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ON THE q-PELL SEQUENCES AND SUMS OF TAILS

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Abstract. We examine the q-Pell sequences and their applications to weighted partition theorems and values of L-functions. We also put them into perspective with sums of tails. It is shown that there is a deeper structure between two-variable generalizations of Rogers-Ramanujan identities and sums of tails, by offering examples of an operator equation considered in a paper published by the present author. The paper starts with the classical example offered by Ramanujan and studied by previous authors noted in the introduction. Showing that simple combinatorial manipulations give rise to an identity published by the present author, a weighted form of a Lebesgue partition theorem is given as the main application to partitions. The conclusion of the paper summarizes some directions for further research, pointing out that certain conditions on the q-polynomial would be desired, and also possibly looking at the operator equation in the present paper from the position of using modular forms.

Keywords: sum of tails; q-series; partition; L-function

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1. Introduction

In [9] we find that Chen and Ji offered weighted partitions theorems relating partitions into distinct parts to partitions into odd parts by paraphrasing some special identities due to Ramanujan (see the section on Special identities, [4], page 149). Namely, we have [9], equation (4.13)

$$\sum_{\mu\in D}\Big(\mu_1+n(\mu)+\frac{1-(-1)^{r(\mu)}}{2}\Big)q^{|\mu|}=2\sum_{\lambda\in O}n(\lambda)q^{|\lambda|}. \tag{1.1}$$

Here D is the set of partitions into distinct parts, and O is the set of partitions into odd parts. Recall that a partition λ of n is a nonincreasing sequence $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_k)$, where the λ_i, for $1\leqslant i\leqslant k$, are parts that sum to n. The sum

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of these parts is $|\lambda|$. We denote by $n(\lambda)$ the number of parts of a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, where λ_1 denotes the largest part. We denote by $r(\lambda)$ the largest part minus the number of parts, or the "rank" of the partition. As it stands, equation (1.1) is equivalent to the identity

$$\sum_{n\geqslant 0}((-q)_\infty - (-q)_n) = (-q)_\infty\left(-\frac{1}{2} + \sum_{n\geqslant 1}\frac{q^n}{1-q^n}\right) + \frac{1}{2}\sum_{n\geqslant 0}\frac{q^{n(n+1)/2}}{(-q)_n}. \tag{1.2}$$

Here the q-series on the far right side of (1.2) is the distinct rank parity function $\sigma(q)$ and has been studied extensively, see [5], [4], [10], [9], [17], [20]. (See [8] for similar functions.) We put $(V;q)_n := (1-V)(1-Vq)\ldots(1-Vq^{n-1})$.

Now if we subtract $2\sum_{\mu\in D} n(\mu)q^{|\mu|}$ from both sides of (1.1) we may write

$$\sum_{\mu\in D}\left(r(\mu) + \frac{1-(-1)^{r(\mu)}}{2}\right)q^{|\mu|} = 2\sum_{\lambda\in O} n(\lambda)q^{|\lambda|} - 2\sum_{\mu\in D} n(\mu)q^{|\mu|}. \tag{1.3}$$

It is not difficult to see that

$$2(-q)_\infty\sum_{n\geqslant 1}\frac{q^{2n}}{1-q^{2n}} = 2\sum_{\lambda\in O} n(\lambda)q^{|\lambda|} - 2\sum_{\mu\in D} n(\mu)q^{|\mu|}. \tag{1.4}$$

Hence, we have shown combinatorially the q-series identity, see [20],

$$\sum_{n\geqslant 0}((-q)_\infty - V_n(q)) = (-q)_\infty\left(-\frac{1}{2} + 2\sum_{n\geqslant 1}\frac{q^{2n}}{1-q^{2n}}\right) + \frac{1}{2}\sum_{n\geqslant 0}\frac{q^{n(n+1)/2}}{(-q)_n}, \tag{1.5}$$

where $V_m(q)$ is the generating function for the number of partitions of n with rank m. In fact, Fine in [12], page 8, has studied this function which is given by

$$V_m(q) = \sum_{n\geqslant 0}\begin{bmatrix} n+m \\ n \end{bmatrix}_q q^{n(n+1)/2}, \tag{1.6}$$

$$\sum_{n\geqslant 0} V_n(q)t^n = \sum_{n\geqslant 0}\frac{q^{n(n+1)/2}}{(t)_{n+1}}.$$

Here

$$\begin{bmatrix} n \\ m \end{bmatrix}_q := \frac{(q)_n}{(q)_m(q)_{n-m}}. \tag{1.7}$$

The relation between (1.5) and Ramanujan's other identity related to $\sigma(q)$, see [4],

$$\sum_{n\geqslant 0}\Big(\frac{1}{(q;q^2)_\infty}-\frac{1}{(q;q^2)_{n+1}}\Big) = \frac{1}{(q;q^2)_\infty}\Big(-\frac{1}{2}+\sum_{n\geqslant 1}\frac{q^{2n}}{1-q^{2n}}\Big)+\frac{1}{2}\sum_{n\geqslant 0}\frac{q^{n(n+1)/2}}{(-q)_n} \tag{1.8}$$

was shown analytically in [20].

This study offers two more examples of two-variable Rogers-Ramanujan-type q-series $f(t,q)$ that possess the following properties:

(i) $\lim\limits_{t\to 1^-}(1-t)f(t,q)=f(q)$ *is a weight* 0 *modular form.*

(ii) $f(-1,q)$ *is a false mock theta function.*

(iii) $\lim\limits_{t\to 1^-}\frac{\partial}{\partial t}(1-t)f(t,q)=f(q)D(q)+f(-1,q)$, *where* $D(q)$ *is some linear combination of generating functions for divisor functions.*

For (ii) we refer the reader to the description mentioned in [8], page 2192. This allows us to put sums of tails into perspective with two variable generalizations of Rogers-Ramanujan type series, see [1]. The first example of a q-series in the literature that satisfies (i)–(iii) appears to be $\sum\limits_{n\geqslant 0}V_n(q)t^n$, where for (i) we have the modular form $\eta(2z)/\eta(z)$, where as usual we define the Dedekind eta function

$$\eta(z):=\mathrm{e}^{2\pi\mathrm{i}z/24}\prod_{n\geqslant 1}^{\infty}(1-\mathrm{e}^{2\pi\mathrm{i}nz}),$$

where $\Im(z)>0$.

The main theorems of this paper offer two more examples satisfying (i)–(iii), for the weight 0 modular forms

$$\frac{\eta(4z)}{\eta(z)}\quad\text{and}\quad\frac{\eta^2(2z)}{\eta^2(z)}\frac{\eta(2z)}{\eta(4z)}.$$

See, especially, [6], Theorem 3, and the commentary associated with that theorem. For more material on sums of tails see [4], [7], [14], [19], [18].

2. The q-Pell sequences

In [21] Sills and Santos study the sequences

$$\omega_n(q) = \sum_{n\geqslant j\geqslant 0}\sum_{j\geqslant k\geqslant 0} \begin{bmatrix} j \\ k \end{bmatrix}_q \begin{bmatrix} n-k \\ j \end{bmatrix}_q q^{j(j+1)/2+k(k+1)/2} \tag{2.1}$$

and

$$\Theta_n(q) = \sum_{n\geqslant j\geqslant 0}\sum_{j\geqslant k\geqslant 0} \begin{bmatrix} j \\ k \end{bmatrix}_q \begin{bmatrix} n-k \\ j \end{bmatrix}_q q^{j(j+1)/2+k(k-1)/2}, \tag{2.2}$$

where

$$\sum_{n\geqslant 0} \omega_n(q)t^n = \sum_{n\geqslant 0} \frac{(-tq)_n t^n q^{n(n+1)/2}}{(t)_{n+1}} \tag{2.3}$$

and

$$\sum_{n\geqslant 0} \Theta_n(q)t^n = \sum_{n\geqslant 0} \frac{(-t)_n t^n q^{n(n+1)/2}}{(t)_{n+1}}. \tag{2.4}$$

Here we shall define $L_1(t,q)$ to be (2.3), and $L_2(t,q)$ to be (2.4). That is, two-variable generalizations of the Lebesgue identities, see [13]. Then $\lim\limits_{t\to 1^-}(1-t)L_1(t,q) = (-q^2;q^2)_\infty/(q;q^2)_\infty$, and $\lim\limits_{t\to 1^-}(1-t)L_2(t,q) = (-q;q^2)_\infty/(q;q^2)_\infty$ by [13].

The function $L_1(-1,q)$ appeared in [11], and $L_2(-1,q)$ appeared in Lovejoy work [15], where they related these functions to the arithmetic of $\mathbb{Q}(\sqrt{2})$. As usual, let $N(a)$ be the norm of an ideal $a \subset \mathbb{Z}[\sqrt{2}]$. Paper [11] showed that

$$\sum_{n\geqslant 0} \frac{(q)_n(-1)^n q^{n(n+1)/2}}{(-q)_n} = \sum_{\substack{a\subset\mathbb{Z}[\sqrt{2}] \\ N(a)\equiv 1 \pmod 8}} (-q)^{(N(a)-1)/8}, \tag{2.5}$$

and Lovejoy in [15] showed that

$$\sum_{n\geqslant 1} \frac{(q)_{n-1}(-1)^n q^{n(n+1)/2}}{(-q)_n} = \sum_{\substack{a\subset\mathbb{Z}[\sqrt{2}] \\ N(a)<0}} \mathrm{i}^{N(a)^2+N(a)} q^{|N(a)|}. \tag{2.6}$$

The functions (2.5) and (2.6) appear as "error" series in the following *sums of tails* identities. These are the cases ([6], Theorem 1, $a = q^{3/2}$, $t = -q$), and ([6], Theorem 1, $a = q^{3/2}$, $t = -q^{1/2}$). Equation (2.8) of the below theorem involves the "error" series (2.10).

Lemma 1. *We have the following sums of tails:*

$$\sum_{n\geqslant 0}\left(\frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty}-\frac{(-q^2;q^2)_n}{(q;q^2)_{n+1}}\right) = \frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty}\left(-\frac{1}{2}+\sum_{n\geqslant 1}\frac{q^{2n}}{1-q^{2n}}+\sum_{n\geqslant 1}\frac{q^{2n+1}}{1+q^{2n+1}}\right) + \frac{1}{2}\sum_{n\geqslant 0}\frac{(q)_n(-1)^n q^{n(n+1)/2}}{(-q)_n}, \tag{2.7}$$

$$\sum_{n\geqslant 0}\left(\frac{(-q;q^2)_\infty}{(q;q^2)_\infty}-\frac{(-q;q^2)_n}{(q;q^2)_{n+1}}\right) = \frac{(-q;q^2)_\infty}{(q;q^2)_\infty}\left(\sum_{n\geqslant 1}\frac{q^{2n}}{1-q^{2n}}+\sum_{n\geqslant 1}\frac{q^{2n}}{1+q^{2n}}\right) + \sum_{n\geqslant 1}\frac{(q)_{n-1}(-1)^n q^{n(n+1)/2}}{(-q)_n}. \tag{2.8}$$

We also need a well-known transformation of Fine in [12], page 5, equation (6.3).

Lemma 2. *We have,*

$$\sum_{n\geqslant 0}\frac{(aq)_n}{(bq)_n}t^n = \frac{1-b}{1-t}\sum_{n\geqslant 0}\frac{(atq/b)_n}{(tq)_n}b^n. \tag{2.9}$$

We will also need the special case of Lemma 2 ($a=-q^{-1/2}$, $b=q^{1/2}$, $t=-q$) coupled with a result due to Lovejoy in [16]:

$$q\sum_{n\geqslant 0}\frac{(-q;q^2)_n}{(q;q^2)_{n+1}}(-q^2)^n = -\sum_{n\geqslant 1}\frac{(q^2;q^2)_{n-1}}{(-q^2;q^2)_n}q^n = -\sum_{n\geqslant 1}\frac{(-q;-q)_{n-1}(-1)^n(-q)^{n(n+1)/2}}{(q;-q)_n} = -L_2(-1,-q). \tag{2.10}$$

Lemma 3 ([4], page 30)**.** *Let $b\to -t$, $a=xt$ then*

$$\sum_{n\geqslant 0}\frac{(xq;q^2)_n}{(t;q^2)_{n+1}}(tq)^n = \sum_{n\geqslant 0}\frac{(xq)_n t^n q^{n(n+1)/2}}{(t)_{n+1}}. \tag{2.11}$$

We are now ready to prove the main q-series identities of the paper.

Theorem 1.

$$\sum_{n\geqslant 0}\left(\frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty}-\omega_n(q)\right) = \frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty}\left(-\frac{1}{2}+2\sum_{n\geqslant 1}\frac{q^{2n-1}}{1-q^{2n-1}}\right) + \frac{1}{2}\sum_{n\geqslant 0}\frac{(q)_n(-1)^n q^{n(n+1)/2}}{(-q)_n}, \tag{2.12}$$

$$\sum_{n\geqslant 0}\Big(\frac{(-q;q^2)_\infty}{(q;q^2)_\infty}-\Theta_n(q)\Big)=2\frac{(-q;q^2)_\infty}{(q;q^2)_\infty}\Bigg(\sum_{n\geqslant 1}\frac{q^{2n-1}}{1-q^{2n-1}}\Bigg)+\sum_{n\geqslant 1}\frac{(q)_{n-1}(-1)^n q^{n(n+1)/2}}{(-q)_n}. \tag{2.13}$$

Proof. In keeping with [6] we shall set ε to be the differential operator $\lim\limits_{t\to 1^-}\partial/\partial t$. Applying Proposition 2.1 of [6] to the left hand side of (2.3), and putting $x=-t$ in Lemma 3 gives

$$\varepsilon(1-t)\sum_{n\geqslant 0}\omega_n(q)t^n \tag{2.14}$$

$$=\varepsilon\sum_{n\geqslant 0}\frac{(-tq)_n t^n q^{n(n+1)/2}}{(tq)_n} \tag{2.15}$$

$$=\varepsilon(1-t)\sum_{n\geqslant 0}\frac{(-tq;q^2)_n}{(t;q^2)_{n+1}}(tq)^n \tag{2.16}$$

$$=\varepsilon(1-t)\sum_{n\geqslant 0}\frac{(-q^2;q^2)_n t^n}{(q;q^2)_{n+1}}+\sum_{n\geqslant 1}\frac{(-q;q^2)_n}{(q^2;q^2)_n}nq^n \tag{2.17}$$

$$=\varepsilon(1-t)\sum_{n\geqslant 0}\frac{(-q^2;q^2)_n t^n}{(q;q^2)_{n+1}}+\varepsilon\frac{(-tq^2;q^2)_\infty}{(tq;q^2)_\infty} \tag{2.18}$$

$$=\varepsilon(1-t)\sum_{n\geqslant 0}\frac{(-q^2;q^2)_n t^n}{(q;q^2)_{n+1}}+\frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty}\Bigg(\sum_{n\geqslant 1}\frac{q^{2n}}{1+q^{2n}}+\sum_{n\geqslant 1}\frac{q^{2n+1}}{1-q^{2n+1}}\Bigg). \tag{2.19}$$

Equation (2.17) follows from the specialization of Lemma 2 with $q\to q^2$, $b=q$, $a=-1$, and standard properties of ε. The final step to prove (2.12) requires invoking Proposition 2.1 of [6] coupled with (2.7) of Lemma 1.

The proof of (2.13) is very similar, but we use (2.8) of Lemma 1 coupled with (2.4) and (2.10). □

The methods employed in [6] and [7] to obtain special values of L-functions may be applied to our main theorem. We consider equation (2.13) after noting that $(-q;q^2)_\infty/(q;q^2)_\infty$ vanishes to infinite order when $q\to -1$. Define the real quadratic field $K:=\mathbb{Q}(\sqrt{2})$.

Theorem 2. *As a formal power series in t we have,*

$$-\sum_{n\geqslant 0}\Theta_n(-\mathrm{e}^{-t})=\sum_{n\geqslant 0}\frac{(-t)^n L_K(-n)}{n!}, \tag{2.20}$$

where

$$L_K(s) = \sum_{\substack{a \subset \mathbb{Z}[\sqrt{2}] \\ N(a) < 0}} \frac{\mathrm{i}^{N(a)^2 + N(a) + 2|N(a)|}}{|N(a)|^s}. \tag{2.21}$$

The method to obtain this result can be found in [6], which amounts to using Mellin transforms and calculating the residue $s = -n$ of $\Gamma(s)L_K(s)$, so we omit the details.

Next we consider a combinatorial interpretation of (2.12) of Theorem 1. Let $D_{\leqslant n}$ be the set of partitions into distinct parts "$\leqslant n$", and let D_n be the set of partitions into n distinct parts. Let DE be the set of partitions with distinct evens (i.e. even parts do not repeat), see [3].

Theorem 3. *Define $\gamma(\pi_1, \pi_2) := \pi_2' + n(\pi_1)$, where π_2' denotes the largest part of π_2. Let $n_o(\pi)$ be the number of odd parts in a partition π. Then*

$$\sum_{n \geqslant 0} \sum_{(\pi_1, \pi_2) \in D_{\leqslant n} \times D_n} \Big(\gamma(\pi_1, \pi_2) + \frac{1 - (-1)^{\gamma(\pi_1, \pi_2)}}{2}\Big) q^{|\pi_1| + |\pi_2|} = 2 \sum_{\pi \in DE} n_o(\pi) q^{|\pi|}. \tag{2.22}$$

P r o o f. We consider the combinatorial interpretation related to that given in [11], page 400. Note that $(-aq)_n$ generates partitions into distinct parts "$\leqslant n$" and a keeps track of the number of parts. The function $b^n q^{n(n+1)/2}/(bq)_n$ generates partitions into n distinct parts and b keeps track of the largest part. Hence,

$$\sum_{n \geqslant 0} \frac{(-aq)_n b^n q^{n(n+1)/2}}{(bq)_n} = \sum_{n \geqslant 0} \sum_{(\pi_1, \pi_2) \in D_{\leqslant n} \times D_n} a^{n(\pi_1)} b^{\pi_2'} q^{|\pi_1| + |\pi_2|}. \tag{2.23}$$

Putting $a = b = t$ in (2.23) and differentiating with respect to t and setting $t = 1$ gives the left hand side of (2.12). On the other hand, putting $a = b = -1$ in (2.23) gives the far right hand side of (2.12). The proof is complete after noting that differentiating

$$\frac{(-q^2; q^2)_\infty}{(tq; q^2)_\infty} = \sum_{\pi \in DE} t^{n_o(\pi)} q^{|\pi|} \tag{2.24}$$

with respect to t and then setting $t = 1$ gives the middle term in (2.12). $\square$

3. Some remarks for further research

It is well-known, see [6], that

$$\sum_{n\geqslant 0}\Big(\frac{1}{(q)_\infty}-\frac{1}{(q)_n}\Big)=\frac{1}{(q)_\infty}\sum_{n\geqslant 1}\frac{q^n}{1-q^n}. \tag{3.1}$$

The left hand side is $\varepsilon(1-t)/(t)_\infty$, which is $\sum\limits_{n,m\geqslant 1} mp(n,m)q^n$, where $p(n,m)$ is the number of partitions of n into m parts. Since the number of partitions of n into m parts is equal to the number of partitions of n with largest part m, we may write

$$\sum_{n\geqslant 0}\Big(\frac{1}{(q)_\infty}-j_n(q)\Big)=\frac{2}{(q)_\infty}\sum_{n\geqslant 1}\frac{q^n}{1-q^n}, \tag{3.2}$$

where $j_n(q)$ is the generating function for the number of partitions wherein the largest plus the number of parts is at most n. In fact, Andrews studied $j_n(q)$ in [2], and it is given by

$$j_n(q)=\sum_{0\leqslant m\leqslant n-1} q^m\begin{bmatrix} n-1\\ m\end{bmatrix}_q, \tag{3.3}$$

for $n>0$, and $j_0(q)=1$. This leads us to consider the series, see [2],

$$M(q,t)=\sum_{n\geqslant 0}\frac{t^{2n}q^{n^2}}{(t)_{n+1}(tq)_n},$$

which has properties different from those listed in the first section. Furthermore, if we subtract $2\sum\limits_{m\geqslant 1} mp(m,n)$ from both sides of (3.2), we find $\sum\limits_{m} mN(m,n)=0$. This is a well-known result concerning $N(m,n)$, the number of partitions of n with rank m.

At this point it is clear $M(q,t)$ does not appear to satisfy any of (i)–(iii). Indeed, we would need to replace the condition (ii) with the statement "$M(q,-1)$ is a mock modular form of weight $1/2$", and it happens that (iii) satisfies $\lim\limits_{t\to 1^-}\frac{\partial}{\partial t}(1-t)f(t,q)=f(q)D(q)$, which is less interesting without the "error series" $M(q,-1)$. This is due to the fact that (i) $\lim\limits_{t\to 1^-}(1-t)M(q,t)=M(q)=1/(q)_\infty$ is a weight $-1/2$ modular form. In fact, we would require a sum of tails over $j_{2n}(q)$ to amend this issue, and satisfy the (iii) in the first section:

$$\sum_{n\geqslant 0}\Big(\frac{1}{(q)_\infty}-j_{2n}(q)\Big)=\frac{1}{(q)_\infty}\Big(\frac{1}{4}+\sum_{n\geqslant 1}\frac{q^n}{1-q^n}\Big)-\frac{1}{4}\sum_{n\geqslant 0}\frac{q^{n^2}}{(-q)_n^2}. \tag{3.4}$$

The last q-series on the far right side of (3.4) is one of Ramanujan's third order mock theta functions, see [2], which generates $\sum_m (-1)^m N(m,n)$. This leaves us with a question: Is there an explanation using modular forms to explain the correlation between the weight of a given modular form and the operator equation in (iii)? What conditions are needed on the associated q-polynomial?

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