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SOME MEAN VALUE THEOREMS AS CONSEQUENCES OF THE DARBOUX PROPERTY

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Abstract. The aim of the paper is to present some mean value theorems obtained as consequences of the intermediate value property. First, we will prove that any nonextremum value of a Darboux function can be represented as an arithmetic, geometric or harmonic mean of some different values of this function. Then, we will present some extensions of the Cauchy or Lagrange Theorem in classical or integral form. Also, we include similar results involving divided differences. The paper was motivated by some problems published in mathematical journals.

Keywords: Darboux function; mean value theorem; continuous function; integrable function; differentiable function; arithmetic mean; geometric mean; harmonic mean

MSC 2010: 26A15, 26A24, 26A42

1. Introduction

The mean value theorems represent some of the most useful mathematical analysis tools. The first result is due to Lagrange (1736–1813). The mean value theorem in its modern form was stated by Cauchy (1789–1857). In the years that followed, more mathematicians investigated this subject. As consequences of this fact, now we can find similar results, more generalizations or extensions. Sahoo and Riedle's book [14] presents a large collection of old and new mean value theorems.

The authors of this paper decided to investigate another direction. Our idea is coming from the next two problems, posed recently by Pangsriiam [10], and Plaza and Rodrigues [11], respectively.

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P r o b l e m 1.1. Let $f\colon [0,1] \to \mathbb{R}$ be a function continuous on $[0,1]$ and differentiable on $(0,1)$, with $f(0)=0$ and $f(1)=1$. Show that for each positive integer n, there exist distinct numbers $c_1, c_2, \ldots, c_n \in (0,1)$ such that

$$f'(c_1)f'(c_2)\ldots f'(c_n) = 1.$$

P r o b l e m 1.2. Let $f\colon [0,1] \to \mathbb{R}$ be a function continuous on $[0,1]$ such that $\int_0^1 f(x)\,\mathrm{d}x = 1$ and let n be a positive integer. Show:

(a) There are distinct $c_1, c_2, \ldots, c_n \in (0,1)$ such that

$$f(c_1) + f(c_2) + \ldots + f(c_n) = n.$$

(b) There are distinct $c_1, c_2, \ldots, c_n \in (0,1)$ such that

$$\frac{1}{f(c_1)} + \frac{1}{f(c_2)} + \ldots + \frac{1}{f(c_n)} = n.$$

The solution of the first problem has not been published yet. Garcia and Suarez [2] proposed a solution to Problem 1.2. Our searches in other mathematical journals or books led us to other problems of the same type. We will enumerate a few:

P r o b l e m 1.3 (Precupanu, [12], page 146). Let $f\colon [a,b] \to \mathbb{R}$ be an integrable function on $[a,b]$, continuous on (a,b), such that $\int_a^b f(t)\,\mathrm{d}t \neq 0$. Then, for any positive integer n, there exist distinct numbers $c_1, c_2, \ldots, c_n \in (a,b)$ satisfying the equality

$$\int_a^b f(t)\,\mathrm{d}t = \frac{n(b-a)}{1/f(c_1) + 1/f(c_2) + \ldots + 1/f(c_n)}.$$

P r o b l e m 1.4 (Orno, [9]). Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function on $[0,1]$, differentiable on $(0,1)$. If $f(0)=0$ and $f(1)=1$, then, for any positive integer n, there exist distinct numbers $c_1, c_2, \ldots, c_n \in (0,1)$ such that

$$\frac{1}{f(c_1)} + \frac{1}{f(c_2)} + \ldots + \frac{1}{f(c_n)} = n.$$

P r o b l e m 1.5 (Marinescu, [7]). Let $f\colon [0,1] \to \mathbb{R}$ be a continuous nonconstant function on $[0,1]$. We choose $\alpha \in (\min f, \max f)$. Prove: for any positive integer n, there exist distinct numbers $c_1, c_2, \ldots, c_n \in (0,1)$ such that

$$\alpha = \frac{f(c_1) + f(c_2) + \ldots + f(c_n)}{n}.$$

P r o b l e m 1.6 (Thong, [15]). Let $f\colon [0,1] \to \mathbb{R}$ be a strictly monotonic and continuous function on $[0,1]$ such that $\int_0^1 f(x)\,\mathrm{d}x = 1$. Prove there exist $\alpha, \beta, \gamma \in (0,1)$ with $\alpha < \beta < \gamma$ such that

$$f(\alpha)f(\beta)f(\gamma) = 1.$$

The solutions can be found in the journal or book where the problems have been published. Problem 1.6 enjoyed the attention of many mathematicians. A solution is due to Herman, Lampakis and Witkowski [3]. Moreover, Rocca Jr. presented a short note about this problem to a seminar [13].

The aim of this paper is to include these problems in some general results. In fact, the conclusions of the Problems 1.1–1.6 are particular cases of some mean value theorems. In this paper, we present and prove these theorems. We obtain some extensions of Lagrange or Cauchy Theorem, in classical form (see Theorems 2.2 and 2.3) or integral form (see Theorems 2.4 and 2.5). Also, we prove new results involving divided differences (see Theorems 2.6 and 2.7). The main tool, which we will use, is Theorem 2.1 from the next section. It is shown that any nonextremum value of a Darboux function can be obtained as an arithmetic, geometric or harmonic mean of some distinct values of the same function.

For the sake of clearness, the proofs will be presented in a separate section of this paper.

2. The results

Let $I \subset \mathbb{R}$ be an interval. We use the following notation throughout this section. Recall that a function $f\colon I \to \mathbb{R}$ has the Darboux property on I if for any $x, y \in I$, $x < y$ and for any value η between $f(x)$ and $f(y)$, there exists $c \in (x, y)$ such that $f(c) = \eta$. Any function satisfying this property is called a *Darboux function.*

Theorem 2.1. *Let $f\colon I \to \mathbb{R}$ be a Darboux function. Let $c \in I$ be an interior point, which is not an extremum point of f.*

(a) *For any positive integer n, there exist distinct $c_1, c_2, \ldots, c_n \in I$ such that*

$$f(c) = \frac{f(c_1) + f(c_2) + \ldots + f(c_n)}{n}.$$

(b) *For any positive integer n, there exist distinct $c_1, c_2, \ldots, c_n \in I$ such that*

$$(f(c))^n = f(c_1)f(c_2)\ldots f(c_n).$$

(c) *If $f(c) \neq 0$, then for any positive integer n, there exist distinct $c_1, c_2, \ldots, c_n \in I$ such that*

$$f(c) = \frac{n}{1/f(c_1) + 1/f(c_2) + \ldots + 1/f(c_n)}.$$

We know that any continuous function on a real interval has the Darboux property. Hence, the condition $\alpha \in (\min f, \max f)$ from Problem 1.5 shows us that α is not an extremum value of the function f. Now, Problem 1.5 is a consequence of the assertion (a) of the previous theorem.

Darboux property is difficult to explore since it is not compatible with the algebraic operations. For example, the sum of two Darboux functions is not necessarily a function of the same type. In this context, Jarník's Theorem is very important. This result can be found in [4] or [8] and it says that the function f'/g' has the Darboux property, for any functions $f, g\colon [a,b] \to \mathbb{R}$, differentiable on $[a,b]$, such that $g'(x) \neq 0$ ($x \in [a,b]$).

We use this result to prove an extension of the Cauchy Mean Value Theorem.

Theorem 2.2. *Let $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b] \to \mathbb{R}$ be functions continuous on $[a,b]$ and differentiable on (a,b). Suppose that $g'(x) \neq 0$ for any $x \in (a,b)$.*

(a) *For any positive integer n, there exist distinct $c_1, c_2, \ldots, c_n \in (a,b)$ such that*

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{1}{n}\Big[\frac{f'(c_1)}{g'(c_1)} + \frac{f'(c_2)}{g'(c_2)} + \ldots + \frac{f'(c_n)}{g'(c_n)}\Big].$$

(b) *For any positive integer n, there exist distinct $c_1, c_2, \ldots, c_n \in (a,b)$ such that*

$$\Big(\frac{f(b)-f(a)}{g(b)-g(a)}\Big)^n = \frac{f'(c_1)}{g'(c_1)}\frac{f'(c_2)}{g'(c_2)}\cdots\frac{f'(c_n)}{g'(c_n)}.$$

(c) *If $f(b) \neq f(a)$, then for any positive integer n, there exist distinct $c_1, c_2, \ldots, c_n \in (a,b)$ such that*

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{n}{g'(c_1)/f'(c_1) + g'(c_2)/f'(c_2) + \ldots + g'(c_n)/f'(c_n)}.$$

By choosing $g(x) = x$ for any $x \in [a,b]$, we obtain the following particular case.

Theorem 2.3. *Let $f\colon [a,b] \to \mathbb{R}$ be a function continuous on $[a,b]$ and differentiable on (a,b).*

(a) *For any positive integer n, there exist distinct $c_1, c_2, \ldots, c_n \in (a,b)$ such that*

$$\frac{f(b)-f(a)}{b-a} = \frac{f'(c_1) + f'(c_2) + \ldots + f'(c_n)}{n}.$$

(b) *For any positive integer n, there exist distinct $c_1, c_2, \ldots, c_n \in (a,b)$ such that*

$$\Big(\frac{f(b)-f(a)}{b-a}\Big)^n = f'(c_1)f'(c_2)\ldots f'(c_n).$$

(c) *If $f(b) \neq f(a)$, then for any positive integer n, there exist distinct $c_1, c_2, \ldots, c_n \in (a,b)$ such that*

$$\frac{f(b)-f(a)}{b-a} = \frac{n}{1/f'(c_1) + 1/f'(c_2) + \ldots + 1/f'(c_n)}.$$

We remark that Problem 1.1 is a particular case of the assertion (b) of Theorem 2.3 and Problem 1.4 is a particular case of the assertion (c) of the same theorem.

Further, we transform the previous results to obtain similar theorems in the case of integrable functions. First, let $f, g\colon [a,b] \to \mathbb{R}$ be functions integrable on $[a,b]$ and continuous on (a,b). Then the functions $F, G\colon [a,b] \to \mathbb{R}$, defined by $F(x) = \int_a^x f(t)\,\mathrm{d}t$ and $G(x) = \int_a^x g(t)\,\mathrm{d}t$ for any $x \in [a,b]$, are differentiable on (a,b). If $g(x) \neq 0$ for any $x \in (a,b)$, then G is strictly monotonic and $G(b) \neq 0 = G(a)$. Under these assumptions, the integral form of the Cauchy Theorem says that there exists $c \in (a,b)$ such that

$$\frac{\int_a^b f(t)\,\mathrm{d}t}{\int_a^b g(t)\,\mathrm{d}t} = \frac{f(c)}{g(c)}.$$

We obtain the next theorem by applying Theorem 2.2 to the functions F and G, defined above.

Theorem 2.4. *Let $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b] \to \mathbb{R}$ be functions integrable on $[a,b]$ and continuous on (a,b). Suppose that $g(x) \neq 0$, for any $x \in (a,b)$.*

(a) *For any positive integer n, there exist distinct $c_1, c_2, \ldots, c_n \in (a,b)$ such that*

$$\frac{\int_a^b f(t)\,\mathrm{d}t}{\int_a^b g(t)\,\mathrm{d}t} = \frac{1}{n}\Big[\frac{f(c_1)}{g(c_1)} + \frac{f(c_2)}{g(c_2)} + \ldots + \frac{f(c_n)}{g(c_n)}\Big].$$

(b) *For any positive integer n, there exist distinct $c_1, c_2, \ldots, c_n \in (a,b)$ such that*

$$\left(\frac{\int_a^b f(t)\,\mathrm{d}t}{\int_a^b g(t)\,\mathrm{d}t}\right)^n = \frac{f(c_1)}{g(c_1)}\frac{f(c_2)}{g(c_2)}\cdots\frac{f(c_n)}{g(c_n)}.$$

(c) *If $\int_a^b f(t)\,\mathrm{d}t \neq 0$, then for any positive integer n, there exist distinct $c_1, c_2, \ldots, c_n \in (a,b)$ such that*

$$\frac{\int_a^b f(t)\,\mathrm{d}t}{\int_a^b g(t)\,\mathrm{d}t} = \frac{n}{g(c_1)/f(c_1) + g(c_2)/f(c_2) + \ldots + g(c_n)/f(c_n)}.$$

If g is a nonzero constant, then Theorem 2.4 becomes

Theorem 2.5. *Let $f\colon [a,b] \to \mathbb{R}$ be a function integrable on $[a,b]$ and continuous on (a,b).*

(a) *For any positive integer n, there exist distinct $c_1, c_2, \ldots, c_n \in (a,b)$ such that*

$$\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t = \frac{f(c_1)+f(c_2)+\ldots+f(c_n)}{n}.$$

(b) *For any positive integer n, there exist distinct $c_1, c_2, \ldots, c_n \in (a,b)$ such that*

$$\left(\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right)^n = f(c_1)f(c_2)\ldots f(c_n).$$

(c) *If $\int_a^b f(t)\,\mathrm{d}t \neq 0$, then for any positive integer n, there exist distinct $c_1, c_2, \ldots,$ $c_n \in (a,b)$ such that*

$$\int_a^b f(t)\,\mathrm{d}t = \frac{n(b-a)}{1/f(c_1)+1/f(c_2)+\ldots+1/f(c_n)}.$$

Now, we can observe that the assertion (c) represents Problem 1.3. Moreover, if we choose $a=0$, $b=1$ and a function f such that $\int_0^1 f(t)\,\mathrm{d}t = 1$, we obtain Problem 1.2 by using the assertions (a) and (c) of Theorem 2.5. Problem 1.6 is a particular case of the assertion (b) of Theorem 2.5.

The assertion (c) of Theorem 2.2 or Theorem 2.4 remains valid if we change the harmonic mean to the ponderate harmonic mean. This version can be found in [6], but in that paper the above mentioned results were obtained in other way.

The final part of this section is reserved to similar results, involving the divided differences. Recall that for an arbitrary function $f\colon I \to \mathbb{R}$, for any positive integer p and for any $x_0, x_1, \ldots, x_p \in I$ such that $x_0 < x_1 < \ldots < x_p$, we denote

$$f[x_0, x_1, \ldots, x_p] = \sum_{k=0}^{p} \frac{f(x_k)}{\prod\limits_{s=0, s\neq k}^{p} (x_k - x_s)}.$$

This expression is called the *divided difference* of the function f associated with the points $x_0, x_1, \ldots, x_p \in I$. One of the most important results is the Lagrange Theorem for divided differences (Theorem 2.10, [14]) and it says that there exists $c \in (x_0, x_p)$ such that

$$f[x_0, x_1, \ldots, x_p] = \frac{f^{(p)}(c)}{p!},$$

where $f\colon I \to \mathbb{R}$ is a p-times differentiable function.

We extend this result in the following form:

Theorem 2.6. *Let $f\colon [a,b] \to \mathbb{R}$ be a function continuous on $[a,b]$ and p-times differentiable on (a,b). Let $x_0, x_1, \ldots, x_p \in [a,b]$ be such that*

$$x_0 = a < x_1 < \ldots < x_p = b.$$

(a) *For any positive integer n, there exist distinct $c_1, c_2, \ldots, c_n \in (a,b)$ such that*

$$f[x_0, x_1, \ldots, x_p] = \frac{1}{n} \frac{f^{(p)}(c_1) + f^{(p)}(c_2) + \ldots + f^{(p)}(c_n)}{p!}.$$

(b) *For any positive integer n, there exist distinct $c_1, c_2, \ldots, c_n \in (a,b)$ such that*

$$(f[x_0, x_1, \ldots, x_p])^n = \frac{f^{(p)}(c_1) f^{(p)}(c_2) \ldots f^{(p)}(c_n)}{(p!)^n}.$$

(c) *If $f[x_0, x_1, \ldots, x_p] \neq 0$, then for any positive integer n, there exist distinct $c_1, c_2, \ldots, c_n \in (a,b)$ such that*

$$f[x_0, x_1, \ldots, x_p] = \frac{1}{p!} \frac{n}{1/f^{(p)}(c_1) + 1/f^{(p)}(c_2) + \ldots + 1/f^{(p)}(c_n)}.$$

The Lagrange Theorem for divided differences was generalized due to Kowalewski [5]. He proved that for any functions $f, g\colon [a,b] \to \mathbb{R}$, p-times differentiable on $[a,b]$, satisfying the condition $g^{(p)}(x) \neq 0$, $x \in [a,b]$, there exists $c \in (a,b)$ such that

$$\frac{f[x_0, x_1, x_2, \ldots, x_p]}{g[x_0, x_1, x_2, \ldots, x_p]} = \frac{f^{(p)}(c)}{g^{(p)}(c)},$$

where $x_0, x_1, \ldots, x_{p-1}, x_p \in [a,b]$ are such that $x_0 = a < x_1 < \ldots < x_{p-1} < x_p = b$. Recently, Chen and Ding [1] showed that it is sufficient for the functions f and g to be continuous on $[a,b]$ and p-times differentiable on (a,b). An extension of Kowalewski's result concludes this section.

Theorem 2.7. *Let $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b] \to \mathbb{R}$ be functions continuous on $[a,b]$ and p-times differentiable on (a,b). Suppose that $g^{(p)}(x) \neq 0$ for every $x \in (a,b)$. Let $x_0, x_1, \ldots, x_p \in [a,b]$ be such that $x_0 = a < x_1 < \ldots < x_p = b$.*

(a) *For any positive integer n, there exist distinct $c_1, c_2, \ldots, c_n \in (a,b)$ such that*

$$\frac{f[x_0, x_1, \ldots, x_p]}{g[x_0, x_1, \ldots, x_p]} = \frac{1}{n} \Big[\frac{f^{(p)}(c_1)}{g^{(p)}(c_1)} + \frac{f^{(p)}(c_2)}{g^{(p)}(c_2)} + \ldots + \frac{f^{(p)}(c_n)}{g^{(p)}(c_n)} \Big].$$

(b) *For any positive integer n, there exist distinct $c_1, c_2, \ldots, c_n \in (a,b)$ such that*

$$\Big(\frac{f[x_0, x_1, \ldots, x_p]}{g[x_0, x_1, \ldots, x_p]} \Big)^n = \frac{f^{(p)}(c_1)}{g^{(p)}(c_1)} \frac{f^{(p)}(c_2)}{g^{(p)}(c_2)} \cdots \frac{f^{(p)}(c_n)}{g^{(p)}(c_n)}.$$

(c) *If $f[x_0, x_1, \ldots, x_p] \neq 0$ and $f^{(p)}(x) \neq 0$ for every $x \in (a, b)$, then for any positive integer n, there exist distinct $c_1, c_2, \ldots, c_n \in (a, b)$ such that*

$$\frac{f[x_0, x_1, \ldots, x_p]}{g[x_0, x_1, \ldots, x_p]} = \frac{n}{g^{(p)}(c_1)/f^{(p)}(c_1) + g^{(p)}(c_2)/f^{(p)}(c_2) + \ldots + g^{(p)}(c_n)/f^{(p)}(c_n)}.$$

3. The proofs

First, we present the proofs of Theorem 2.1 and Theorem 2.2.

Proof of Theorem 2.1. If c is not an extremum point, we can find $a, b \in I$ such that $f(a) < f(c) < f(b)$. The function f has the Darboux property, so we can find $\gamma \in (a, b)$ such that $f(y) = f(c)$. We can assume $a < \gamma < b$.

First, we prove all three assertions in the case that n is an even number. There exists $k \in \mathbb{N}$, $k \geqslant 1$ with $n = 2k$.

(a) Let $\varepsilon > 0$ be such that

$$f(a) < f(\gamma) - \varepsilon < f(\gamma) < f(\gamma) + \varepsilon < f(b).$$

We define real numbers $x_1, x_2, \ldots, x_k$ such that $x_1 = f(\gamma) - \varepsilon$ and $x_{s+1} \in (x_s, f(\gamma))$ for any $s \in \{1, 2, \ldots, k-1\}$. We observe that $x_1 < x_2 < \ldots < x_k$ and

$$2f(\gamma) - x_{s+1} \in (f(\gamma), 2f(\gamma) - x_s)$$

for any $s \in \{1, 2, \ldots, k-1\}$. We denote $y_s = 2f(\gamma) - x_s$ for any $s \in \{1, 2, \ldots, k\}$. We obtain $y_1 > y_2 > \ldots > y_k$. More, we have $x_s + y_s = 2f(\gamma)$ for any $s \in \{1, 2, \ldots, k\}$.

The function f has the Darboux property. There exist $c_1, c_2, \ldots, c_k$ such that $c_1 \in (a, \gamma)$ and $c_{s+1} \in (c_s, \gamma)$ for any $s \in \{1, 2, \ldots, k-1\}$ and satisfying the condition $f(c_s) = x_s$ for any $s \in \{1, 2, \ldots, k-1\}$. Similarly, there exist $c_{k+1}, c_{k+2}, \ldots, c_{2k}$ such that $c_{k+1} \in (\gamma, b)$ and $c_{k+s+1} \in (\gamma, c_{k+s})$ for any $s \in \{1, 2, 3, \ldots, k-1\}$, satisfying the condition $f(c_{k+s}) = y_s$ for any $s \in \{1, 2, \ldots, k\}$. In this context, we have

$$c_1 < c_2 < \ldots < c_k < c_{2k} < c_{2k-1} < \ldots < c_{k+1}$$

and

$$f(c) = \frac{f(c_1) + f(c_2) + \ldots + f(c_{2k})}{2k}.$$

Hence $n = 2k$, and we obtain the conclusion.

(b) If $f(c) = 0$, we choose $c_1, c_2, \ldots, c_{n-1} \in I - \{c\}$, distinct, and $c_n = c$, else we can assume $f(c) > 0$. If $f(c) < 0$, we make the same reasoning for the function $-f$. Then there exist $a, b \in I$ such that $0 < f(a) < f(c) < f(b)$. Similarly to the assertion (a), we find $\gamma \in (a, b)$ with $f(a) < f(\gamma) = f(c) < f(b)$. Let $\varepsilon > 1$ be such that

$$f(a) < \frac{f(\gamma)}{\varepsilon} < f(\gamma) < f(\gamma)\varepsilon < f(b).$$

We define real numbers $x_1, x_2, \ldots, x_k$ such that $x_1 = f(\gamma)/\varepsilon$ and $x_{s+1} \in (x_s, f(\gamma))$ for any $s \in \{1, 2, \ldots, k-1\}$. We have $x_1 < x_2 < \ldots < x_k$. In the same way as in the previous assertion, we obtain numbers $c_1, c_2, \ldots, c_k$ such that $c_1 \in (a, \gamma)$ and $c_{s+1} \in (c_s, \gamma)$ for any $s \in \{2, 3, \ldots, k-1\}$, and satisfying the conditions $f(c_s) = x_s$ for any $s \in \{1, 2, \ldots, k\}$.

Further, $f^2(\gamma)/x_{s+1} \in (f(\gamma), f^2(\gamma)/x_s)$ for any $s \in \{1, 2, \ldots, k-1\}$. For any $s \in \{1, 2, \ldots, k\}$, we denote $y_s = f^2(\gamma)/x_s$. We can observe that $x_s y_s = f^2(\gamma)$ for any $s \in \{1, 2, \ldots, k\}$. This implies $y_1 > y_2 > \ldots > y_k$. More, there exist $c_{k+1}, c_{k+2}, \ldots, c_{2k}$ such that $c_{k+1} \in (\gamma, b)$ and $c_{k+s+1} \in (\gamma, c_{k+s})$ for any $s \in \{1, 2, \ldots, k-1\}$, satisfying the conditions $f(c_{k+s}) = y_s$ for any $s \in \{1, 2, \ldots, k\}$. In this context, we have

$$c_1 < c_2 < \ldots < c_k < c_{2k} < c_{2k-1} < \ldots < c_{k+1}$$

and

$$f^{2k}(c) = f(c_1)f(c_2)\ldots f(c_{2k}).$$

The last equality is equivalent to the conclusion, hence $n = 2k$.

(c) We suppose that $f(c) > 0$. If $f(c) < 0$, we obtain a similar proof by using the function $-f$. Then there exist $a, b \in I$ such that $0 < f(a) < f(c) < f(b)$. We find $\gamma \in (a, b)$ with $f(a) < f(\gamma) = f(c) < f(b)$. Let $\varepsilon > 0$ be such that $0 < \varepsilon f(\gamma) < 1$ and

$$f(a) < \frac{f(\gamma)}{1 + \varepsilon f(\gamma)} < f(\gamma) < \frac{f(\gamma)}{1 - \varepsilon f(\gamma)} < f(b).$$

We define real numbers $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k$ such that

$$0 < \varepsilon_k < \varepsilon_{k-1} < \ldots < \varepsilon_2 < \varepsilon_1 = \varepsilon.$$

For any $s \in \{1, 2, \ldots, k\}$, we denote

$$x_s = \frac{f(\gamma)}{1 + \varepsilon_s f(\gamma)} \quad \text{and} \quad y_s = \frac{f(\gamma)}{1 - \varepsilon_s f(\gamma)}.$$

Then $x_1 < x_2 < \ldots < x_k$ and $y_1 > y_2 > \ldots > y_k$. More, for any $s \in \{1, 2, \ldots, k\}$, we have

$$\frac{2}{f(\gamma)} = \frac{1}{x_s} + \frac{1}{y_s}.$$

Similarly to the other assertions, we find numbers $c_1, c_2, \dots, c_{2k}$ such that $f(c_s) = x_s$, $f(c_{k+s}) = y_s$ for any $s \in \{1, 2, \dots, k\}$. We have $c_1 \in (a, \gamma)$, $c_{k+1} \in (\gamma, b)$, $c_{s+1} \in (c_s, \gamma)$ and $c_{k+s+1} \in (\gamma, c_{k+s})$ for any $s \in \{1,\ 2, \dots,\ k-1\}$. Then

$$c_1 < c_2 < \dots < c_k < c_{2k} < c_{2k-1} < \dots < c_{k+1}$$

and

$$\frac{2k}{f(c)} = \frac{1}{f(c_1)} + \frac{1}{f(c_2)} + \dots + \frac{1}{f(c_{2k})}.$$

Now $n = 2k$, then we conclude the proof.

If n is an odd number, there exists $k \in \mathbb{N}$ with $n = 2k + 1$. For the assertions (a) and (c), we define numbers $c_1, c_2, \dots, c_{2k}$ similarly to the other case and $c_{2k+1} = \gamma$. We make the same choice for the assertion (b) if $f(c) \neq 0$. If $f(c) = 0$, we choose $c_1, c_2, \dots, c_{2k} \in I - \{c\}$ distinct and $c_{2k+1} = c$. Now, we obtain all the conclusions.

$\square$

P r o o f of Theorem 2.2. From the Cauchy Theorem, we find $c \in (a, b)$ such that $(f(b) - f(a))/(g(b) - g(a)) = f'(c)/g'(c)$. The function f'/g' has the Darboux property and, if c is not an extremum point for this function, we obtain the conclusion of the assertions (a) and (b) by applying the similar assertions of Theorem 2.1. If $f(b) \neq f(a)$, the point c from the Cauchy Theorem satisfies the condition $f'(c)/g'(c) \neq 0$. Now, we obtain the conclusion of the assertion (c) as a consequence of the similar assertion of Theorem 2.1.

If c is an extremum point of the function f'/g', then we can assume that it is a maximum point. In this context, for any $x \in (a, b)$ we have

$$\frac{f'(x)}{g'(x)} \leqslant \frac{f'(c)}{g'(c)}.$$

Hence $g'(x) \neq 0$ for any $x \in (a, b)$, and we can suppose $g'(x) > 0$ for any $x \in (a, b)$. Then

$$f'(x) \leqslant \frac{f(b) - f(a)}{g(b) - g(a)} g'(x).$$

We define a function $h\colon [a, b] \to \mathbb{R}$ by

$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)} (g(x) - g(a))$$

for any $x \in [a, b]$. It is a continuous function on $[a, b]$, differentiable on (a, b) and $h(a) = h(b) = 0$. Moreover, $h'(x) \leqslant 0$ for any $x \in (a, b)$. Hence, the function h is constant. We obtain $h(x) = 0$ for any $x \in [a, b]$. Now, $h'(x) = 0$ for any $x \in [a, b]$ and we have

$$\frac{f'(x)}{g'(x)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

The function f'/g' is constant. The conclusion of all three assertions is obtained if we choose any distinct $c_1, c_2, \ldots, c_n \in (a, b)$. $\square$

The proofs of the following theorems need some preparation. First, we prove a useful result.

Lemma 3.1. *Let p be a positive integer and let $x_0, x_1, \ldots, x_p \in \mathbb{R}$ be such that $x_0 < x_1 < \ldots < x_p$. Let $u\colon [x_0, x_p] \to \mathbb{R}$ be a continuous function on $[x_0, x_p]$, p-times differentiable on (x_0, x_p) such that*

$$u(x_0) = u(x_1) = \ldots = u(x_p).$$

If $u^{(p)}(x) \geqslant 0$ for any $x \in (x_0, x_p)$, then the function u is constant on $[x_0, x_p]$.

Proof. We can assume that $u(x_0) = 0$. Let $x \in [x_0, x_p] \setminus \{x_0, x_1, \ldots, x_p\}$. It suffices to prove that $u(x) = 0$. There exists $k \in \{1, 2, \ldots, p\}$ such that $x \in (x_{k-1}, x_k)$. We apply the Lagrange Theorem for divided differences associated with the points $x_0, \ldots, x_{k-1}, x, x_{k+1}, \ldots, x_p$ and we find $c \in (x_0, x_p)$ such that

$$u[x_0, \ldots, x_{k-1}, x, x_{k+1}, \ldots, x_p] = \frac{u^{(p)}(c)}{p!}.$$

Then

$$\frac{u(x)}{(x - x_0)\ldots(x - x_{k-1})(x - x_{k+1})\ldots(x - x_p)} \geqslant 0. \tag{3.1}$$

Similarly, for the points $x_0, \ldots, x_{k-2}, x, x_k, \ldots, x_p$, we find $d \in (x_0, x_p)$ such that

$$u[x_0, \ldots, x_{k-2}, x, x_k, \ldots, x_p] = \frac{u^{(p)}(d)}{p!}.$$

Then

$$\frac{u(x)}{(x - x_0)\ldots(x - x_{k-2})(x - x_k)\ldots(x - x_p)} \geqslant 0. \tag{3.2}$$

The product of (3.1) and (3.2) gives

$$\frac{u^2(x)}{(x - x_0)^2\ldots(x - x_{k-2})^2(x - x_{k-1})(x - x_k)(x - x_{k+1})^2\ldots(x - x_p)^2} \geqslant 0.$$

The inequality $(x - x_{k-1})(x - x_k) < 0$ implies $u^2(x) \leqslant 0$. Then $u(x) = 0$ which completes the proof. $\square$

P r o o f of Theorem 2.6. The Lagrange Theorem for divided differences gives us $c \in (a,b)$ such that $f^{(p)}(c) = p!f[x_0, x_1, \ldots, x_p]$. The function $f^{(p)}$ has the Darboux property. If c is not an extremum point of $f^{(p)}$, we can apply Theorem 2.1 and obtain the conclusion.

If c is an extremum point of $f^{(p)}$, we can assume that it is a maximum point. Then $f^{(p)}(x) \leqslant f^{(p)}(c)$ for any $x \in [a,b]$. We define a function $u\colon [a,b] \to \mathbb{R}$ by

$$u(x) = \sum_{k=0}^{n} f(x_k)\left(\prod_{s=0, s\neq k}^{n} \frac{x - x_s}{x_k - x_s}\right) - f(x)$$

for any $x \in [a,b]$. It is p-times differentiable on (a,b) and

$$u^{(p)}(x) = p!f[x_0, x_1, \ldots, x_p] - f^{(p)}(x)$$

Then $u^{(p)}(x) \geqslant 0$ for any $x \in (a,b)$. Moreover, $u(x_k) = 0$ for any $k \in \{0, 1, \ldots, p\}$. By using Lemma 3.1, we obtain $u(x) = 0$ for any $x \in [a,b]$. Then

$$f^{(p)}(x) = p!f[x_0, x_1, \ldots, x_p]$$

for any $x \in [a,b]$, so $f^{(p)}$ is constant. Now we obtain the conclusion if we choose any distinct $c_1, c_2, \ldots, c_n \in (a,b)$. $\square$

We conclude our paper with the proof of Theorem 2.7. First, for any $p+1$ pairwise different points $x_0, x_1, \ldots, x_p \in \mathbb{R}$ we denote

$$V[x_0, x_1, \ldots, x_p] = \begin{vmatrix} 1 & 1 & \ldots & 1 \\ x_0 & x_1 & \ldots & x_p \\ \ldots & \ldots & \ldots & \ldots \\ x_0^{p-1} & x_1^{p-1} & \ldots & x_p^{p-1} \\ x_0^p & x_1^p & \ldots & x_p^p \end{vmatrix},$$

the classical Vandermonde determinant. Now we have the equality:

$$V[x_0, x_1, \ldots, x_p] = \prod_{0\leqslant i<j\leqslant p} (x_j - x_i).$$

Let $I \subset \mathbb{R}$ be an interval. For any function $f\colon I \to \mathbb{R}$ and for any points $x_0, x_1, \ldots, x_p \in I$, we denote

$$\Delta_f[x_0, x_1, x_2, \ldots, x_p] = \begin{vmatrix} 1 & 1 & \ldots & 1 \\ x_0 & x_1 & \ldots & x_p \\ \ldots & \ldots & \ldots & \ldots \\ x_0^{p-1} & x_1^{p-1} & \ldots & x_p^{p-1} \\ f(x_0) & f(x_1) & \ldots & f(x_p) \end{vmatrix}.$$

Clearly, we have

$$f[x_0, x_1, x_2, \dots, x_p] = \frac{\Delta_f[x_0, x_1, \dots, x_p]}{V[x_0, x_1, \dots, x_p]}.$$

Proof of Theorem 2.7. By using the Kowalewski Theorem, we find $c \in (a, b)$ such that

$$\frac{f[x_0, x_1, x_2, \dots, x_p]}{g[x_0, x_1, x_2, \dots, x_p]} = \frac{f^{(p)}(c)}{g^{(p)}(c)}.$$

The Jarník Theorem shows that the function $f^{(p)}/g^{(p)}$ has the Darboux property. If c is not an extremum point of $f^{(p)}/g^{(p)}$, we obtain the conclusion by using Theorem 2.1.

Also, we can assume that $g^{(p)}(x) > 0$ ($x \in (a, b)$) and c is a maximum point. Then

$$\frac{f^{(p)}(x)}{g^{(p)}(x)} \leqslant \frac{f^{(p)}(c)}{g^{(p)}(c)}, \quad x \in (a, b). \tag{3.3}$$

We introduce a function $u\colon [a, b] \to \mathbb{R}$ defined for every $x \in [a, b]$ by

$$u(x) = \Delta_f[x_0, x_1, \dots, x_p]\Delta_g[x_0, \dots, x_{p-1}, x] - \Delta_g[x_0, x_1, \dots, x_p]\Delta_f[x_0, \dots, x_{p-1}, x].$$

It is a continuous function on $[a, b]$, p-times differentiable on (a, b) and $u(x_k) = 0$ for any $k \in \{0, 1, 2, \dots, p\}$. For any $x \in (a, b)$ we have

$$\begin{aligned} u^{(p)}(x) &= \Delta_f[x_0, x_1, \dots, x_p]g^{(p)}(x) - \Delta_g[x_0, x_1, \dots, x_p]f^{(p)}(x) \\ &= V[x_0, x_1, \dots, x_p](f[x_0, x_1, \dots, x_p]g^{(p)}(x) - g[x_0, x_1, \dots, x_p]f^{(p)}(x)) \\ &= V[x_0, x_1, \dots, x_p](f^{(p)}(c)g^{(p)}(x) - g^{(p)}(c)f^{(p)}(x)). \end{aligned}$$

Further, $x_0 < x_1 < \dots < x_p$ and $V[x_0, x_1, \dots, x_p] > 0$. By using (3.3), we obtain $u^{(p)}(x) \geqslant 0$. Lemma 3.1 yields $u(x) = 0$ for any $x \in [a, b]$. Then

$$\frac{f^{(p)}(x)}{g^{(p)}(x)} = \frac{f^{(p)}(c)}{g^{(p)}(c)}, \quad x \in (a, b).$$

and $f^{(p)}/g^{(p)}$ is constant. Now, we obtain the conclusion if we choose any distinct $c_1, c_2, \dots, c_n \in (a, b)$. $\square$

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