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STATISCH PAIRS IN ATOMISTIC POSETS

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Abstract. We introduce statisch pairs in atomistic posets and study its relationships with some known concepts in posets such as biatomic and dual modular pairs, perspectivity and subspaces of atom space of an atomistic poset. We generalize the notion of exchange property in posets and with the help of it we prove the equivalence of dual modular, biatomic and statisch pairs in atomistic posets. Also, we prove that the set of all finite elements of a statisch poset with such property forms an ideal. ∇-relation is partly studied by means of statisch pairs.

Keywords: atomistic poset; statisch pair; finite element

MSC 2010: 06A06, 06A11

1. Introduction

The concept of a statisch (static) lattice as a subclass of atomistic lattices was introduced by Wille [12] and studied by Janowitz [1], [2] and Maeda and Maeda [3] in connection with modularity. Statisch lattices include the lattice of all subspaces of any vector space and the lattice of convex subsets of a poset, etc. These lattices are called biatomic lattices. Biatomicity in atomistic posets was introduced by Waphare and Joshi in [9]. On the other hand, Wasadikar and Thakare introduced in [11] statisch pairs in atomistic join-semilattices. Now, we introduce statisch pairs in atomistic posets and study its relationships with dual modular and biatomic pairs. We also study subspaces of atom space of an atomistic poset with the help of statisch pairs and we investigate the set of finite elements of a statisch poset to be an ideal. We begin with the necessary definitions and terminologies in a poset P.

An element x of a poset P is an *upper bound* of $A \subseteq P$ if $a \leqslant x$ for all $a \in A$. A *lower bound* is defined dually. The set of all upper bounds of A is denoted by A^{u} (read as A *upper cone*), so $A^{\mathrm{u}} = \{x \in P \colon x \geqslant a$ for every $a \in A\}$ and dually,

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we have the concept of a *lower cone* A^{l} of A. A^{ul} shall mean $\{A^{\mathrm{u}}\}^{\mathrm{l}}$ and A^{lu} shall mean $\{A^{\mathrm{l}}\}^{\mathrm{u}}$. The upper cone $\{a\}^{\mathrm{u}}$ is simply denoted by a^{u} and $\{a,b\}^{\mathrm{u}}$ is denoted by $(a,b)^{\mathrm{u}}$. Similar notations are used for lower cones. Further, for $A, B \subseteq P$, $\{A \cup B\}^{\mathrm{u}}$ is denoted by $\{A,B\}^{\mathrm{u}}$ and for $x \in P$, the set $\{A \cup \{x\}\}^{\mathrm{u}}$ is denoted by $\{A,x\}^{\mathrm{u}}$. Similar notations are used for lower cones. We note that $A \subseteq A^{\mathrm{ul}}$ and $A \subseteq A^{\mathrm{lu}}$. If $A \subseteq B$, then $B^{\mathrm{l}} \subseteq A^{\mathrm{l}}$ and $B^{\mathrm{u}} \subseteq A^{\mathrm{u}}$. Moreover, $A^{\mathrm{lul}} = A^{\mathrm{l}}$, $A^{\mathrm{ulu}} = A^{\mathrm{u}}$ and $\{a^{\mathrm{u}}\}^{\mathrm{l}} = \{a\}^{\mathrm{l}} = a^{\mathrm{l}}$.

If P contains a finite number of elements, it is called a *finite poset*. In a poset P we say that x is *covered* by y and write $x \prec y$ if $x \leqslant z \leqslant y$ implies $x = z$ or $z = y$. An element p of a poset P with 0 is called an *atom* if $0 \prec p$. The set of atoms of P is denoted by $A(P)$. For a nonzero element $a \in P$, $\omega(a)$ denotes the set of atoms contained in a, that is $\omega(a) = \{p \in A(P) \colon\ p \leqslant a\}$.

A subset I of a poset P is called an *ideal* if $a, b \in I$ implies that $(a,b)^{\mathrm{ul}} \subseteq I$. The set of ideals of a poset P is denoted by $\mathrm{Id}(P)$, which is a lattice under set inclusion.

A poset P with 0 is called *atomistic* if every element of P is a join of atoms contained in it (that is, the set of atoms of P is join-dense in P). Equivalently, for any two elements $a, b \in P$ with $a \nleqslant b$, there is a $p \in A(P)$ such as $p \leqslant a$, $p \nleqslant b$.

Let P be a poset. A pair of elements $a, b \in P$ is called a *distributive pair*, denoted by $(a,b)\mathrm{D}$, if the following condition holds:

$$\{(a,b)^{\mathrm{u}}, x\}^{\mathrm{l}} = \{(a,x)^{\mathrm{l}}, (b,x)^{\mathrm{l}}\}^{\mathrm{ul}} \quad \text{for every } x \in P.$$

A pair of elements a, b in a poset P is said to be a *modular pair*, denoted by $(a,b)\mathrm{M}$, if the following condition holds:

$$\{\{a, (b,c)^{\mathrm{l}}\}^{\mathrm{u}}, b\}^{\mathrm{l}} = \{(a,b)^{\mathrm{l}}, (b,c)^{\mathrm{l}}\}^{\mathrm{ul}} \quad \text{for every } c \in P.$$

Dually, we have the concept of a *dual modular pair*, denoted by $(a,b)\mathrm{M}^*$, if the following condition holds:

$$\{\{a, (b,c)^{\mathrm{u}}\}^{\mathrm{l}}, b\}^{\mathrm{u}} = \{(a,b)^{\mathrm{u}}, (b,c)^{\mathrm{u}}\}^{\mathrm{lu}} \quad \text{for every } c \in P.$$

A pair of nonzero elements a, b of an atomistic poset P is called a *biatomic pair* and is denoted by $(a,b)P$ if for every atom $p \in (a,b)^{\mathrm{ul}}$ there exist atoms $q, r \in P$ such that $p \in (q,r)^{\mathrm{ul}}$, $q \leqslant a$ and $r \leqslant b$. An atomistic poset P is called *biatomic* if $(a,b)P$ holds for all $a, b \in P$. We also need the following definitions given by Waphare in [7].

Let P be an atomistic poset. A subset ω of the atom space $A(P)$ is called a *subspace* when it satisfies the following condition:

If $p \in A(P)$, $q_i \in \omega$, $i = 1, 2, \ldots, n$ for some natural number n and $p \leqslant z$ for all $z \in \{q_1, \ldots, q_n\}^{\mathrm{u}}$, then $p \in \omega$. Denote by $L(A(P))$ the set of all subspaces of $A(P)$.

Let S be a subset of an atomistic poset P. The set of atoms contained in S^{ul}, which is a subspace of $A(P)$, is denoted by $\omega(S)$. If $S = \{a, b\}$ for some $a, b \in P$, then $\omega(\{a, b\})$ may be simply denoted by $\omega(a, b)$.

An atomistic poset P is called *statisch* when $p \in (a, b)^{\mathrm{ul}}$ for $p \in A(P)$ implies the existence of a finite subset $S \subseteq \omega(a) \cup \omega(b)$ such that $p \in S^{\mathrm{ul}}$.

Shewale introduced in [4] the concept of a finite element in posets. An element a of a poset P with 0 is called a *finite element* if either $a = 0$ or $a \in \{p\colon\ p \in \omega(a)\}^U$, that is, a belongs to the set of minimal elements of the set of upper bounds of $\omega(a)$; see [6]. Therefore, a finite element in an atomistic poset P is determined as the join of finitely many atoms of P. The set of all finite elements of an atomistic poset P is denoted by $F(P)$.

A poset P is said to have the *exchange property* if $p \in (q, a)^{\mathrm{ul}}$ and $p \nleqslant a$ together imply that $q \in (p, a)^{\mathrm{ul}}$ for atoms $p, q \in P$.

A poset P is said to have the *weak exchange property* if $p \in (q, r)^{\mathrm{ul}}$ and $p \neq r$ together imply that $q \in (p, r)^{\mathrm{ul}}$ for atoms $p, q, r \in P$.

Let $a, b \in P$ with 0. We say that a and b are *perspective* and write $a \sim_x b$ (or simply $a \sim b$) if $(a, x)^{\mathrm{ul}} = (b, x)^{\mathrm{ul}}$ and $(a, x)^{\mathrm{l}} = (b, x)^{\mathrm{l}} = \{0\}$ for some $x \in P$. The element x is called the *axis of perspectivity.* We say that a is *subperspective* to b if $a \in (b, x)^{\mathrm{ul}}$ and $(a, x)^{\mathrm{l}} = \{0\}$ for some $x \in P$. Let P be a poset with 0 and $a, b \in P$. We write $a \mathrel{\nabla} b$ when $\{(a, x)^{\mathrm{u}}, b\}^{\mathrm{l}} = (x, b)^{\mathrm{l}}$ for every $x \in P$.

An element s of a poset P with 0 is called ∇*-standard* if $(s, a)^{\mathrm{l}} = \{0\}$ implies that $s \mathrel{\nabla} a$ for $a \in P$. ∇-relation is called *symmetric* if $a \mathrel{\nabla} b$ implies $b \mathrel{\nabla} a$ for $a, b \in P$.

2. Statisch Pairs

Definition 2.1. A pair a, b of nonzero elements of an atomistic poset P is called a *statisch pair*, denoted by $(a, b)\mathrm{S}$, if for an atom $p \in (a, b)^{\mathrm{ul}}$ there exist finite elements $a_1, b_1 \in P$ such that $a_1 \leqslant a$, $b_1 \leqslant b$ and $p \in (a_1, b_1)^{\mathrm{ul}}$.

In other words, $(a, b)\mathrm{S}$ holds if $p \in \omega(a, b)$ implies $p \in \omega(a_1, b_1)$ for finite elements $a_1 \leqslant a$, $b_1 \leqslant b$. If P is a finite atomistic poset and $a, b \in P$, then clearly $(a, b)\mathrm{S}$ holds since every element is finite.

Remark 2.2. Let P be an atomistic poset. Observe that for $a, b \in P$, $(a, b)\mathrm{S}$ implies $(b, a)\mathrm{S}$ and if $a \leqslant b$ in P, then $(a, b)\mathrm{S}$ holds. We note that for an atom p of P if $p \in (a, b)^{\mathrm{ul}}$ implies that $p \leqslant a$ or $p \leqslant b$, then $(a, b)\mathrm{S}$ holds. For, without loss of generality, assume that $p \in (a, b)^{\mathrm{ul}}$ and $p \leqslant a$. Since $0 \leqslant b$ and both 0 and p are finite elements, we have $p \in (p, 0)^{\mathrm{ul}}$ and hence $(a, b)\mathrm{S}$ holds. We also note that if $(a, b)\mathrm{S}$ holds for all nonzero elements $a, b \in P$, then P is a statisch poset.

Lemma 2.3. *In an atomistic poset P, $(a,b)\mathrm{D}$ implies $(a,b)\mathrm{S}$.*

P r o o f. Let p be an atom of P and $p \in (a,b)^{\mathrm{ul}}$. By $(a,b)\mathrm{D}$ we have $p^{\mathrm{l}} = \{p,(a,b)^{\mathrm{u}}\}^{\mathrm{l}} = \{(p,a)^{\mathrm{l}},(p,b)^{\mathrm{l}}\}^{\mathrm{ul}}$. Since p is an atom, we must have $p \leqslant a$ or $p \leqslant b$. Hence $(a,b)\mathrm{S}$ holds by Remark 2.2. $\square$

The converse of Lemma 2.3 need not be true in general; see Figure 1 and observe that every pair is a statisch pair since every element is finite, but $(x,z)D$ does not hold since $\{(x,z)^{\mathrm{u}},p\}^{\mathrm{l}} = \{0,p\} \neq \{(x,p)^{\mathrm{l}},(z,p)^{\mathrm{l}}\}^{\mathrm{ul}} = \{0\}$.

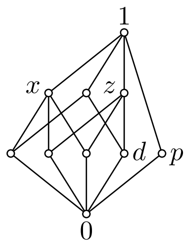

Figure 1.

Lemma 2.4. *Let a,b be elements of an atomistic poset and $(a,b)\mathrm{S}$ hold. Then $(a,b)\mathrm{D}$ holds whenever $(a_1,b_1)\mathrm{D}$ holds for all finite elements a_1,b_1 such that $a_1 \leqslant a$, $b_1 \leqslant b$.*

P r o o f. In view of Waphare and Joshi (see [10]), to prove $(a,b)\mathrm{D}$ it is enough to show that if p is an atom and $p \in (a,b)^{\mathrm{ul}}$, then $p \leqslant a$ or $p \leqslant b$. Consider an atom $p \in (a,b)^{\mathrm{ul}}$ and $p \nleqslant b$. We prove that $p \leqslant a$. By $(a,b)\mathrm{S}$ there exist finite elements a_1,b_1 such that $a_1 \leqslant a$, $b_1 \leqslant b$ and $p \in (a_1,b_1)^{\mathrm{ul}}$. Clearly $p \nleqslant b_1$. Since $p \in \{(a_1,b_1)^{\mathrm{u}},p\}^{\mathrm{l}}$ and by $(a_1,b_1)\mathrm{D}$ we get $\{(a_1,b_1)^{\mathrm{u}},p\}^{\mathrm{l}} = \{(p,a_1)^{\mathrm{l}},(p,b_1)^{\mathrm{l}}\}^{\mathrm{ul}} = (p,a_1)^{\mathrm{l}}$, since $(p,b_1)^{\mathrm{l}} = \{0\}$. Thus $p \in (p,a_1)^{\mathrm{l}}$ and we get $p \leqslant a_1$ and consequently $p \leqslant a$. $\square$

In order to obtain similar type of results using finite elements instead of atoms, we give the following definition.

Definition 2.5. Let P be a poset with 0. We say that P satisfies the *generalized exchange property* if $x \in (y,a)^{\mathrm{ul}}$ and $x \nleqslant a$ for $x,y \in F(P)$ and $a \in P$ together imply $y \in (x,a)^{\mathrm{ul}}$. We say that P satisfies the *generalized weak exchange property* if $x \in (y,z)^{\mathrm{ul}}$ and $x \nleqslant z$ together imply $y \in (x,z)^{\mathrm{ul}}$ for $x,y,z \in F(P)$.

R e m a r k 2.6. In a poset P generalized weak exchange property implies weak exchange property since every atom is a finite element. The converse need not be true; see Figure 1 in which the generalized weak exchange property does not hold for $z,p,d \in F(P)$. In fact, $z \in (p,d)^{\mathrm{ul}}$ and $z \nleqslant d$ but $p \notin (z,d)^{\mathrm{ul}}$.

Lemma 2.7. *Let P be an atomistic poset with exchange property such that $(r, a)\mathrm{S}$ holds for all $r \in A(P)$, $a \in P$ and let p, q be atoms of P. Then $p \sim_x q$ for $x \in P$ implies that $p \sim_{x_1} q$ for some $x_1 \in F(P)$ with $x_1 \leqslant x$.*

Proof. Let $p \sim_x q$ in P. We have $(p, x)^{\mathrm{ul}} = (q, x)^{\mathrm{ul}}$ and $(p, x)^{\mathrm{l}} = (q, x)^{\mathrm{l}} = \{0\}$ for some $x \in P$. Since $p \in (q, x)^{\mathrm{ul}}$ and $(q, x)S$, we have $p \in (q, x_1)^{\mathrm{ul}}$ for some finite element $x_1 \leqslant x$. Therefore $(p, x_1)^{\mathrm{ul}} \subseteq (q, x_1)^{\mathrm{ul}}$. On the other hand, since $p \nleqslant x$, we have $p \nleqslant x_1$, too. Similarly $q \nleqslant x_1$. Hence $(p, x_1)^{\mathrm{l}} = (q, x_1)^{\mathrm{l}} = \{0\}$. By the exchange property we have $q \in (p, x_1)^{\mathrm{ul}}$ and consequently $(q, x_1)^{\mathrm{ul}} \subseteq (p, x_1)^{\mathrm{ul}}$. Hence $(p, x_1)^{\mathrm{ul}} = (q, x_1)^{\mathrm{ul}}$. Therefore $p \sim_{x_1} q$. $\square$

Lemma 2.8. *Let P be an atomistic poset with weak exchange property and p, q be atoms of P. If $(p, q)^{\mathrm{ul}}$ contains a third atom r, then $p \sim_x q$. The converse holds if the following condition is satisfied:*

If p, q are atoms and $p \in (q, a)^{\mathrm{ul}}$, then there exists an atom r such that $p \in (q, r)^{\mathrm{ul}}$, $r \leqslant a$.

Proof. If r is an atom such that $r \in (p, q)^{\mathrm{ul}}$, $r \neq p$, $r \neq q$, then we have $(p, r)^{\mathrm{ul}} \subseteq (p, q)^{\mathrm{ul}}$. By the weak exchange property we have $p \in (r, q)^{\mathrm{ul}}$, hence $(p, q)^{\mathrm{ul}} \subseteq (p, r)^{\mathrm{ul}}$. Also $r \in (p, q)^{\mathrm{ul}}$ implies that $(r, p)^{\mathrm{ul}} \subseteq (p, q)^{\mathrm{ul}}$ and $(r, q)^{\mathrm{ul}} \subseteq (p, q)^{\mathrm{ul}}$. But $p \in (r, q)^{\mathrm{ul}}$ which implies that $(p, q)^{\mathrm{ul}} \subseteq (r, q)^{\mathrm{ul}}$ and $(p, r)^{\mathrm{ul}} \subseteq (r, q)^{\mathrm{ul}}$. We infer that $(p, r)^{\mathrm{ul}} = (p, q)^{\mathrm{ul}} = (q, r)^{\mathrm{ul}}$. Obviously $(p, r)^{\mathrm{l}} = (q, r)^{\mathrm{l}} = \{0\}$, hence $p \sim_x q$ for $x = r$.

Conversely, suppose that $p \sim_x q$ for some $x \in P$. Hence $(p, x)^{\mathrm{ul}} = (q, x)^{\mathrm{ul}}$ and $(p, x)^{\mathrm{l}} = (q, x)^{\mathrm{l}} = \{0\}$. Since $p \in (q, x)^{\mathrm{ul}}$, by the given condition there exists an atom r such that $p \in (q, r)^{\mathrm{ul}}$, $r \leqslant x$. By the weak exchange property we have $r \in (p, q)^{\mathrm{ul}}$. As $(p, x)^{\mathrm{l}} = \{0\}$, we get $(p, r)^{\mathrm{l}} = \{0\}$. This means that $r \neq p$. Similarly, $r \neq q$ and the result follows. $\square$

Lemma 2.9. *Let P be an atomistic poset with generalized weak exchange property and let a, b be nonzero elements of P. If $(a, b)\mathrm{S}$, then $(a, b)\mathrm{M}^*$ and the converse is true if b is a finite element.*

Proof. Let P be an atomistic poset with generalized weak exchange property. Suppose that $(a, b)\mathrm{S}$ holds for $a, b \in P$ and we claim that $(a, b)\mathrm{M}^*$ also holds. By the definition of $(a, b)\mathrm{M}^*$ it suffices to prove that for arbitrary elements $x \in \{(a, b)^{\mathrm{u}}, (b, c)^{\mathrm{u}}\}^{\mathrm{l}}$ and $y \in \{\{a, (b, c)^{\mathrm{u}}\}^{\mathrm{l}}, b\}^{\mathrm{u}}$ for $c \in P$, we have $x \leqslant y$. Since P is atomistic, it is sufficient to prove that every atom contained in x is also contained in y.

Let p be an atom with $p \leqslant x$.

Case 1: if $p \leqslant b$, then $p \leqslant y$ since $b \leqslant y$.

Case 2: suppose that $p \nleqslant b$. Note that $p \leqslant x \in (a,b)^{\mathrm{ul}}$, therefore $p \in (a,b)^{\mathrm{ul}}$ and since $(a,b)\mathrm{S}$ holds, there exist finite elements $a_1, b_1 \in P$ such that $a_1 \leqslant a$, $b_1 \leqslant b$ and $p \in (a_1,b_1)^{\mathrm{ul}}$. Clearly, $p \nleqslant b_1$; otherwise we would have $p \leqslant b$ which is not possible. Now, $p \nleqslant b_1$ and by the generalized weak exchange property we must have $a_1 \in (p,b_1)^{\mathrm{ul}}$. Moreover, $p \in (b,c)^{\mathrm{ul}}$ as $p \leqslant x \in (b,c)^{\mathrm{ul}}$ and $b_1 \in (b,c)^{\mathrm{ul}}$ as $b_1 \leqslant b \in (b,c)^{\mathrm{ul}}$ and so $(p,b_1)^{\mathrm{ul}} \subseteq (b,c)^{\mathrm{ul}}$. Since $a_1 \leqslant a$ and $a_1 \in (p,b_1)^{\mathrm{ul}}$, we have $a_1 \in \{(p,b_1)^{\mathrm{u}}, a\}^{\mathrm{l}} \subseteq \{(b,c)^{\mathrm{u}}, a\}^{\mathrm{l}}$. From this together with $p \in (a_1,b_1)^{\mathrm{ul}}$ and $b_1 \leqslant b$ we conclude that $\{\{a,(b,c)^{\mathrm{u}}\}^{\mathrm{l}}, b\}^{\mathrm{u}} \subseteq (a_1,b_1)^{\mathrm{u}} \subseteq p^{\mathrm{u}}$. Now, $y \in \{\{a,(b,c)^{\mathrm{u}}\}^{\mathrm{l}}, b\}^{\mathrm{u}}$ and so $y \in p^{\mathrm{u}}$ concluding that $p \leqslant y$. Hence, every atom contained in x is essentially contained in y.

Conversely, assume that $(a,d)\mathrm{M}^*$ holds, with d being a finite element. We prove that $(a,d)\mathrm{S}$ also holds. If $d \leqslant a$, then $(a,d)\mathrm{S}$ clearly holds, therefore assume that $d \nleqslant a$. Let p be an atom such as $p \in (a,d)^{\mathrm{ul}}$. If $p \leqslant d$, then $(a,b)\mathrm{S}$ holds by Remark 3.2.1, so suppose that $p \nleqslant d$. Now, by $(a,d)\mathrm{M}^*$ we have $\{\{a,(d,p)^{\mathrm{u}}\}^{\mathrm{l}}, d\}^{\mathrm{ul}} = \{(d,p)^{\mathrm{u}}, (a,d)^{\mathrm{u}}\}^{\mathrm{l}}$. Clearly, $\{a,(d,p)^{\mathrm{u}}\}^{\mathrm{l}} \neq \{0\}$; otherwise $p \leqslant d$, since $p \in (p,d)^{\mathrm{ul}}$ and $p \in (a,d)^{\mathrm{ul}}$ and so $p \in \{(p,d)^{\mathrm{u}}, (a,d)^{\mathrm{u}}\}^{\mathrm{l}} = \{\{a,(d,p)^{\mathrm{u}}\}^{\mathrm{l}}, d\}^{\mathrm{ul}}$, which is not possible. By the atomisticity there exists an atom q (which is necessarily a finite element) contained in $\{a,(d,p)^{\mathrm{u}}\}^{\mathrm{l}}$, that is, $q \leqslant a$ as well as $q \in (p,d)^{\mathrm{ul}}$. Observe that $q \neq d$, otherwise we get $q \nleqslant a$ which is not true. Now, by the generalized weak exchange property we have $p \in (q,d)^{\mathrm{ul}}$ and since q and d are finite elements with $q \leqslant a$, we conclude that $(a,d)\mathrm{S}$ holds. □

Lemma 2.10. *In an atomistic poset,* $(a,b)P$ *implies* $(a,b)\mathrm{S}$.

P r o o f. Let $p \in (a,b)^{\mathrm{ul}}$. By $(a,b)P$ there exist atoms $q \leqslant a$, $r \leqslant b$ such that $p \in (q,r)^{\mathrm{ul}}$. As $q, r \in F(P)$, we have $(a,b)\mathrm{S}$. □

The converse of Lemma 2.10 need not be true in general. Consider the poset depicted in Figure 1 and observe that every pair is a statisch pair since every element is finite, but $(x,z)P$ does not hold. In fact, $p \in (x,z)^{\mathrm{ul}}$ but the required condition of biatomicity is not satisfied.

Lemma 2.11. *Let* a, b *be nonzero elements of an atomistic poset* P *with generalized weak exchange property. If* $(b,d)\mathrm{M}^*$ *holds for every finite element* $d \in P$*, then* $(a,b)\mathrm{S}$ *if and only if* $(a,b)\mathrm{M}^*$.

P r o o f. By Lemma 2.9, $(a,b)\mathrm{S}$ implies $(a,b)\mathrm{M}^*$. Since generalized weak exchange property implies weak exchange property and every biatomic pair is a statisch pair and every atom is a finite element, we have $(a,b)\mathrm{M}^*$ implies $(a,b)P$ and hence $(a,b)\mathrm{S}$ holds by Lemma 2.10. □

Combining Lemma 2.9 and Lemma 2.11, we obtain:

Theorem 2.12. *Let P be an atomistic poset with generalized weak exchange property. Then $(a,b)\mathrm{M}^*$ if and only if $(a,b)\mathrm{S}$.*

In atomistic posets with weak exchange property, Waphare and Joshi obtained the equivalence of $(a,b)P$ and $(a,b)\mathrm{M}^*$ (see [9]). Thus, we have the following result.

Theorem 2.13. *The following statements are equivalent in an atomistic poset P with generalized weak exchange property.*
(1) $(a,b)\mathrm{M}^*$.
(2) $(a,b)\mathrm{S}$.
(3) $(a,b)P$.

We now study statisch pairs in connection with subspaces of atomistic posets.

Theorem 2.14 ([7]). *Let P be an atomistic poset. The set $L(A(P))$ of all subspaces of $A(P)$ is a compactly atomistic complete lattice ordered by set inclusion. The meet of two subspaces ω_1, ω_2 is their intersection and the join is*

$$\omega_1 \vee \omega_2 = \{p \in A(P)\colon\ p \in S^{\mathrm{ul}},\ S \text{ is a finite subset of } \omega_1 \cup \omega_2\}.$$

For any element $a \in P$, the set $\omega(a)$ is a subspace of $A(P)$ and the map $f\colon P \to L(A(P))$ defined by $a \mapsto \omega(a)$ is a biorder preserving mapping of P onto $f(P) \subseteq L(A(P))$. This mapping satisfies the following properties:
(1) $\omega(0) = 0$, ($\omega(1) = 1$ *if P has* 1);
(2) $f(F(P)) \subseteq F(L(A(P)))$.

For the sake of completeness we give the skeleton of the proof.

Proof. The singletons of the set of atoms is an atom of $L(A(P))$ and arbitrary intersection of subspaces is again a subspace and so $L(A(P))$ is a complete atomistic lattice. Observe that $\omega(a)$, $a \in P$ is a subspace. If $p \leqslant z$ for all $z \in S^{\mathrm{u}}$, where S is a finite subset of $\omega(a)$, then $a \in S^{\mathrm{u}}$ and $p \leqslant a$ as well. If $a \leqslant b$, then $\omega(a) \subseteq \omega(b)$ and conversely. As P is atomistic, the mapping $a \mapsto \omega(a)$ of P into $L(A(P))$ is one-one order preserving. Observe that if $a \in P$ is finite, then $\omega(a)$ is finite; for, let $a = \bigvee\{p_1, \ldots, p_n\}$ for some natural number n and $p_i \in A(P)$, then $\omega(a) = \omega(p_1) \vee \ldots \vee \omega(p_n)$, since $\omega(p_i) = \{p_i\}$. Therefore, $F(L(A(P)))$ is isomorphic to $F(P)$ by the mapping $a \mapsto \omega(a)$. $\square$

Proposition 2.15. *Let P be an atomistic poset and $a, b \in P$. Then $(a,b)\mathrm{S}$ implies that $\omega(a,b) = \omega(a) \vee \omega(b)$.*

P r o o f. As it is always true that $\omega(a) \vee \omega(b) \subseteq \omega(a,b)$, we prove that $\omega(a) \vee \omega(b) \supseteq \omega(a,b)$. Suppose that $p \in \omega(a,b)$, then $p \in (a,b)^{\mathrm{ul}}$ and by $(a,b)\mathrm{S}$ we have $p \in (a_1,b_1)^{\mathrm{ul}}$, where $a_1, b_1 \in F(P)$ and $a_1 \leqslant a$, $b_1 \leqslant b$. We note that $a_1 = \bigvee\{p_i \colon i = 1,2,\ldots,n\}$ and $b_1 = \bigvee\{q_j \colon j = 1,2,\ldots,m\}$. It means that there exists a finite subset $S = \{p_i \colon i = 1,2,\ldots,n\} \cup \{q_j \colon j = 1,2,\ldots,m\}$ such that $p \in S^{\mathrm{ul}}$. Since $S \subseteq \omega(a) \cup \omega(b)$ and S is essentially a finite set, by Theorem 2.14 we have $p \in \omega(a) \vee \omega(b)$. □

Lemma 2.16. *In an atomistic poset P if every pair is a statisch pair, then we have $(\omega(a), \omega(b))\mathrm{S}$ for all $a, b \in P$.*

P r o o f. Let $\omega(p) \leqslant \omega(a) \vee \omega(b)$ for an atom p and $a, b \in P$. Since $(a,b)\mathrm{S}$ holds, by Proposition 2.15 we have $\omega(a) \vee \omega(b) = \omega(a,b)$. Hence $p \in (a,b)^{\mathrm{ul}}$ and there exist finite elements $a_1 \leqslant a$, $b_1 \leqslant b$ such that $p \in (a_1,b_1)^{\mathrm{ul}}$. This means that $p \in \omega(a_1,b_1) = \omega(a_1) \vee \omega(b_1)$ and $\omega(a_1) \leqslant \omega(a)$, $\omega(b_1) \leqslant \omega(b)$. Therefore $\omega(p) \leqslant \omega(a_1) \vee \omega(b_1)$, that is, $(\omega(a), \omega(b))\mathrm{S}$. □

It is well-known that the set of finite elements of an AC-lattice forms an ideal. An AC-*lattice* is an atomistic lattice with *covering property*, that is, in a lattice with 0, if p is an atom and $a \wedge p = 0$, then $a \prec a \vee p$. Maeda and Maeda in [3] essentially proved that in an AC-lattice if a is a finite element and $b \leqslant a$, then b is also a finite element and hence, the set of finite elements forms an ideal.

Definition 2.17. A poset P is said to have *strong exchange property* if the following condition holds.

For atoms p, q and a subset $A \subseteq P$ if $p \in \{A, \{q\}\}^{\mathrm{ul}}$ and $p \notin A^{\mathrm{ul}}$, then $q \in \{A, \{p\}\}^{\mathrm{ul}}$.

Lemma 2.18. *Let P be a statisch poset with strong exchange property. Then $L(A(P))$ is an AC-lattice.*

P r o o f. By Theorem 2.14, $L(A(P))$ is atomistic. We show that $L(A(P))$ has the covering property. For an atom p let $\omega(p) \nleqslant \omega$ in $L(A(P))$. Let $\omega < \omega_1 \leqslant \omega \vee \omega(p)$. By definition,

$$\omega \vee \omega(p) = \{t \in A(P) \colon t \in S^{\mathrm{ul}},\ S \text{ is a finite subset of } \omega \cup \omega(p)\}.$$

Therefore there exists a finite set of atoms $\{q_1, \ldots, q_n\} \subseteq \omega$ such that $\omega \vee \omega(p) = \{t \in A(P) \colon t \in \{q_1, \ldots, q_n, p\}^{\mathrm{ul}}\}$. Take an atom $r \in \omega_1$ with $r \notin \omega$ (such atom exists since $L(A(P))$ is atomistic). As $\omega_1 \leqslant \omega \vee \omega(p)$, we have $r \in \{q_1, \ldots, q_n, p\}^{\mathrm{ul}}$. Since $r \notin \omega$ and ω is a subspace, we have $r \notin \{q_1, \ldots, q_n\}^{\mathrm{ul}}$. Now, by the strong exchange property we have $p \in \{q_1, \ldots, q_n, r\}^{\mathrm{ul}}$ and therefore $p \in \omega_1$. This means that every atom contained in $\omega \vee \omega(p)$ is also contained in ω_1 and hence $\omega_1 = \omega \vee \omega(p)$ and consequently P has the covering property. $\square$

Proposition 2.19. *The set of finite elements of a statisch poset with strong exchange property forms an ideal.*

Proof. For $a, b \in F(P)$ of a statisch poset P we prove that $(a,b)^{\mathrm{ul}} \subseteq F(P)$. For $x \in (a,b)^{\mathrm{ul}}$ we show that $x \in F(P)$. Observe that $\omega(x) \subseteq \omega(a,b)$. Since P is statisch, $\omega(a,b) = \omega(a) \vee \omega(b)$ and hence $\omega(x) \leqslant \omega(a) \vee \omega(b)$ in $L(A(P))$. Since a, b are finite, so are $\omega(a)$, $\omega(b)$ and therefore $\omega(a) \vee \omega(b)$ is finite. Since $L(A(P))$ is an AC-lattice by Lemma 2.18, we conclude that $\omega(x)$ is finite in $L(A(P))$ and since for every atom p we have $\omega(p) = \{p\}$, we conclude that x is finite in P. $\square$

We now study statisch pairs in connection with ∇-relation.

Theorem 2.20 ([8]). *Let P be an atomistic poset. The following statements are equivalent for $a \in P$.*

(α) *a is a standard element.*

(β) *a is a ∇-standard element.*

(γ) *a is a distributive element.*

(δ) *If p is an atom and $p \nleqslant a$, then $a \nabla p$.*

(ε) *If an atom p is subperspective to a, then $p \leqslant a$.*

(ζ) *If an atom $p \in (a,x)^{\mathrm{ul}}$ for $x \in P$, then $p \leqslant a$ or $p \leqslant x$.*

Remark 2.21. We note that condition (ζ) implies that $(a,x)\mathrm{S}$ holds. Therefore if P is an atomistic poset satisfying any of the conditions of Theorem 2.20, then $(a,x)\mathrm{S}$ holds for all $x \in P$.

Theorem 2.22. *Let P be an atomistic poset. Then the following statements are equivalent.*

(α) *Every atom of P is ∇-standard.*

(β) *If $p, q \in A(P)$ and p is subperspective to q, then $p = q$.*

(γ) *$(p,x)P$ holds for any $p \in A(P)$ and $(0 \neq)\, x \in P$ and $\omega(p,q) = \{p,q\}$.*

Proof. (α) $\Leftrightarrow$ (β): follows from Theorem 2.20.

(β) $\Rightarrow$ (γ): Let $q \in A(P)$ and $q \in (p,x)^{\mathrm{ul}}$. If $q \nleqslant x$, then q is subperspective to p and by assumption $q = p$. Hence, taking any atom r with $r \leqslant x$ we have $q \in (p,r)^{\mathrm{ul}}$ and $(p,x)P$ holds. If $q \leqslant x$, we may take $r = q$ and $(p,x)P$ holds. Next, if $p \neq q$, then for any atom $r \in (p,q)^{\mathrm{ul}}$ we have $r = p$ or $r = q$ by Theorem 2.20. Therefore $\omega(p,q)^{\mathrm{ul}} = \{p,q\}$.

(γ) $\Rightarrow$ (β): Let $p, q \in A(P)$ and $p \in (q,x)^{\mathrm{ul}}$ and $(p,x)^{\mathrm{l}} = \{0\}$. If $x = 0$, then $p = q$. If $x \neq 0$, then since $(q,x)P$ holds, there exists an atom $r \leqslant x$ and $p \in (q,r)^{\mathrm{ul}}$. Then

$q \neq r$ since otherwise $p \leqslant x$, a contradiction. Hence, by assumption $p \in \omega(q, r) = \{q, r\}$. But $p \neq r$ and hence $p = q$. □

Corollary 2.23. *Let P be an atomistic poset whose every atom is ∇-standard. Then $(p, x)\mathrm{S}$ holds for every $p \in A(P)$ and nonzero element $x \in P$.*

Lemma 2.24. *Let P be an atomistic poset whose every atom is ∇-standard. Consider the following statements.*

(α) *P is ∇-symmetric.*

(β) *Every element of P is ∇-standard.*

(γ) *$(a, b)\mathrm{S}$ holds for every $a, b \in P$.*

Then $(\alpha) \Rightarrow (\beta) \Rightarrow (\gamma)$.

Proof. $(\alpha) \Rightarrow (\beta)$: Let $a \in P$. If an atom $p \nleqslant a$, then since p is ∇-standard, we have $p \nabla a$ and therefore $a \nabla p$ by (α). Hence, a is ∇-standard by Theorem 2.20.

$(\beta) \Rightarrow (\gamma)$: For arbitrary elements $a, b \in P$ let $p \in (a, b)^{\mathrm{ul}}$. By Theorem 2.20, we have either $p \leqslant a$ or $p \leqslant b$ and consequently $(a, b)\mathrm{S}$ holds. □

We remark that the validity of the implication $(\gamma) \Rightarrow (\alpha)$ is still questionable.

Corollary 2.25. *If every element of an atomistic poset P is ∇-standard, then $(a, b)\mathrm{S}$ holds for every $a, b \in P$.*

Proof. Let every element of P be ∇-standard and $p \in (a, b)^{\mathrm{ul}}$ with $p \leqslant a$. Then $(a, b)\mathrm{S}$ holds by Remark 2.2. □

Lemma 2.26. *Let q be an atom of an atomistic poset P. If q is ∇-standard element, then $(q, x)\mathrm{S}$ holds for all $x \in P$.*

Proof. Let $p \in (q, x)^{\mathrm{ul}}$, then by Theorem 2.20 we have either $p \leqslant x$ or $p = q$. If $p = q$, then $p \in (p, 0)^{\mathrm{ul}}$ and $(q, x)\mathrm{S}$ holds. If $p \neq q$, then $p \in (p, q)^{\mathrm{ul}}$ for $p \leqslant x$ and again $(q, x)\mathrm{S}$ holds. □

Lemma 2.27 ([5])**.** *In a poset P for $a, b \in P$:*

(1) *If $a \nabla b$ and $b \in (x, a)^{\mathrm{ul}}$ for $x \in P$, then $b \leqslant x$.*

(2) *$a \nabla b$, $a_1 \leqslant a$ and $b_1 \leqslant b$ imply that $a_1 \nabla b_1$.*

Using Lemma 2.27, we obtain the following result.

Lemma 2.28. *Let P be an atomistic poset and $a, b \in P$. Let $(a, x)\mathrm{S}$ holds for all $x \in P$. Then $a \nabla b$ if and only if $a_1 \nabla b_1$ for all finite elements a_1, b_1 such that $a_1 \leqslant a$, $b_1 \leqslant b$.*

P r o o f. By Lemma 2.27, $a \nabla b$ implies $a_1 \nabla b_1$ for all finite elements a_1, b_1 such that $a_1 \leqslant a$, $b_1 \leqslant b$.

Conversely, suppose that $a_1 \nabla b_1$ holds for all finite elements a_1, b_1 such that $a_1 \leqslant a$, $b_1 \leqslant b$. Let $y \in \{(a,x)^{\mathrm{u}}, b\}^{\mathrm{l}}$. To prove $a \nabla b$, it is sufficient to show that $y \leqslant x$.

Assume on the contrary that $y \nleqslant x$. By the atomisticity of P, there exists an atom p such that $p \leqslant y \in (a,x)^{\mathrm{ul}}$ and $p \nleqslant x$. Since $p \leqslant y$, and using (a,x)S we have finite elements $a_2, b_2 \in P$ such that $p \in (a_2, b_2)^{\mathrm{ul}}$, $a_2 \leqslant a$, $b_2 \leqslant x$. Evidently $p \nleqslant b_2$ as $b_2 \leqslant x$ and $p \nleqslant x$. But then $p \in \{(a_2,b_2)^{\mathrm{u}}, p\}^{\mathrm{l}} = (b_2, p)^{\mathrm{l}} = \{0\}$ by $a_2 \nabla p$, a contradiction. Hence $y \leqslant x$ as required. □

Statisch lattices with covering property have connections with modularity, symmetricity and matroid lattices. Besides, the ideal of finite elements of AC-lattices has connections with parallelism and Wilcox lattices (recently, these two concepts have been generalized to posets by Shewale in [4]). As we have shown that the set of finite elements of a statisch poset with strong exchange property forms an ideal, the investigation and generalization of above mentioned connections appear a wide open research area.

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