have $n \leqslant C^2$ if $1 \leqslant p < 2$, or $n \leqslant C^p$ if $2 \leqslant p < \infty$, which is impossible. Let us note that a contradiction can be obtained if we use (iii) or (v). If $\Pi_2(C[0,1],X) = \mathcal{N}(C[0,1],X)$ then, since $\mathcal{N}(C[0,1],X) \subseteq \Pi_1(C[0,1],X) \subseteq \Pi_2(C[0,1],X)$, it follows that $\Pi_1(C[0,1],X) = \Pi_2(C[0,1],X)$, which as we have shown above is impossible. □

As a natural consequence of Theorem 4, we recover the folklore result that if X is an infinite-dimensional Banach space then the spaces $\Pi_2(C[0,1],X)$ and $\Pi_1(C[0,1],X)$ are distinct. This extends the well-known result that the spaces $\Pi_2(C[0,1],X)$ and $\mathcal{N}(C[0,1],X)$ are distinct, see [10], Theorem 4.2.

Corollary 5. *Let X be an infinite dimensional Banach space. Then:*
(i) $\Pi_2(C[0,1],X)$ *contains $\lambda\sqrt{2}$-uniformly copies of l_∞^n's for all $\lambda > 1$.*
(ii) $\Pi_1(C[0,1],X)$ *contains λ-uniformly copies of l_2^n's for all $\lambda > 1$.*
(iii) *The spaces $\Pi_2(C[0,1],X)$ and $\Pi_1(C[0,1],X)$ are distinct; in particular, $\Pi_2(C[0,1],X) \neq \mathcal{N}(C[0,1],X)$.*

Proof. Since X is infinite-dimensional, by the famous Dvoretzky theorem, see [3], Chapter 19, X contains l_2^n's λ-uniformly for all $1 < \lambda < \infty$. The statement follows by taking $p = 2$ in Theorem 4. □

Let us note that for $p = \infty$ in Theorem 4 ((ii) and (iv)) it follows that if $1 < \lambda < \infty$ and X is a Banach space which contains l_∞^n's λ-uniformly, then $\Pi_2(C[0,1],X)$ contains $\lambda\sqrt{2}$-uniformly copies of l_∞^n's and $\Pi_1(C[0,1],X)$ contains λ-uniformly copies of l_∞^n's, so in this case, there is no distinction between these classes.

We prove now a natural completion of Theorem 4. It shows that for $p = \infty$ in Theorem 4 we have also a distinction if we use the first and the second average.

Theorem 6. *Let $1 < \lambda < \infty$ and let X be a Banach space which contains l_∞^n's λ-uniformly. Then:*

(i) *$\Pi_2(C[0,1], X)$ contains $\lambda\sqrt{2}$-uniformly copies of l_2^n's in the real case ($2\lambda\sqrt{2}$-uniformly copies of l_2^n's in the complex case).*

(ii) *$\Pi_1(C[0,1], X)$ contains λ-uniformly copies of l_1^n's in the real case (2λ-uniformly copies of l_1^n's in the complex case).*

(iii) *$\Pi_2(C[0,1], X)$ contains 2λ-uniformly copies of l_∞^n's in the real case (4λ-uniformly copies of l_∞^n's in the complex case).*

(iv) *$\Pi_1(C[0,1], X)$ contains $\lambda\sqrt{2}$-uniformly copies of l_2^n's in the real case ($2\lambda\sqrt{2}$-uniformly copies of l_2^n's in the complex case).*

Proof. (i) and (ii). Let $n \in \mathbb{N}$ be arbitrary. By hypothesis there exists a bounded linear operator $J_{2^n}\colon l_\infty^{2^n} \to X$ such that

$$\|\xi\|_\infty \leqslant \|J_{2^n}(\xi)\|_X \leqslant \lambda\|\xi\|_\infty, \quad \xi \in l_\infty^{2^n}. \tag{9}$$

We define $Av_n\colon \mathbb{K}^n \to L(C[0,1], X)$ by $Av_n(\alpha) = J_{2^n} \circ Av_\alpha^n$, where $Av_\alpha^n\colon C[0,1] \to l_\infty^{2^n}$ is the operator from Lemma 3. Let us note (not used in the sequel) the explicit expression,

$$[Av_n(\alpha)](f) = \sum_{(\varepsilon_1,\ldots,\varepsilon_n)\in D_n} \biggl(\varepsilon_1\alpha_1 \int_0^1 f(t)r_1(t)\,\mathrm{d}t + \ldots + \varepsilon_n\alpha_n \int_0^1 f(t)r_n(t)\,\mathrm{d}t\biggr) J_{2^n}(e_{(\varepsilon_1,\ldots,\varepsilon_n)})$$

where $\alpha = (\alpha_i)_{1\leqslant i\leqslant n} \in \mathbb{K}^n$ (see also the equality (1)). Let $\alpha \in \mathbb{K}^n$. For every $f \in C[0,1]$ by (9) we have

$$\|Av_\alpha^n(f)\|_\infty \leqslant \|[Av_n(\alpha)](f)\|_X = \|J_{2^n}(Av_\alpha^n(f))\|_X \leqslant \lambda\|Av_\alpha^n(f)\|_\infty,$$

and by the definition of p-summing operators we deduce that

$$\pi_2(Av_\alpha^n) \leqslant \pi_2(Av_n(\alpha)) \leqslant \lambda\pi_2(Av_\alpha^n) \tag{10}$$

and

$$\pi_1(Av_\alpha^n) \leqslant \pi_1(Av_n(\alpha)) \leqslant \lambda\pi_1(Av_\alpha^n).$$

Since by Lemma 3

$$\frac{c_{\mathbb{K}}}{\sqrt{2}}\|\alpha\|_2 \leqslant \pi_2(Av_n(\alpha)) \leqslant \|\alpha\|_2 \quad \text{and} \quad c_{\mathbb{K}}\|\alpha\|_1 \leqslant \pi_1(Av_n(\alpha)) \leqslant \|\alpha\|_1,$$

from (10) we obtain

$$\|\alpha\|_2 \leqslant \pi_2\Big(\frac{\sqrt{2}}{c_{\mathbb{K}}}Av_n(\alpha)\Big) \leqslant \frac{\lambda\sqrt{2}}{c_{\mathbb{K}}}\|\alpha\|_2 \quad \text{and} \quad \|\alpha\|_1 \leqslant \pi_1\Big(\frac{Av_n(\alpha)}{c_{\mathbb{K}}}\Big) \leqslant \frac{\lambda}{c_{\mathbb{K}}}\|\alpha\|_1$$

which ends the proof of (i) and (ii).

(iii) and (iv). Let $n \in \mathbb{N}$ be arbitrary. By hypothesis there exists a bounded linear operator $J_{2^{2^n}} : l_\infty^{2^{2^n}} \to X$ such that

$$\|\xi\|_\infty \leqslant \|J_{2^{2^n}}(\xi)\|_X \leqslant \lambda\|\xi\|_\infty, \quad \xi \in l_\infty^{2^{2^n}}. \tag{11}$$

We define $\operatorname{Sav}_n : \mathbb{K}^n \to L(C[0,1], X)$ by $\operatorname{Sav}_n(\alpha) = J_{2^{2^n}} \circ \operatorname{Sav}_\alpha^n$ where $\operatorname{Sav}_\alpha^n : C[0,1] \to l_\infty^{2^{2^n}}$ is the operator from Lemma 3. We leave for the interested reader to write the explicit expresion for $[\operatorname{Sav}_n(\alpha)](f)$, which again is not used in the sequel. Let $\alpha \in \mathbb{K}^n$. For every $f \in C[0,1]$ by (11) we have

$$\|\operatorname{Sav}_\alpha^n(f)\|_\infty \leqslant \|[\operatorname{Sav}_n(\alpha)](f)\|_X = \|J_{2^{2^n}}(\operatorname{Sav}_\alpha^n(f))\|_X \leqslant \lambda\|\operatorname{Sav}_\alpha^n(f)\|_\infty$$

and by the definition of p-summing operators we deduce that

$$\pi_2(\operatorname{Sav}_\alpha^n) \leqslant \pi_2(\operatorname{Sav}_n(\alpha)) \leqslant \lambda\pi_2(\operatorname{Sav}_\alpha^n) \tag{12}$$

and

$$\pi_1(\operatorname{Sav}_\alpha^n) \leqslant \pi_1(\operatorname{Sav}_n(\alpha)) \leqslant \lambda\pi_1(\operatorname{Sav}_\alpha^n).$$

Since by Lemma 3

$$\frac{c_{\mathbb{K}}}{2}\|\alpha\|_\infty \leqslant \pi_2(\operatorname{Sav}_n(\alpha)) \leqslant \|\alpha\|_\infty \quad \text{and} \quad \frac{c_{\mathbb{K}}}{\sqrt{2}}\|\alpha\|_2 \leqslant \pi_1(\operatorname{Sav}_n(\alpha)) \leqslant \|\alpha\|_2,$$

from (12) we obtain

$$\|\alpha\|_\infty \leqslant \pi_2\Big(\frac{2}{c_{\mathbb{K}}}\operatorname{Sav}_n(\alpha)\Big) \leqslant \frac{2\lambda}{c_{\mathbb{K}}}\|\alpha\|_\infty \text{ and } \|\alpha\|_2 \leqslant \pi_1\Big(\frac{\sqrt{2}\operatorname{Sav}_n(\alpha)}{c_{\mathbb{K}}}\Big) \leqslant \frac{\lambda\sqrt{2}}{c_{\mathbb{K}}}\|\alpha\|_2,$$

which ends the proof of (iii) and (iv). □

In [5] was shown that the space $\Pi_1(C[0,1], X)$ can be identified with the so called space $l_1^{\text{tree}}(X)$; we refer the reader to the paper [5] for the definition of this space and more details. From Theorems 4, 6 and Corollary 5 we get

Corollary 7. (a) *Let $1 \leqslant p \leqslant \infty$, $1 < \lambda < \infty$ and let X be a Banach space which contains l_p^n's λ-uniformly. Then:*

(i) *$l_1^{\rm tree}(X)$ contains λ-uniformly copies of l_p^n's.*

(ii) *For $1 \leqslant p < \infty$, $l_1^{\rm tree}(X)$ contains $\lambda B_p/A_p$-uniformly copies of l_2^n's.*

(b) *Let $1 < \lambda < \infty$ and let X be a Banach space which contains l_∞^n's λ-uniformly. Then:*

(i) *$l_1^{\rm tree}(X)$ contains $\lambda\sqrt{2}$-uniformly copies of l_1^n's in the real case ($2\lambda\sqrt{2}$-uniformly copies of l_1^n's in the complex case).*

(ii) *$l_1^{\rm tree}(X)$ contains λ-uniformly copies of l_2^n's in the real case (2λ-uniformly copies of l_2^n's in the complex case).*

(c) *Let X be an infinite dimensional Banach space. Then $l_1^{\rm tree}(X)$ contains λ-uniformly copies of l_2^n's for all $\lambda > 1$.*

Acknowledgement. We thank the referee for his/her constructive comments and remarks which have improved the quality of presentation.

References

[1] *C. Costara, D. Popa*: Exercises in Functional Analysis. Kluwer Texts in the Mathematical Sciences 26, Kluwer Academic Publishers Group, Dordrecht, 2003.

[2] *A. Defant, K. Floret*: Tensor Norms and Operator Ideals. North-Holland Mathematics Studies 176, North-Holland Publishing, Amsterdam, 1993.

[3] *J. Diestel, H. Jarchow, A. Tonge*: Absolutely Summing Operators. Cambridge Studies in Advanced Mathematics 43, Cambridge University Press, Cambridge, 1995.

[4] *J. Diestel, J. J. Uhl,* Jr.: Vector Measures. Mathematical Surveys 15, American Mathematical Society, Providence, 1977.

[5] *Å. Lima, V. Lima, E. Oja*: Absolutely summing operators on $C[0,1]$ as a tree space and the bounded approximation property. J. Funct. Anal. *259* (2010), 2886–2901.

[6] *A. Pietsch*: Operator Ideals. Mathematische Monographien 16, VEB Deutscher der Wissenschaften, Berlin, 1978.

[7] *D. Popa*: Examples of operators on $C[0,1]$ distinguishing certain operator ideals. Arch. Math. *88* (2007), 349–357.

[8] *D. Popa*: Khinchin's inequality, Dunford-Pettis and compact operators on the space $C([0,1],X)$. Proc. Indian Acad. Sci., Math. Sci. *117* (2007), 13–30.

[9] *D. Popa*: Averages and compact, absolutely summing and nuclear operators on $C(\Omega)$. J. Korean Math. Soc. *47* (2010), 899–924.

[10] *M. A. Sofi*: Factoring operators over Hilbert-Schmidt maps and vector measures. Indag. Math., New Ser. *20* (2009), 273–284.

Author's address: Dumitru Popa, Department of Mathematics, Ovidius University of Constanţa, Bd. Mamaia 124, 900527 Constanţa, Romania, e-mail: `dpopa@univ-ovidius.ro`.