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INITIAL DATA STABILITY AND ADMISSIBILITY OF SPACES FOR ITÔ LINEAR DIFFERENCE EQUATIONS

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Abstract. The admissibility of spaces for Itô functional difference equations is investigated by the method of modeling equations. The problem of space admissibility is closely connected with the initial data stability problem of solutions for Itô delay differential equations. For these equations the p-stability of initial data solutions is studied as a special case of admissibility of spaces for the corresponding Itô functional difference equation. In most cases, this approach seems to be more constructive and expedient than other traditional approaches. For certain equations sufficient conditions of solution stability are given in terms of parameters of those equations.

Keywords: Itô functional difference equation; stability of solutions; admissibility of spaces

MSC 2010: 39A60, 39A30, 60H25, 37H10, 93E15

1. Introduction

Stability of solutions of the stochastic difference equations with aftereffect is not a well studied issue. One of our papers [10] is devoted to some questions of stability of solutions for Itô functional difference equations. The issue of stability for deterministic functional difference equations was generally studied on the basis of a classical method of Lyapunov-Krasovsky-Razumikhin. This method assumes the existence of a suitable Lyapunov function (Lyapunov-Krasovsky's functional) which provides a desirable stability property (asymptotic behavior) of solutions of the equations under study. However, in the theory of solution stability for deterministic functional differential equations and functional difference equations the method of the auxiliary

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or "model" equations, also known as "W-method" by Azbelev [2], is very effective and widely used. Earlier in [6], [7], [8], [9] we used this method for the case of stochastic functional differential equations of Itô. The major objective of this research is the extension of this method to the case of Itô difference equations with aftereffect. For these equations, initial data solution stability is studied as a special case of admissibility of spaces for the corresponding Itô functional difference equation. Let us note that this approach in many cases turns out to be more constructive than other traditional approaches. In the end of the paper we present some examples of equations for which sufficient conditions of solution stability are given in terms of parameters of these equations.

2. Preliminary data and object of research

Assume that $(\Omega, \Im, (\Im_t)_{t\geqslant 0}, P)$ is the stochastic basis; k^n is the linear space of n-dimensional $\Im_0$-measurable random variables; B_i, $i = 2, \dots, m$ are the scalar independent standard Wiener processes; $1 \leqslant p < \infty$; E is the mathematical expectation; $|\cdot|$ is the norm in $\mathbb{R}^n$; $\|\cdot\|$ is the norm of an $n \times m$-matrix consistent with the norm in $\mathbb{R}^m$, $\mathbb{N}$ is the set of natural numbers; $\mathbb{N}_+ = \{0\} \cup \mathbb{N}$.

The main object of research is an Itô linear difference system with the following form of aftereffect

$$(2.1)\quad x(s+1) = x(s) + \Big[\sum_{j=-\infty}^{s} A_1(s,j)x(j) + f_1(s)\Big]h \\ + \sum_{i=2}^{m}\Big[\sum_{j=-\infty}^{s} A_i(s,j)x(j) + f_i(s)\Big](B_i((s+1)h) - B_i(sh)), \quad s \in \mathbb{N}_+,$$

where $f_i(s)$ is the $\Im_s$-measurable n-dimensional random variable at $s \in \mathbb{N}_+$, $i = 1, \dots, m$, h is a sufficiently small real number, $A_i(s,j)$ is an $n\times m$-matrix whose elements are $\Im_s$-measurable random variables at $i = 1, \dots, m$, $j = 0, \dots, s$, $s \in \mathbb{N}_+$ and B_i are $\Im_0$-measurable random variables at $i = 1, \dots, m$, $j = -\infty, \dots, -1$, $s \in \mathbb{N}_+$.

For equation (2.1), let us consider the problem

$$(2.2)\qquad x(j) = \varphi(j); \quad j \leqslant 0,$$

where $\varphi(j)$ is the $\Im_0$-measurable n-dimensional random variable for all $j \leqslant 0$.

Definition 2.1. The solution of problem (2.1), (2.2) is the sequence of random variables $x(s)$, $s \in \mathbb{N}_+$, where $x(s)$ is the $\Im_s$-measurable n-dimensional random variable satisfying equation (2.1) P-almost everywhere under condition (2.2). Let us denote this solution by $x_\varphi(s)$, $s \in \mathbb{N}_+$.

Special cases of (2.1) are:

a) Linear system of “ordinary” Itô difference equations

$$
\begin{aligned}
(2.3)\qquad x(s+1) = x(s) + [a_1(s)x(s) + f_1(s)]h \\
+ \sum_{i=2}^{m}[a_i(s)x(s) + f_i(s)](B_i((s+1)h) - B_i(sh)), \quad s \in \mathbb{N}_+,
\end{aligned}
$$

where $f_i(s)$ is an $\Im_s$-measurable n-dimensional random variable at $s \in \mathbb{N}_+$, $i = 1,\dots,m$, $h > 0$, $a_i(s)$ is an $n\times m$-matrix whose elements are $\Im_s$-measurable random variables at $i = 1,\dots,m$, $s \in \mathbb{N}_+$.

b) Linear system of Itô difference equations with bounded delay

$$
\begin{aligned}
(2.4)\quad x(s+1) = x(s) + \Bigg[\sum_{j=s-d}^{s} a_1(s,j)x(j) + f_1(s)\Bigg]h \\
+ \sum_{i=2}^{m}\Bigg[\sum_{j=s-d}^{s} a_i(s,j)x(s) + f_i(s)\Bigg](B_i((s+1)h) - B_i(sh)), \quad s \in \mathbb{N}_+,
\end{aligned}
$$

where $d \in \mathbb{N}$, $f_i(s)$ is an $\Im_s$-measurable n-dimensional random variable at $s \in \mathbb{N}_+$, $i = 1,\dots,m$, $h > 0$, $a_i(s,j)$ is an $n\times m$-matrix whose elements are $\Im_s$-measurable random variables at $i = 1,\dots,m$, $s \in \mathbb{N}_+$, $j = 0,\dots,s$, and B_i are $\Im_0$-measurable random variables at $i = 1,\dots,m$, $s = 0,\dots,d-1$, $j = -d,\dots,-1$.

Remark 2.1. By analogy with the determinate case, let equation (2.1) with condition (2.2) be called the initial task. In the determinate case, Azbelev and his followers applied the other approach when studying issues of stability for differential equations with aftereffect [6], [7], [8], [9]. Similarly to this approach, *equation* is thought of as equation (2.1) with condition (2.2) with $j < 0$.

As noted in Remark 2.1 *equation* means equation (2.1) with condition (2.2) at $j < 0$. Let us rewrite this equation in the following form:

$$
(2.5)\qquad x(s+1) = x(s) + [f(s) + (Vx)(s)]Z(s), \quad s \in \mathbb{N}_+,
$$

where

$$
\begin{aligned}
(Vx)(s) &= ((V_1x)(s),\dots,(V_mx)(s)) = \Bigg(\sum_{j=0}^{s} A_1(s,j)x(j),\dots,\sum_{j=0}^{s} A_m(s,j)x(j)\Bigg),\\
Z(s) &= (h, B_2((s+1)h) - B_2(sh),\dots,B_m((s+1)h) - B_m(sh)),\\
f(s) &= \Bigg(f_1(s) + \sum_{j=-\infty}^{-1} A_1(s,j)\varphi(j),\dots,f_m(s) + \sum_{j=-\infty}^{-1} A_m(s,j)\varphi(j)\Bigg).
\end{aligned}
$$

By analogy with the determinate case, we call equation (2.5) the *Itô functional difference equation*. For this equation, let us consider the problem

$$x(0) = x_0, \tag{2.6}$$

where x_0 is an $\Im_0$-measurable n-dimensional random variable.

Definition 2.2. The solution of problem (2.5), (2.6) is the sequence of random variables $x(s)$, $s \in \mathbb{N}_+$ where $x(s)$ is the $\Im_s$-measurable n-dimensional random variable satisfying condition (2.6) and equation (2.5) P-almost everywhere. Let us denote this solution by $x^f(s, x_0)$, $s \in \mathbb{N}_+$.

Remark 2.2. In passing to equation (2.5), initial conditions (2.2) at $j < 0$ become the right part f of this equation. Thus, initial problem (2.1), (2.2) is equivalent to problem (2.5), (2.6) in the case when $x_0 = \varphi(0)$.

Definition 2.3. Equation (2.5) is homogeneous, if $f(s) = 0$ holds P-almost everywhere with all $s \in \mathbb{N}_+$.

Remark 2.3. Homogeneous equation (2.5) corresponds to equation (2.1) in the case when $f_i \equiv 0$ with $i = 1, \dots, m$ and to condition (2.2) when $j < 0$ in the case when $\varphi(j) = 0$ $(j < 0)$, i.e.

$$\begin{aligned} x(s+1) = x(s) &+ \sum_{j=0}^{s} A_1(s,j)x(j)h \\ &+ \sum_{i=2}^{m}\sum_{j=0}^{s} A_i(s,j)x(j)(B_i((s+1)h) - B_i(sh)), \quad s \in \mathbb{N}_+, \\ x(j) = 0, &\quad j < 0. \end{aligned} \tag{2.7}$$

Let d^n be the linear sequence space of n-matrices $x(s)$, $s \in \mathbb{N}_+$ with elements $x(s)$ being $\Im_s$-measurable n-dimensional random variables; let l^n be the linear sequence space of $n \times m$-matrices $H(s)$, $s \in \mathbb{N}_+$ where elements of the matrix $H(s)$ are $\Im_s$-measurable random variables. It is easy to see that V is an additive operator acting from the space d^n into the space l^n. Assume $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$, $\gamma(s)$, $s \in \mathbb{N}_+$ is the sequence of positive real numbers. In the sequel the following normalized linear subspaces of the spaces d^n, l^n, k^n are used:

$$m_p^\gamma \overset{\text{def}}{=} \Big\{x\colon\ x \in d^n,\ \|x\|_{m_p^\gamma} = \sup_{s\in\mathbb{N}_+} (E|\gamma(s)x(s)|^p)^{1/p} < \infty\Big\}, \quad m_p^1 = m_p;$$

$$k_p^n \overset{\text{def}}{=} \{\alpha\colon\ \alpha \in k^n,\ \|\alpha\|_{k_p^n} = (E|\alpha|^p)^{1/p} < \infty\};$$

$$\lambda^{\gamma}_{p,q} \stackrel{\text{def}}{=} \Big\{ f\colon\ f \in l^n,\ \|f\|_{\lambda^{\gamma}_{p,q}} = \Big(\sum_{s\in\mathbb{N}_+} (E\|\gamma(s)f(s)\|^p)^{q/p} \Big)^{1/q} < \infty \Big\};$$

$$\lambda^{\gamma}_{p,\infty} \stackrel{\text{def}}{=} \Big\{ f\colon\ f \in l^n,\ \|f\|_{\lambda^{\gamma}_{p,\infty}} = \sup_{s\in\mathbb{N}_+} (E\|\gamma(s)f(s)\|^p)^{1/p} < \infty \Big\}.$$

3. Initial data stability and admissibility of spaces

Other researchers apparently did not study the issues of space admissibility even in the case of Itô stochastic differential equations. We should note that the problem of space admissibility is closely connected with a problem of initial data stability of solutions for Itô delay differential equations.

Definition 3.1. The trivial solution of homogeneous equation (2.7) is

- ▷ p-stable if for any $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that for any $\varphi(j)$, $j < 0$ the estimate $E|x_\varphi(s)|^p \leqslant \varepsilon$ at $s \in \mathbb{N}_+$ follows from the inequality $\sup\limits_{j<0} E|\varphi(j)|^p < \delta$;
- ▷ asymptotically p-stable if it is p-stable and, besides, for any $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that $\lim\limits_{s\to\infty} E|x_\varphi(s)|^p = 0$ follows from the inequality $\sup\limits_{j<0} E|\varphi(j)|^p < \delta$ for any $\varphi(j)$, $j < 0$;
- ▷ exponentially p-stable if there are numbers $\overline{c} > 0$, $\beta > 0$ such that the inequality $E|x_\varphi(s)|^p \leqslant \overline{c} \sup\limits_{j<0} E|\varphi(j)|^p \exp\{-\beta s\}$, $s \in \mathbb{N}_+$ holds.

Suppose J, S, R are linear normalized subspaces of the spaces k^n, d^n, l^n, respectively.

Definition 3.2. For equation (2.5) the triplet (J, S, R) is admissible if $x_f(\cdot\,, x_0) \in S$ for any $x_0 \in J$, $f \in R$, and there exists $\overline{c} \in \mathbb{R}^1_+$ at which the inequality

$$\|x_f(\cdot\,, x_0)\|_S \leqslant \overline{c}(\|x_0\|_J + \|f\|_R) \tag{3.1}$$

is valid.

Theorem 3.1. *Suppose $J = k^n_p$, $S = m^{\gamma}_p$, $R = \lambda^{\gamma}_{p,q}$ for certain p, q, $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$ and a positive sequence $\gamma(s)$, $s \in \mathbb{N}_+$, for equation* (2.5) *the triplet (J, S, R) is admissible,*

$$\Phi \stackrel{\text{def}}{=} \Big(\sum_{j=-\infty}^{-1} A_1(s,j)\varphi(j), \ldots, \sum_{j=-\infty}^{-1} A_m(s,j)\varphi(j) \Big)$$

belongs to the space R for any φ such that $\sup\limits_{j<0} E|\varphi(j)|^p < \infty$,

$$\|\Phi\|_R \leqslant K \sup_{j<0} (E|\varphi(j)|^p)^{1/p},$$

where K is a positive number. Then:

1) *if $\gamma(s) = 1$, $s \in \mathbb{N}_+$, then the trivial solution of equation* (2.7) *is p-stable;*
2) *if $\gamma(s) \geqslant \delta$, $s \in \mathbb{N}_+$ for a certain $\delta > 0$ and $\lim\limits_{s\to\infty} \gamma(s) = \infty$, then the trivial solution of equation* (2.7) *is asymptotically p-stable;*
3) *if $\gamma(s) = \exp\{\beta s\}$, $s \in \mathbb{N}_+$ for a certain $\beta > 0$, then the trivial solution of equation* (2.7) *is exponentially p-stable.*

The validity of the theorem is obvious.

4. Method of auxiliary equations or "W-method"

We need the following lemma.

Lemma 4.1. *For the solution of equation* (2.5) *passing through $x_0 \in k^n$, the representation*

$$x^f(s, x_0) = X(s)x_0 + (Cf)(s), \quad s \in \mathbb{N}_+ \tag{4.1}$$

is true, where $X(s)$, $s \in \mathbb{N}_+$ ($X(0) = \overline{E}$ is the unit matrix) is the matrix whose columns are solutions of equation (2.5) (*fundamental matrix*)*, and $C\colon l^n \to d^n$ is the linear operator* (*Cauchy operator*) *such that $(Cf)(0) = 0$ and $(Cf)(s)$, $s \in \mathbb{N}_+$ is the solution of equation* (2.5).

Note that for the case of deterministic functional difference equations, representation (4.1) is obtained in [1] (see [3], [4]).

P r o o f. It is easy to verify that $X(s)x_0$, $s \in \mathbb{N}_+$ where $x_0 \in k^n$ is the solution of equation (2.5).

For equation (2.5), let us consider the Cauchy problem

$$x(0) = 0. \tag{4.2}$$

Cauchy problem (2.5), (4.2) is uniquely solvable with any $f \in l^n$. Therefore, this problem yields a certain operator acting from the space l^n into the space d^n. Let us denote this operator by C. It is obvious that $(Cf)(0) = 0$, and it is possible to verify that this operator is linear by a direct check using the unique solvability of problem (2.5), (4.2). This implies that (4.1) is the solution of equation (2.5). Since the unique solution (with an accuracy up to P-equivalence) of equation (2.5) passes through any $x_0 \in k^n$, we can see that any solution $x^f(s, x_0)$, $s \in \mathbb{N}_+$ of equation (2.5) is represented as (4.1).

The lemma is proved. $\square$

Suppose (as in Section 3) J, S, R are the linear normalized subspaces of the spaces k^n, d^n, l^n, respectively.

In order to verify the admissibility of the triplet (J, S, R) for equation (2.5) it is necessary to check whether the solution $x^f(\cdot\,, x_0)$ of equation (2.5) belongs to the space S for any $x_0 \in J$, $f \in R$, and whether (3.1) is true for this equation. Let us check the feasibility of these conditions using the equivalence conversion of equation (2.5).

Let us consider a "model" equation (a "modeling" equation) with well-known asymptotic properties of its solutions. Suppose the model equation has the form

$$x(s+1) = x(s) + [(Qx)(s) + g(s)]Z(s), \quad s \in \mathbb{N}_+, \tag{4.3}$$

where $Q\colon d^n \to l^n$ is an additive operator, $g \in l^n$. It is assumed that the only solution x (with an accuracy up to P-equivalence) of equation (4.3) passes through any $x_0 \in k^n$. Then owing to Lemma 4.1, for this solution x there is a representation $x(s) = U(s)x_0 + (Wg)(s)$, $s \in \mathbb{N}_+$, where U is the fundamental matrix and W is the Cauchy operator for equation (4.3).

Using model equation (4.3), let us rewrite equation (2.5) as

$$x(s+1) = x(s) + [(Qx)(s) + ((V-Q)x)(s) + f(s)]Z(s), \quad s \in \mathbb{N}_+$$

or

$$x(s) = U(s)x_0 + (W(V-Q)x)(s) + (Wf)(s), \quad s \in \mathbb{N}_+.$$

Let $W(V-Q) = \Theta_l$, then

$$((I - \Theta_l)x)(s) = U(s)x_0 + (Wf)(s), \quad s \in \mathbb{N}_+.$$

It should be noted that in the sequel the reversibility of $I - \Theta_l\colon S \to S$ means that the operator $I - \Theta_l$ takes the space S to itself in a one-to-one manner.

Theorem 4.1. *Suppose the triplet (J, S, R) is admissible for model equation (4.3), and the operator Θ_l acts in the space S. Then, if the operator $I - \Theta_l\colon S \to S$ is continuously invertible, the triplet (J, S, R) is admissible for equation (2.5).*

Proof. Owing to invertibility of $I - \Theta_l\colon S \to S$ the equation $(I - \Theta_l)x = g$ where $g \in S$ has the only solution from S, i.e. $x = (I - \Theta_l)^{-1} g \in S$. From this and from the theorem conditions we can deduce that $(I - \Theta_l)^{-1}(Ux_0 + Wf) \in S$ for any $x_0 \in J$. But, on the other hand, $x^f(s, x_0) = ((I - \Theta_l)^{-1}(Ux_0 + Wf))(s)$, $s \in \mathbb{N}_+$. Owing to the assumptions of the theorem we have $x^f(\cdot\,, x_0) \in S$ for any $x_0 \in J$, $f \in R$. Feasibility of inequality (3.1) for $x^f(s, x_0)$, $s \in \mathbb{N}_+$ follows from continuous invertibility of the operator $I - \Theta_l\colon S \to S$ and admissibility of (J, S, R) for equation (4.3), i.e. from the theorem conditions. Therefore, the triplet (J, S, R) is admissible for equation (2.5).

The theorem is proved. □

Let us note that at a priori arbitrary choice of model equation (4.3) for which the triplet (J, S, R) is admissible there are cases when the operator Θ_l does not act in the corresponding functional space at all while for equation (2.5) the triplet (J, S, R) is admissible. However, if for equation (2.5) the triplet (J, S, R) is admissible, there is always at least one model equation for which the triplet (J, S, R) is admissible and the operator Θ_l acts in the corresponding functional space, and at that the operator $I - \Theta_l$ is continuously invertible in the same space. Equation (2.5) can be considered as such model equation as (4.3).

When using Theorem 4.1 the most difficult question is that of finding continuous invertibility conditions of $I - \Theta_l\colon S \to S$. It is possible to check the continuous invertibility of the operator $I - \Theta_l\colon S \to S$ by estimating the norm of the operator Θ_l in the space S. If it is less than 1, then the continuous invertibility is guaranteed.

5. Sufficient stability conditions

Now the following inequality is necessary:

$$\left(E\left|\int_t^{t+h} \psi(\tau)\,\mathrm{d}B(\tau)\right|^{2p}\right)^{1/(2p)} \leqslant c_p\left(E\left|\int_t^{t+h} |\psi(\tau)|^2\,\mathrm{d}\tau\right|^{p}\right)^{1/(2p)}, \tag{5.1}$$

where $1 \leqslant p < \infty$, $h > 0$, $\psi(\tau)$ is a locally integrable martingale, c_p is a certain number depending on p, $c_1 = 1$ whose validity follows from [5], Chapter III, Section 3, Theorem 3.1.

In order to determine the p-stability of the trivial solution of (2.7) or admissibility of the triplet $(k_p^n, m_p^\gamma, \lambda_{p,q}^\gamma)$ for equation (2.5) using Theorem 4.1 it is necessary that for model equation (4.3) the triplet $(k_p^n, m_p^\gamma, \lambda_{p,q}^\gamma)$ should be admissible. In this regard, first of all we consider the question of admissibility of $(k_{2p}^n, m_{2p}^\gamma, \lambda_{2p,q}^\gamma)$, where $\gamma(s) = \exp\{\beta s\}$, $s \in \mathbb{N}_+$ for a certain positive number β, for the equation often taken as a model equation.

Suppose for model equation (4.3) we have $(Qx)(s) = (-\alpha x(s), 0, \ldots, 0)$ where α is the certain positive number such that $0 < \alpha < 1/h$. The following lemma is true.

Lemma 5.1. *For model equation* (4.3) *the triplet* $(k_{2p}^n, m_{2p}^\gamma, \lambda_{2p,q}^\gamma)$ *is admissible for all* $0 \leqslant \beta < -\ln(1 - \alpha h)$.

P r o o f. Let us note that for the solution $x(s)$ of (4.3) under the previous assumptions the representation $x(s) = U(s)x(0) + (Wg)(s)$, $s \in \mathbb{N}_+$, where $U(s) = \exp\{\ln(1 - \alpha h)s\}\overline{E}$, $(Wg)(s) = \sum\limits_{\tau=0}^{s} U(s)(U(\tau))^{-1}g(\tau)Z(\tau)$ is true. As $Ux(0) \in m_{2p}^\gamma$ at any $x(0) \in k_{2p}^n$ and $\|Ux(0)\|_{m_{2p}^\gamma} \leqslant \|x(0)\|_{k_{2p}^n}$, for proving the lemma it is enough to show that for any $g \in \lambda_{2p,q}^\gamma$ we have $Wg \in m_{2p}^\gamma$ and $\|Wg\|_{m_{2p}^\gamma} \leqslant \overline{c}\|g\|_{\lambda_{2p,q}^\gamma}$, where $\overline{c}$ is a positive number.

First, let us prove that for any $\tau \in \mathbb{N}$ and $n \times m$-matrix ϑ with $\Im_0$-measurable columns ϑ_i, $i = 1, \ldots, m$, the inequality is true

$$E|\vartheta(\tau)Z(\tau)|^{2p} \leqslant c_1 E\|\vartheta(\tau)\|^{2p}, \tag{5.2}$$

where c_1 is a positive number depending on p and h.

Validity of inequality (5.2) follows from

$$\begin{aligned}
E|\vartheta(\tau)Z(\tau)|^{2p} &= E\left|\vartheta_1(\tau)h + \sum_{i=2}^{m} \vartheta_i(\tau)(B_i((\tau+1)h) - B_i(\tau h))\right|^{2p} \\
&= E\left|\vartheta_1(\tau)h + \sum_{i=2}^{m} \vartheta_i(\tau) \int_{\tau h}^{(\tau+1)h} \mathrm{d}B_i(\varsigma)\right|^{2p} \\
&\leqslant c_1^{(1')} E|\vartheta_1(\tau)h|^{2p} + c_2^{(2)} \sum_{i=2}^{m} E|\vartheta_i(\tau)|^{2p} \leqslant c_1 E\|\vartheta(\tau)\|^{2p}.
\end{aligned}$$

In the previous estimates inequality (5.1) is used.

We have

$$\begin{aligned}
\|Wg\|_{m_{2p}^\gamma} = \|\gamma Wg\|_{m_{2p}} &= \left\|\gamma \sum_{\tau=0}^{(\cdot)} U(s)(U(\tau))^{-1}g(\tau)Z(\tau)\right\|_{m_{2p}} \\
&= \sup_{s\in\mathbb{N}_+} \left(E\left|\gamma(s)\sum_{\tau=0}^{s} U(s)(U(\tau))^{-1}g(\tau)Z(\tau)\right|^{2p}\right)^{1/(2p)} \\
&\leqslant \sup_{s\in\mathbb{N}_+} \left(\sum_{\tau=0}^{s} (E|\gamma(s)U(s)(U(\tau))^{-1}g(\tau)Z(\tau)|^{2p})^{1/(2p)}\right)
\end{aligned}$$

$$\leqslant c_1^{1/(2p)} \sup_{s\in\mathbb{N}_+} \left(\sum_{\tau=0}^{s} (E\|\gamma(s)U(s)(U(\tau))^{-1} g(\tau)\|^{2p})^{1/(2p)} \right)$$

$$\leqslant c_2 \sup_{s\in\mathbb{N}_+} \left(\sum_{\tau=0}^{s} \exp\{(\ln(1-\alpha h)+\beta)(s-\tau)\} (E\|\gamma(\tau)g(\tau)\|^{2p})^{1/(2p)} \right),$$

where c_2 is a positive number.

When $q=1$ by virtue of $\exp\{(\ln(1-\alpha h)+\beta)(s-\tau)\} \leqslant 1$ we obtain

$$\|Wg\|_{m_{2p}^{\gamma}} \leqslant c_3 \|g\|_{\lambda_{2p,q}^{\gamma}},$$

i.e. for any $g \in \lambda_{2p,q}^{\gamma}$ there is $Wg \in m_{2p}^{\gamma}$ with $\|Wg\|_{m_{2p}^{\gamma}} \leqslant c_3 \|g\|_{\lambda_{2p,q}^{\gamma}}$ where c_3 is a positive number.

As for any positive number $a = -\ln(1-\alpha h) - \beta$ the equality

$$\sum_{\tau=0}^{s} \exp\{-a(s-\tau)\} = 1 + \exp\{-a\} + \ldots + \exp\{-as\} = \frac{1-\exp\{-a(s+1)\}}{1-\exp\{-a\}} \tag{5.3}$$

is true, the theorem becomes obvious when $q=1$.

For $1 < q < \infty$ we obtain

$$\|Wg\|_{m_{2p}^{\gamma}} \leqslant c_4 \sup_{s\in\mathbb{N}_+} \left[\left(\sum_{\tau=0}^{s} \exp\{\overline{q}(\ln(1-\alpha h)+\beta)(s-\tau)\} \right)^{1/\overline{q}} \times \left(\sum_{\tau=0}^{s} (E\|\gamma(\tau)g(\tau)\|^{2p})^{q/(2p)} \right)^{1/q} \right],$$

where $\overline{q} = q/(1-q)$, and c_4 is a positive number. Taking (5.3) into account we obtain $Wg \in m_{2p}^{\gamma}$ for any $g \in \lambda_{2p,q}^{\gamma}$ and $\|Wg\|_{m_{2p}^{\gamma}} \leqslant c_5 \|g\|_{\lambda_{2p,q}^{\gamma}}$, where c_5 is a positive number.

The lemma is proved. □

Let us consider some examples. First let us consider the Itô scalar ordinary difference equation

$$x(s+1) = x(s) + [a_1 x(s) + f_1(s)]h + \sum_{i=2}^{m} [a_i x(s) + f_i(s)](B_i((s+1)h) - B_i(sh)), \quad s \in \mathbb{N}_+, \tag{5.4}$$

where $f_i(s)$ is the $\Im_s$-measurable n-dimensional random variable and $s \in \mathbb{N}_+$, $i = 1,\ldots,m$, $h > 0$, a_i, $i = 1,\ldots,m$, are real numbers.

Theorem 5.1. *Suppose the conditions*

$$-1 < a_1 h < 0, \quad c_p \sum_{i=2}^{m} |a_i| < -a_1\sqrt{h}$$

are satisfied for equation (5.4). *Then the trivial solution of homogeneous equation* (5.4) *is* $2p$*-stable at* $1 \leqslant p < \infty$.

P r o o f. For proving this theorem we use Theorems 4.1 and 3.1. As a model equation we take equation (4.3) where $(Qx)(s) = ((1 + a_1 h)x(s), 0, \dots, 0)$, $s \in \mathbb{N}_+$, and as spaces J, S, R we take the spaces k^n_{2p}, m_{2p}, $\lambda_{2p,q}$, respectively. Then, owing to Lemma 5.1, the triplet $(k^n_{2p}, m_{2p}, \lambda_{2p,q})$ is admissible for the model equation and

$$(\Theta_l x)(s) = \lambda^s \sum_{\tau=0}^{s-1} \lambda^{-\tau-1}(Kx)(\tau), \quad s \in \mathbb{N}_+,$$

where $\lambda = 1 + a_1 h$, $(Kx)(\tau) = \sum\limits_{i=2}^{m} a_i x(\tau) \int_{\tau h}^{(\tau+1)h} \mathrm{d}B_i(\zeta)$.

Using inequality (5.1) it is easy to verify directly that $\|\Theta_l\|_{m_{2p}} \leqslant \overline{c}\|x\|_{m_{2p}}$ where $\overline{c} = \Bigl(c_p \sum\limits_{i=2}^{m} |a_i|\Bigr)/(1-\lambda)$. This implies that the conditions of Theorem 4.1 are satisfied. Now by virtue of $\overline{c} < 1$, according to the conditions of Theorem 4.1 it follows that for equation (5.4) the triplet $(k^n_{2p}, m_{2p}, \lambda_{2p,q})$ is admissible. Therefore, owing to Theorem 3.1 (in this case $\Phi = 0$) the trivial solution of homogeneous equation (5.4) is $2p$-stable for $1 \leqslant p < \infty$.

The theorem is proved. □

Finally, let us consider the Itô scalar difference equation with bounded delays

$$\begin{aligned}(5.5)\quad x(s+1) &= x(s) + [a_1 x(s) + f_1(s)]h \\ &\quad + \sum_{i=2}^{m} \Bigl[\sum_{j=0}^{d} a_{ij} x(s-j) + f_i(s)\Bigr](B_i((s+1)h) - B_i(sh)), \quad s \in \mathbb{N}_+, \\ x(j) &= \varphi(j), \quad j < 0,\end{aligned}$$

where $d \in \mathbb{N}$, $f_i(s)$ is a $\Im_s$-measurable n-dimensional random variable for $s \in \mathbb{N}_+$, $i = 1, \dots, m$, $h > 0$, a_1, a_{ij}, $j = 0, \dots, d$, $i = 2, \dots, m$, are real numbers, $\varphi(j)$ is an $\Im_0$-measurable random variable for all $j < 0$.

Theorem 5.2. *Suppose for equation* (5.5) *the conditions* $-1 < a_1 h < 0$, $c_p \sum\limits_{i=2}^{m} \sum\limits_{j=0}^{d} |a_{ij}| < -a_1\sqrt{h}$ *are satisfied. Then the trivial solution of homogeneous equation* (5.5) *is* $2p$*-stable at* $1 \leqslant p < \infty$.

Theorem 5.2 is proved similarly to Theorem 5.1. Model equation (4.3) and the spaces J, S, R are the same as in the proof of Theorem 5.1. In the case of equation (5.5) it is easy to show that for any φ such that $\sup\limits_{j<0} E|\varphi(j)|^p < \infty$, Φ belongs to the space R and $\|\Phi\|_R \leqslant K \sup\limits_{j<0}(E|\varphi(j)|^p)^{1/p}$, where K is a positive number.

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