Mojgan Afkhami; Kazem Khashyarmanesh; Zohreh Rajabi
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SOME RESULTS ON THE ANNIHILATOR GRAPH OF A COMMUTATIVE RING

Mojgan Afkhami, Neyshabur,
Kazem Khashyarmanesh, Zohreh Rajabi, Mashhad

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Abstract. Let R be a commutative ring. The annihilator graph of R, denoted by $\mathrm{AG}(R)$, is the undirected graph with all nonzero zero-divisors of R as vertex set, and two distinct vertices x and y are adjacent if and only if $\mathrm{ann}_R(xy) \neq \mathrm{ann}_R(x) \cup \mathrm{ann}_R(y)$, where for $z \in R$, $\mathrm{ann}_R(z) = \{r \in R\colon rz = 0\}$. In this paper, we characterize all finite commutative rings R with planar or outerplanar or ring-graph annihilator graphs. We characterize all finite commutative rings R whose annihilator graphs have clique number 1, 2 or 3. Also, we investigate some properties of the annihilator graph under the extension of R to polynomial rings and rings of fractions. For instance, we show that the graphs $\mathrm{AG}(R)$ and $\mathrm{AG}(T(R))$ are isomorphic, where $T(R)$ is the total quotient ring of R. Moreover, we investigate some properties of the annihilator graph of the ring of integers modulo n, where $n \geqslant 1$.

Keywords: annihilator graph; zero-divisor graph; outerplanar; ring-graph; cut-vertex; clique number; weakly perfect; chromatic number; polynomial ring; ring of fractions

MSC 2010: 05C75, 13A99, 05C99

1. Introduction

Let R be a commutative ring with nonzero identity. We denote the sets of all zero-divisors and nilpotent elements of R by $Z(R)$ and Nil(R), respectively. In 1999, Anderson and Livingston introduced the zero-divisor graph of R, denoted by $\Gamma(R)$, that is the graph with vertices $Z(R)^* = Z(R)\setminus\{0\}$ and distinct vertices x and y being adjacent in $\Gamma(R)$ if and only if $xy = 0$. Beck introduced this concept in 1988 but he allowed all the elements of R as vertices and was mainly interested in colorings. Several other classes of graphs associated with algebraic structures have been defined and studied (cf. [2], [6], [10], [14], [13], [22]). One of the most important class of graphs associated with the algebraic structures is that of Cayley graphs (cf. [19],

[20], [21]). Recently, in [12], the concept of the annihilator graph has been defined and studied. The annihilator graph of R, denoted by $\mathrm{AG}(R)$, is an undirected graph with vertex set $Z(R)^*$, and two distinct vertices x and y are adjacent if and only if $\mathrm{ann}_R(xy) \neq \mathrm{ann}_R(x) \cup \mathrm{ann}_R(y)$, where for $z \in R$, $\mathrm{ann}_R(z) = \{r \in R\colon\ rz = 0\}$. Let G be an additive abelian group and let S be a symmetric subset of G. The Cayley graph $\mathrm{Cay}(\mathrm{G}, S)$ is the graph with vertex set G and two vertices x and y are adjacent if and only if $x - y \in S$. It is easy to see that the induced subgraph of the Cayley graph $\mathrm{Cay}(R, S)$, where $S = R \setminus Z(R)$, with vertex set $Z(R)^*$ is a subgraph of the annihilator graph $\mathrm{AG}(R)$. Indeed, assume that $x - y \in S$, and suppose on the contrary that x and y are not adjacent in $\mathrm{AG}(R)$. Then, by [12], Lemma 2.1, we have $\mathrm{ann}_R(x) \subseteq \mathrm{ann}_R(y)$ or $\mathrm{ann}_R(y) \subseteq \mathrm{ann}_R(x)$. Without loss of generality, we may assume that $\mathrm{ann}_R(x) \subseteq \mathrm{ann}_R(y)$. So $x - y \in Z(R)$, which is a contradiction.

By [12], Lemma 2.1, the zero-divisor graph $\Gamma(R)$ is a (spanning) subgraph of the annihilator graph $\mathrm{AG}(R)$. Many results on zero-divisor graphs of commutative rings have been obtained (cf. [3], [4], [5], [8], [9], [15], [17]). Let $\mathrm{AG}_{\mathrm{N}}(R)$ be the (induced) subgraph of $\mathrm{AG}(R)$ with vertices $\mathrm{Nil}(R)^* = \mathrm{Nil}(R) \setminus \{0\}$. Recall that R is reduced if $\mathrm{Nil}(R) = 0$. Also, $\mathrm{Min}(R)$ is the set of all minimal prime ideals of R. In [2], the authors studied the situations that the unit, unitary and total graphs are ring-graph or outerplanar. Also, in [1], they studied the ring-graph and outerplanarity for co-maximal and zero-divisor graphs. In the second section of this paper, we completely characterize all finite commutative rings with planar or outerplanar or ring-graph annihilator graphs. In the third section we characterize all finite commutative rings R, whose annihilator graphs have clique number $1, 2$ or 3. In the fourth section, we investigate the annihilator graph of the extension of R to polynomial rings and rings of fractions. Also, we show that the graphs $\mathrm{AG}(R)$ and $\mathrm{AG}(T(R))$ are isomorphic, where $T(R)$ is the total quotient ring of R. Finally, in the fifth section, we investigate some properties of the annihilator graph of the ring of integers modulo n, where $n \geqslant 1$. For instance, we study cut-vertices and cut-sets in $\mathrm{AG}(\mathbb{Z}_n)$.

Now, we recall some definitions and notation on graphs. Let G be a simple graph with vertex set $V(G)$ and let C be a cycle of G. A chord in G is any edge joining two nonadjacent vertices in C. A primitive cycle is a cycle without chord. Moreover, if any two primitive cycles intersect in at most one edge, then we say G has the primitive cycle property (PCP). The number of primitive cycles of G is the free rank of G and is denoted by $\mathrm{frank}(G)$. We have $\mathrm{rank}(G) := q - n + r$, where q, n and r are the number of edges of G, the number of vertices of G and the number of connected components of G, respectively.

A graph G is called planar if it can be drawn in the plane without crossing edges. A graph G is an outerplanar graph if it can be drawn in the plane without crossing in such a way that all of the vertices belong to the unbounded face of the drawing.

The precise definition of a ring-graph can be found in section 2 of [18]. Also, in [18], the authors showed that the following conditions are equivalent:

(i) G is a ring-graph,
(ii) $\operatorname{rank}(G) = \operatorname{frank}(G)$,
(iii) G satisfies PCP and G does not contain a subdivision of K^4 as a subgraph.

So every ring-graph is planar. Moreover, in [18], authors showed that every outerplanar graph is a ring-graph. A set $A \subset V(G)$ is said to be a cut-set if its removal increases the number of connected components of G and no proper subset of A satisfies the same condition. A cut-set consisting of only one element is called a cut-vertex of G. Suppose that $x, y \in V(G)$. If x is adjacent to y, then y is a neighbour of x. We use the notation $x - y$ to say that x and y are adjacent in a graph G. The girth of G, denoted by $\operatorname{gr}(G)$, is the length of a shortest cycle in G ($\operatorname{gr}(G) = \infty$ if G contains no cycles). Also we denote the complete graph with n vertices by K^n and we denote the complete bipartite graph by $K^{m,n}$. We denote the star graph by $K^{1,n}$. Let k be a positive integer. For a graph G, a k-coloring of the vertices of G is an assignment of k colors to the vertices of G in such a way that no two adjacent vertices receive the same color. The chromatic number of G, denoted by $\chi(G)$, is the smallest number k such that G admits a k-coloring. Any subgraph of G is called a clique if it is complete and the size of a largest clique in a graph G is denoted by $\operatorname{cl}(G)$. A graph G is called weakly perfect provided $\chi(G) = \operatorname{cl}(G)$ (cf. [23]).

2. Ring-graphs and outerplanar annihilator graphs

In this section, we investigate all finite commutative rings R such that their annihilator graphs are planar or outerplanar or ring-graph. Throughout this section, R is a finite commutative ring with nonzero identity and $\mathbb{F}$ is a finite field. Specially, $\mathbb{F}_4$ is a field with four elements.

Theorem 2.1. *The annihilator graph* $\operatorname{AG}(R)$ *is planar if and only if* R *is isomorphic to one of the following rings:*

(i) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$,
(ii) $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_2 \times \mathbb{F}$, $\mathbb{Z}_3 \times \mathbb{F}$,
(iii) $\mathbb{Z}_4$, $\mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_8$, $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_4[x]/(2x, x^2-2)$, $\mathbb{Z}_2[x,y]/(x,y)^2$, $\mathbb{Z}_9$, $\mathbb{Z}_4[x]/(2x, x^2)$, $\mathbb{Z}_3[x]/(x^2)$, $\mathbb{F}_4[x]/(x^2)$, $\mathbb{Z}_4[x]/(x^2+x+1)$, $\mathbb{Z}_{25}$, $\mathbb{Z}_5[x]/(x^2)$.

P r o o f. Clearly, by [12], Lemma 2.1, the zero-divisor graph $\Gamma(R)$ is a (spanning) subgraph of the annihilator graph $\operatorname{AG}(R)$. Hence if $\Gamma(R)$ is not planar, then $\operatorname{AG}(R)$ is not planar either. So in order to investigate the planarity of $\operatorname{AG}(R)$, we need only to study the rings R whose zero-divisor graphs are planar. In [3] and [16] it was shown that $\Gamma(R)$ is planar if and only if R is isomorphic to one of the following rings:

(i) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$,

(ii) $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_2 \times \mathbb{F}$, $\mathbb{Z}_3 \times \mathbb{F}$, $\mathbb{Z}_3 \times \mathbb{Z}_4$, $\mathbb{Z}_3 \times \mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_2 \times \mathbb{Z}_8$, $\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_2 \times \mathbb{Z}_4[x]/(2x, x^2-2)$, $\mathbb{Z}_2 \times \mathbb{Z}_9$, $\mathbb{Z}_2 \times \mathbb{Z}_3[x]/(x^2)$, $\mathbb{Z}_3 \times \mathbb{Z}_9$, $\mathbb{Z}_3 \times \mathbb{Z}_3[x]/(x^2)$,

(iii) $\mathbb{Z}_4, \mathbb{Z}_2[x]/(x^2), \mathbb{Z}_8, \mathbb{Z}_2[x]/(x^3), \mathbb{Z}_4[x]/(2x, x^2-2), \mathbb{Z}_2[x,y]/(x,y)^2$, $\mathbb{Z}_4[x]/(2x, x^2)$, $\mathbb{Z}_9$, $\mathbb{Z}_3[x]/(x^2)$, $\mathbb{F}_4[x]/(x^2)$, $\mathbb{Z}_4[x]/(x^2-2)$, $\mathbb{Z}_4[x]/(x^2+2x+2)$, $\mathbb{Z}_4[x]/(x^2+x+1)$, $\mathbb{Z}_{25}$, $\mathbb{Z}_5[x]/(x^2)$, $\mathbb{Z}_{16}$, $\mathbb{Z}_2[x]/(x^4)$, $\mathbb{Z}_2[x,y]/(x^2-y^2, xy)$, $\mathbb{Z}_2[x,y]/(x^2, y^2)$, $\mathbb{Z}_4[x]/(2x, x^3-2)$, $\mathbb{Z}_4[x,y]/(x^2-2, xy, y^2-2, 2x)$, $\mathbb{Z}_4[x,y]/(x^2, xy-2, y^2)$, $\mathbb{Z}_4[x]/(x^2)$, $\mathbb{Z}_4[x]/(x^2-2x)$, $\mathbb{Z}_8[x]/(2x, x^2-4)$, $\mathbb{Z}_{27}$, $\mathbb{Z}_9[x]/(x^2-3, 3x)$, $\mathbb{Z}_9[x]/(x^2-6, 3x)$, $\mathbb{Z}_3[x]/(x^3)$.

Now we study the planarity of $\mathrm{AG}(R)$, when R is one of the above rings. By Figure 1, it is easy to see that $\mathrm{AG}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ is planar.

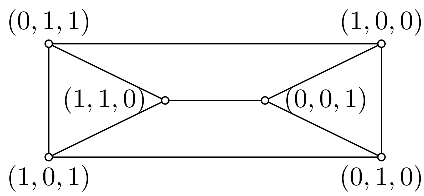

Figure 1. $\mathrm{AG}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$.

In Figure 2, the graph $\mathrm{AG}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3)$ has a copy of $K^{3,3}$, and so it is not planar.

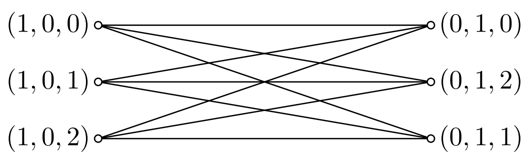

Figure 2.

If R is isomorphic to $\mathbb{Z}_2 \times \mathbb{F}$ with $|\mathbb{F}| = m$, then $\mathrm{AG}(R) \cong K^{1,m-1}$. Hence $\mathrm{AG}(\mathbb{Z}_2 \times \mathbb{F})$ is planar. Also if R is isomorphic to $\mathbb{Z}_3 \times \mathbb{F}$ with $|\mathbb{F}| = m$, then one can easily check that $\mathrm{AG}(R) \cong K^{2,m-1}$. Thus $\mathrm{AG}(\mathbb{Z}_3 \times \mathbb{F})$ is planar.

If $R \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ or $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2)$, then we have $\mathrm{AG}(\mathbb{Z}_2 \times \mathbb{Z}_4) \cong \mathrm{AG}(\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2))$. Let $R \cong \mathbb{Z}_2 \times \mathbb{Z}_4$. Then, by Figure 3, it is obvious that $\mathrm{AG}(\mathbb{Z}_2 \times \mathbb{Z}_4)$ is planar.

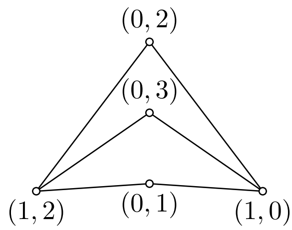

Figure 3. $\mathrm{AG}(\mathbb{Z}_2 \times \mathbb{Z}_4)$.

If $R \cong \mathbb{Z}_3 \times \mathbb{Z}_4$ or $R \cong \mathbb{Z}_3 \times \mathbb{Z}_2[x]/(x^2)$, then we have $\mathrm{AG}(\mathbb{Z}_3 \times \mathbb{Z}_4) \cong \mathrm{AG}(\mathbb{Z}_3 \times \mathbb{Z}_2[x]/(x^2))$. Now, one can easily find a copy of $K^{3,3}$ with vertex set $\{(1,0),(2,0),(1,2),(0,1),(0,2),(0,3)\}$ in $\mathrm{AG}(\mathbb{Z}_3 \times \mathbb{Z}_4)$ (see Figure 4), and so it is not planar.

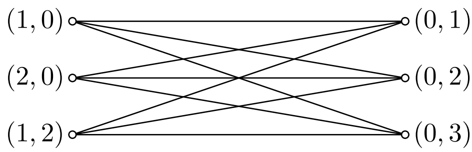

Figure 4.

If R is isomorphic to one of the rings $\mathbb{Z}_2 \times \mathbb{Z}_8$, $\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^3)$ or $\mathbb{Z}_2 \times \mathbb{Z}_4[x]/(2x, x^2-2)$, then we have $\mathrm{AG}(\mathbb{Z}_2 \times \mathbb{Z}_8) \cong \mathrm{AG}(\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^3)) \cong \mathrm{AG}(\mathbb{Z}_2 \times \mathbb{Z}_4[x]/(2x, x^2-2))$. Let $R \cong \mathbb{Z}_2 \times \mathbb{Z}_8$. By Figure 5, the graph $\mathrm{AG}(\mathbb{Z}_2 \times \mathbb{Z}_8)$ has a subdivision of K^5. So $\mathrm{AG}(\mathbb{Z}_2 \times \mathbb{Z}_8)$ is not planar.

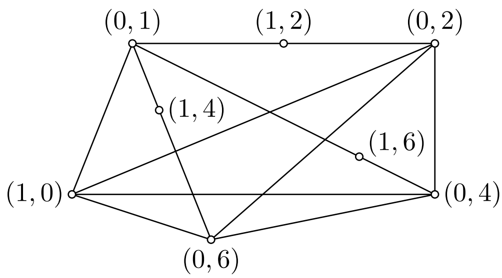

Figure 5.

If $R \cong \mathbb{Z}_2 \times \mathbb{Z}_9$ or $R \cong \mathbb{Z}_2 \times \mathbb{Z}_3[x]/(x^2)$, then we have $\mathrm{AG}(\mathbb{Z}_2 \times \mathbb{Z}_9) \cong \mathrm{AG}(\mathbb{Z}_2 \times \mathbb{Z}_3[x]/(x^2))$. Let $R \cong \mathbb{Z}_2 \times \mathbb{Z}_9$. Then, by Figure 6, one can find a copy of $K^{3,3}$. Hence $\mathrm{AG}(\mathbb{Z}_2 \times \mathbb{Z}_9)$ is not planar.

Also, if $R \cong \mathbb{Z}_3 \times \mathbb{Z}_9$ or $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3[x]/(x^2)$, then $\mathrm{AG}(\mathbb{Z}_3 \times \mathbb{Z}_9) \cong \mathrm{AG}(\mathbb{Z}_3 \times \mathbb{Z}_3[x]/(x^2))$. Let $R \cong \mathbb{Z}_3 \times \mathbb{Z}_9$. Then one can find a copy of K^5 with vertex set $\{(0,3),(0,6),(1,3),(1,6),(2,3)\}$ in $\mathrm{AG}(\mathbb{Z}_3 \times \mathbb{Z}_9)$, so $\mathrm{AG}(\mathbb{Z}_3 \times \mathbb{Z}_9)$ is not planar.

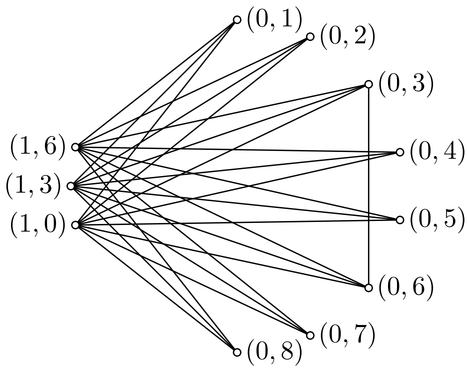

Figure 6. $\mathrm{AG}(\mathbb{Z}_2 \times \mathbb{Z}_9)$.

Now, we study the situation when R is a local ring. Badawi in [12] proved that $\mathrm{AG_N}(R)$ is a complete graph. Since for finite local rings we have $Z(R) = \mathrm{Nil}(R)$ if $|Z(R)| \geqslant 6$, hence $\mathrm{AG}(R)$ contains a copy of K^5, and so it is not planar. It is easy to see that the following rings have $|Z(R)| = 8$, and hence their annihilator graphs are not planar:

$$\mathbb{Z}_{16},\ \mathbb{Z}_2[x]/(x^4),\ \mathbb{Z}_2[x,y]/(x^2-y^2,xy),\ \mathbb{Z}_2[x,y]/(x^2,y^2),\ \mathbb{Z}_4[x]/(2x,x^3-2),$$
$$\mathbb{Z}_8[x]/(2x,x^2-4),\ \mathbb{Z}_4[x,y]/(x^2-2,xy,y^2-2,2x),\ \mathbb{Z}_4[x,y]/(x^2,xy-2,y^2),$$
$$\mathbb{Z}_4[x]/(x^2),\ \mathbb{Z}_4[x]/(x^2-2x),\ \mathbb{Z}_4[x]/(x^2-2),\ \mathbb{Z}_4[x]/(x^2+2x+2).$$

Also in the following rings we have $|Z(R)| = 9$, and hence their annihilator graphs are not planar:

$$\mathbb{Z}_{27},\ \mathbb{Z}_9[x]/(x^2-3,3x),\ \mathbb{Z}_9[x]/(x^2-6,3x),\ \mathbb{Z}_3[x]/(x^3).$$

Now, one can easily check that the following isomorphisms hold:

$$\begin{aligned}
\mathrm{AG}(\mathbb{Z}_4) &\cong \mathrm{AG}(\mathbb{Z}_2[x]/(x^2)) \cong K^1,\\
\mathrm{AG}(\mathbb{Z}_8) &\cong \mathrm{AG}(\mathbb{Z}_2[x]/(x^3)) \cong \mathrm{AG}(\mathbb{Z}_4[x]/(2x,x^2-2))\\
&\cong \mathrm{AG}(\mathbb{Z}_2[x,y]/(x,y)^2) \cong \mathrm{AG}(\mathbb{Z}_4[x]/(2x,x^2))\\
&\cong \mathrm{AG}(\mathbb{F}_4[x]/(x^2)) \cong \mathrm{AG}(\mathbb{Z}_4[x]/(x^2+x+1)) \cong K^3,\\
\mathrm{AG}(\mathbb{Z}_9) &\cong \mathrm{AG}(\mathbb{Z}_3[x]/(x^2)) \cong K^2,\\
\mathrm{AG}(\mathbb{Z}_{25}) &\cong \mathrm{AG}(\mathbb{Z}_5[x]/(x^2)) \cong K^4.
\end{aligned}$$

By the above discussion the result holds. □

In the next theorem, we characterize all rings with ring-graph annihilator graphs.

Theorem 2.2. *The annihilator graph* $\mathrm{AG}(R)$ *is a ring-graph if and only if* R *is isomorphic to one of the following rings:*

(i) $\mathbb{Z}_2 \times \mathbb{F}$, $\mathbb{Z}_3 \times \mathbb{Z}_3$,

(ii) $\mathbb{Z}_4$, $\mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_8$, $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_4[x]/(2x, x^2-2)$, $\mathbb{Z}_2[x,y]/(x,y)^2$, $\mathbb{Z}_4[x]/(2x, x^2)$, $\mathbb{Z}_9$, $\mathbb{Z}_3[x]/(x^2)$, $\mathbb{F}_4[x]/(x^2)$, $\mathbb{Z}_4[x]/(x^2+x+1)$.

P r o o f. Since every ring-graph is planar, it is enough to study the rings with planar annihilator graphs. Since

$$\mathrm{frank}(\mathrm{AG}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)) = 7 \quad \text{and} \quad \mathrm{rank}(\mathrm{AG}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)) = 4,$$

by Figure 1, $\mathrm{AG}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ is not a ring-graph. If R is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4$, by Figure 3, we have

$$\mathrm{rank}(\mathrm{AG}(\mathbb{Z}_2 \times \mathbb{Z}_4)) = 6 - 5 + 1 = 2 \quad \text{and} \quad \mathrm{frank}(\mathrm{AG}(\mathbb{Z}_2 \times \mathbb{Z}_4)) = 3.$$

Thus $\mathrm{AG}(\mathbb{Z}_2 \times \mathbb{Z}_4)$ is not a ring-graph. Also $\mathrm{AG}(\mathbb{Z}_2 \times \mathbb{Z}_4) \cong \mathrm{AG}(\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2))$, and so $\mathrm{AG}(\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2))$ is not a ring-graph. If R is isomorphic to $\mathbb{Z}_2 \times \mathbb{F}$, then it is easy to see that $\mathrm{AG}(\mathbb{Z}_2 \times \mathbb{F})$ is a star graph. Hence $\mathrm{AG}(\mathbb{Z}_2 \times \mathbb{F})$ is a ring-graph. If R is isomorphic to $\mathbb{Z}_3 \times \mathbb{F}$, then $\mathrm{AG}(R)$ is isomorphic to $K^{2,m-1}$, where $|\mathbb{F}| = m$. Thus $\mathrm{rank}(\mathrm{AG}(\mathbb{Z}_3 \times \mathbb{F})) = m - 2$ and $\mathrm{frank}(\mathrm{AG}(\mathbb{Z}_3 \times \mathbb{F})) = (m-1)(m-2)/2$. Therefore $\mathrm{AG}((\mathbb{Z}_3 \times \mathbb{F}))$ is a ring-graph if and only if $(m-1)(m-2)/2 = m-2$, which implies that $m = 2$ or $m = 3$. So $\mathrm{AG}(\mathbb{Z}_3 \times \mathbb{F})$ is a ring-graph if and only if $\mathbb{F} \cong \mathbb{Z}_2$ or $\mathbb{F} \cong \mathbb{Z}_3$. Also, in view of the proof of Theorem 2.1, the annihilator graphs of all rings

$$\mathbb{Z}_4,\ \mathbb{Z}_2[x]/(x^2),\ \mathbb{Z}_8,\ \mathbb{Z}_2[x]/(x^3),\ \mathbb{Z}_4[x]/(2x, x^2-2),\ \mathbb{Z}_2[x,y]/(x,y)^2,$$
$$\mathbb{Z}_4[x]/(2x, x^2),\ \mathbb{Z}_9,\ \mathbb{Z}_3[x]/(x^2),\ \mathbb{F}_4[x]/(x^2) \text{ and } \mathbb{Z}_4[x]/(x^2+x+1)$$

are ring-graphs. The graphs $\mathrm{AG}(\mathbb{Z}_{25})$ and $\mathrm{AG}(\mathbb{Z}_5[x]/(x^2))$ are isomorphic to K^4, and so they are not ring-graphs. □

In the next theorem, by using the fact that every outerplanar graph is a ring-graph in conjunction with Theorem 2.2, we determine all rings R with outerplanar annihilator graphs.

Theorem 2.3. *The annihilator graph* $\mathrm{AG}(R)$ *is outerplanar if and only if* R *is isomorphic to one of the following rings:*

(i) $\mathbb{Z}_2 \times \mathbb{F}$, $\mathbb{Z}_3 \times \mathbb{Z}_3$,

(ii) $\mathbb{Z}_4$, $\mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_8$, $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_4[x]/(2x, x^2-2)$, $\mathbb{Z}_2[x,y]/(x,y)^2$, $\mathbb{Z}_4[x]/(2x, x^2)$, $\mathbb{Z}_9$, $\mathbb{Z}_3[x]/(x^2)$, $\mathbb{F}_4[x]/(x^2)$, $\mathbb{Z}_4[x]/(x^2+x+1)$.

P r o o f. Since a graph G is outerplanar if and only if it does not contain a subdivision of a complete graph K^4 or a complete bipartite graph $K^{2,3}$, one can check that no ring-graph annihilator graphs contain a subdivision of a complete graph K^4 or a complete bipartite graph $K^{2,3}$. Now the result follows immediately from Theorem 2.2. □

3. Clique numbers of the annihilator graphs

We begin this section with a lemma which shows that the annihilator graph of the product of three fields is weakly perfect.

Lemma 3.1. *Let K_1, K_2 and K_3 be fields. Then* $\mathrm{cl}(\mathrm{AG}(K_1 \times K_2 \times K_3)) = \chi(\mathrm{AG}(K_1 \times K_2 \times K_3)) = 3$.

P r o o f. Suppose that (a, b, c) is in $Z(K_1 \times K_2 \times K_3)$. Then at least one of a, b or c is zero. So, if

$$\begin{aligned}
A_1 &= \{(a,b,c) \in K_1 \times K_2 \times K_3 \colon a = 0 \quad \text{and} \quad b \neq 0 \neq c\},\\
A_2 &= \{(a,b,c) \in K_1 \times K_2 \times K_3 \colon b = 0 \quad \text{and} \quad a \neq 0 \neq c\},\\
A_3 &= \{(a,b,c) \in K_1 \times K_2 \times K_3 \colon c = 0 \quad \text{and} \quad a \neq 0 \neq b\},\\
A_4 &= \{(a,b,c) \in K_1 \times K_2 \times K_3 \colon a = b = 0 \quad \text{and} \quad c \neq 0\},\\
A_5 &= \{(a,b,c) \in K_1 \times K_2 \times K_3 \colon a = c = 0 \quad \text{and} \quad b \neq 0\},\\
A_6 &= \{(a,b,c) \in K_1 \times K_2 \times K_3 \colon b = c = 0 \quad \text{and} \quad a \neq 0\},
\end{aligned}$$

then $Z(K_1 \times K_2 \times K_3)^* = \bigcup_{i=1}^{6} A_i$. Now, by the definition of the annihilator graph $\mathrm{AG}(R)$, every vertex in A_1 is adjacent to every vertex in A_2, A_3 and A_6, every vertex in A_2 is adjacent to every vertex in A_1, A_3 and A_5, every vertex in A_3 is adjacent to every vertex in A_1, A_2 and A_4, every vertex in A_4 is adjacent to every vertex in A_3, A_5 and A_6, every vertex in A_5 is adjacent to every vertex in A_2, A_4 and A_6 and every vertex in A_6 is adjacent to every vertex in A_1, A_4 and A_5. Also, each A_i for $i = 1, \ldots, 6$ is an independent set. Hence $\mathrm{cl}(\mathrm{AG}(K_1 \times K_2 \times K_3)) = 3$, and so $\chi(\mathrm{AG}(K_1 \times K_2 \times K_3)) \geqslant 3$. Since each A_i for $i = 1, \ldots, 6$ is an independent set, we can color every vertex in A_1 by λ_1. Now, since every vertex in A_2 is adjacent to every vertex in A_1, we color every vertex in A_2 by λ_2. Also, every vertex in A_3 is adjacent to every vertex in A_1 and A_2. Therefore we need another color λ_3 for every vertex in A_3. Every vertex in A_4 is adjacent to every vertex in A_3, A_5 and A_6, and so we can color the vertices in A_4 by λ_1 or λ_2. Without loss of generality, we color every vertex in A_4 by λ_1. Since every vertex in A_5 is adjacent to A_2 and A_4, we cannot color the vertices in A_5 by λ_1 and λ_2. So we color every vertex in A_5 by λ_3. Finally, since every vertex in A_6 is adjacent to A_1, A_4 and A_5, we color the vertices in A_6 by λ_2. Hence $\chi(\mathrm{AG}(K_1 \times K_2 \times K_3)) = 3$. □

In the next theorem we characterize all finite rings R whose annihilator graphs have clique number $1, 2$ or 3.

Theorem 3.2. *Let R be a finite commutative ring and let K_1, K_2 and K_3 be finite fields. Also, let $\mathbb{F}_4$ be a field with four elements. Then the following statements hold:*

(a) $\mathrm{cl}(\mathrm{AG}(R)) = 1$ *if and only if R is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$.*

(b) $\mathrm{cl}(\mathrm{AG}(R)) = 2$ *if and only if R is isomorphic to one of the following rings:*

$$K_1 \times K_2,\ \ \mathbb{Z}_2 \times \mathbb{Z}_4,\ \ \mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2),\ \ \mathbb{Z}_9,\ \ \mathbb{Z}_3[x]/(x^2).$$

(c) $\mathrm{cl}(\mathrm{AG}(R)) = 3$ *if and only if R is isomorphic to one of the following rings:*

$$K_1 \times K_2 \times K_3,\ \ \mathbb{Z}_3 \times \mathbb{Z}_4,\ \ \mathbb{Z}_3 \times \mathbb{Z}_2[x]/(x^2),\ \ \mathbb{Z}_8,\ \ \mathbb{Z}_2[x]/(x^3),\ \ \mathbb{Z}_4[x]/(2,x)^2,$$
$$\mathbb{F}_4[x]/(x^2),\ \ \mathbb{Z}_4[x]/(x^2+x+1),\ \ \mathbb{Z}_2[x,y]/(x,y)^2,\ \ \mathbb{Z}_4[x]/(2x, x^2-2).$$

P r o o f. (a) Clearly, by [12], Lemma 2.1, the zero-divisor graph $\Gamma(R)$ is a (spanning) subgraph of the annihilator graph $\mathrm{AG}(R)$. Hence if $\mathrm{cl}(\mathrm{AG}(R)) = n$, then $\mathrm{cl}(\Gamma(R)) \leqslant n$. So $\mathrm{cl}(\mathrm{AG}(R)) = 1$ if and only if $\mathrm{cl}(\Gamma(R)) = 1$. Also by [15], Proposition 2.2, $\mathrm{cl}(\Gamma(R)) = 1$ if and only if R is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$.

(b) In order to characterize all rings R with $\mathrm{cl}(\mathrm{AG}(R)) = 2$, we need only to study the rings R with $\mathrm{cl}(\Gamma(R)) = 1$ or 2. It is easy to see that if $\mathrm{cl}(\Gamma(R)) = 1$, then $\mathrm{cl}(\mathrm{AG}(R)) = 1$. Now, by [15], page 226, $\mathrm{cl}(\Gamma(R)) = 2$ if and only if R is isomorphic to one of the following rings:

$$K_1 \times K_2,\ \ K_1 \times \mathbb{Z}_4,\ \ K_1 \times \mathbb{Z}_2[x]/(x^2),\ \ \mathbb{Z}_8,\ \ \mathbb{Z}_9,$$
$$\mathbb{Z}_3[x]/(x^2), \mathbb{Z}_2[x]/(x^3),\ \ \mathbb{Z}_4[x]/(2x, x^2-2).$$

If $R \cong K_1 \times K_2$ with $|K_1| = n$ and $|K_2| = m$, then one can easily check that $\mathrm{AG}(R) \cong K^{n-1,m-1}$. Thus $\mathrm{cl}(\mathrm{AG}(K_1 \times K_2)) = 2$.

If $R \cong K_1 \times \mathbb{Z}_4$ or $K_1 \times \mathbb{Z}_2[x]/(x^2)$ with $|K_1| = n$, then $\mathrm{AG}(K_1 \times \mathbb{Z}_4) \cong \mathrm{AG}(K_1 \times \mathbb{Z}_2[x]/(x^2))$. Let $R \cong K_1 \times \mathbb{Z}_4$. Then $\mathrm{AG}(K_1 \times \mathbb{Z}_4)$ contains a complete graph K^n with vertex set $\{(0,2), (r_1,2), (r_2,2), \ldots, (r_{n-1},2)\}$, where $r_i \neq 0$ for $i = 1, \ldots, n-1$, and so $\mathrm{cl}(\mathrm{AG}(R)) \geqslant n$. Thus, for $n \geqslant 3$ we have that $\mathrm{cl}(\mathrm{AG}(K_1 \times \mathbb{Z}_4)) \geqslant 3$. Now, if $n=2$, then $K_1\cong\mathbb{Z}_2$, and so $\mathrm{cl}(\mathrm{AG}(\mathbb{Z}_2\times\mathbb{Z}_4))=2$. If $R\cong\mathbb{Z}_8$, then we have $\mathrm{AG}(\mathbb{Z}_8)\cong K^3$. Thus $\mathrm{cl}(\mathrm{AG}(\mathbb{Z}_8))=3$.

If $R\cong\mathbb{Z}_9$ or $\mathbb{Z}_3[x]/(x^2)$, then we have $\mathrm{AG}(\mathbb{Z}_9)\cong\mathrm{AG}(\mathbb{Z}_3[x]/(x^2))\cong K^2$. Thus $\mathrm{cl}(\mathrm{AG}(\mathbb{Z}_9))=\mathrm{cl}(\mathrm{AG}(\mathbb{Z}_3[x]/(x^2)))=2$. If $R\cong\mathbb{Z}_2[x]/(x^3)$ or $\mathbb{Z}_4[x]/(2x,x^2-2)$, then we have $\mathrm{AG}(\mathbb{Z}_2[x]/(x^3))\cong\mathrm{AG}(\mathbb{Z}_4[x]/(2x,x^2-2))\cong K^3$. Therefore, for clicque subgraph is $\mathrm{cl}(\mathrm{AG}(\mathbb{Z}_2[x]/(x^3)))=\mathrm{cl}(\mathrm{AG}(\mathbb{Z}_4[x]/(2x,x^2-2)))=3$.

(c) In order to characterize all rings R with $\mathrm{cl}(\mathrm{AG}(R))=3$, we need only to study the rings R with $\mathrm{cl}(\Gamma(R))=1,2$ or 3. In view of the proof of part (b), we have $\mathrm{cl}(\mathrm{AG}(\mathbb{Z}_3\times\mathbb{Z}_4))=\mathrm{cl}(\mathrm{AG}(\mathbb{Z}_3\times\mathbb{Z}_2[x]/(x^2)))=\mathrm{cl}(\mathrm{AG}(\mathbb{Z}_8))=\mathrm{cl}(\mathrm{AG}(\mathbb{Z}_2[x]/(x^3)))=\mathrm{cl}(\mathrm{AG}(\mathbb{Z}_4[x]/(2x,x^2-2)))=3$. Now we study the rings R with $\mathrm{cl}(\Gamma(R))=3$. By [9], Theorem 4.4, $\mathrm{cl}(\Gamma(R))=3$ if and only if R is isomorphic to one of the following rings:

$$\begin{gathered}\mathbb{Z}_4\times\mathbb{Z}_4,\ \ \mathbb{Z}_4\times\mathbb{Z}_2[x]/(x^2),\ \ \mathbb{Z}_2[x]/(x^2)\times\mathbb{Z}_2[x]/(x^2),\\ K_1\times K_2\times K_3,\ \ K_1\times K_2\times\mathbb{Z}_4,\ \ K_1\times K_2\times\mathbb{Z}_2[x]/(x^2),\\ K_1\times\mathbb{Z}_8,\ \ K_1\times\mathbb{Z}_9,\ \ K_1\times\mathbb{Z}_3[x]/(x^2),\ \ K_1\times\mathbb{Z}_2[x]/(x^3),\\ K_1\times\mathbb{Z}_4[x]/(2x,x^2-2),\ \ \mathbb{Z}_{16},\mathbb{Z}_2[x]/(x^4),\ \ \mathbb{Z}_4[x]/(2x,x^3-2),\\ \mathbb{Z}_4[x]/(x^2-2),\ \ \mathbb{Z}_4[x]/(x^2+2x+2),\ \ \mathbb{F}_4[x]/(x^2),\ \ \mathbb{Z}_4[x]/(x^2+x+1),\\ \mathbb{Z}_2[x,y]/(x,y)^2,\ \ \mathbb{Z}_4[x]/(2,x)^2,\mathbb{Z}_{27},\ \ \mathbb{Z}_3[x]/(x^3),\ \ \mathbb{Z}_9[x]/(3x,x^2-3),\\ \mathbb{Z}_9[x]/(3x,x^2-6),\ \ \mathbb{Z}_2[x,y]/(x^2,y^2-xy),\ \ \mathbb{Z}_2[x,y]/(x^2,y^2),\\ \mathbb{Z}_8[x]/(2x-4,x^2),\ \ \mathbb{Z}_4[x]/(x^2),\ \ \mathbb{Z}_4[x]/(x^2-2x),\\ \mathbb{Z}_4[x,y]/(x^2,xy-2,y^2,2x,2y)\quad\text{or}\quad\mathbb{Z}_4[x,y]/(x^2,xy-2,x^2-xy,2x,2y).\end{gathered}$$

If $R\cong\mathbb{Z}_4\times\mathbb{Z}_4$, then $\mathrm{AG}(\mathbb{Z}_4\times\mathbb{Z}_4)$ contains a complete graph K^5 with vertex set $\{(2,1),(2,2),(2,3),(1,2),(3,2)\}$. Thus $\mathrm{cl}(\mathrm{AG}(\mathbb{Z}_4\times\mathbb{Z}_4))\geqslant 5$. If $R\cong\mathbb{Z}_4\times\mathbb{Z}_2[x]/(x^2)$, then $\mathrm{AG}(R)$ contains a complete graph K^5 with vertex set $\{(2,1),(2,x),(2,1+x),(1,x),(3,x)\}$. Therefore $\mathrm{cl}(\mathrm{AG}(\mathbb{Z}_4\times\mathbb{Z}_2[x]/(x^2)))\geqslant 5$. If $R\cong\mathbb{Z}_2[x]/(x^2)\times\mathbb{Z}_2[x]/(x^2)$, then $\mathrm{AG}(R)$ contains a complete graph K^5 with vertex set $\{(x,1),(x,x),(x,1+x),(1+x,x),(1,x)\}$. Thus $\mathrm{cl}(\mathrm{AG}(R))\geqslant 5$. If $R\cong K_1\times K_2\times K_3$, then by Lemma 3.1, $\mathrm{cl}(\mathrm{AG}(R))=3$. If $R\cong K_1\times K_2\times\mathbb{Z}_4$, then $\mathrm{AG}(R)$ contains a complete graph K^4 with vertex set $\{(0,1,2),(1,1,0),(1,0,2),(0,0,2)\}$. Therefore $\mathrm{cl}(\mathrm{AG}(R))\geqslant 4$. If $R\cong K_1\times K_2\times\mathbb{Z}_2[x]/(x^2)$, then $\mathrm{AG}(R)$ contains a complete graph K^4 with vertex set $\{(0,1,x),(1,1,0),(1,0,x),(0,0,x)\}$. Hence $\mathrm{cl}(\mathrm{AG}(R))\geqslant 4$. If $R\cong K_1\times\mathbb{Z}_8$, then $\mathrm{AG}(R)$ contains a complete graph K^4 with vertex set $\{(0,1),(1,2),(1,4),(1,6)\}$. Thus $\mathrm{cl}(\mathrm{AG}(R))\geqslant 4$. If $R\cong K_1\times\mathbb{Z}_9$, then $\mathrm{AG}(R)$ contains a complete graph K^4 with vertex set $\{(1,3),(1,6),(0,3),(0,6)\}$. So $\mathrm{cl}(\mathrm{AG}(R))\geqslant 4$. If $R\cong K_1\times\mathbb{Z}_3[x]/(x^2)$, then $\mathrm{AG}(R)$ contains a complete graph K^4 with vertex set $\{(1,x),(1,2x),(0,x),(0,2x)\}$. Thus $\mathrm{cl}(\mathrm{AG}(R))\geqslant 4$.

Now, if $R \cong K_1 \times \mathbb{Z}_2[x]/(x^3)$, then $\mathrm{AG}(R)$ contains a complete graph K^4 with vertex set $\{(1,x),(1,x^2+x),(0,x),(0,x^2+x)\}$, and so $\mathrm{cl}(\mathrm{AG}(R)) \geqslant 4$. If $R \cong K_1 \times \mathbb{Z}_4[x]/(2x, x^2-2)$, then $\mathrm{AG}(R)$ contains a complete graph K^4 with vertex set $\{(1,2),(1,2+x),(0,2),(0,2+x)\}$. So $\mathrm{cl}(\mathrm{AG}(R)) \geqslant 4$. If R is isomorphic to one of the following rings:

$$\mathbb{Z}_{16},\ \ \mathbb{Z}_2[x]/(x^4),\ \ \mathbb{Z}_4[x]/(2x,x^3-2),\ \ \mathbb{Z}_4[x]/(x^2-2),$$
$$\mathbb{Z}_4[x]/(x^2+2x+2),\ \ \mathbb{Z}_2[x,y]/(x^2,y^2-xy),\ \ \mathbb{Z}_2[x,y]/(x^2,y^2),$$
$$\mathbb{Z}_8[x]/(2x-4,x^2),\ \ \mathbb{Z}_4[x]/(x^2),\ \ \mathbb{Z}_4[x]/(x^2-2x),$$
$$\mathbb{Z}_4[x,y]/(x^2,xy-2,y^2,2x,2y) \quad \text{or} \quad \mathbb{Z}_4[x,y]/(x^2,xy-2,x^2-xy,2x,2y),$$

then its annihilator graph is isomorphic to K^7. Thus, for clicque subgraph is $\mathrm{cl}(\mathrm{AG}(R)) = 7$. If $R \cong \mathbb{F}_4[x]/(x^2)$, $\mathbb{Z}_4[x]/(x^2+x+1)$, $\mathbb{Z}_2[x,y]/(x,y)^2$ or $\mathbb{Z}_4[x]/(2,x)^2$, then its annihilator graph is isomorphic to K^3. Hence $\mathrm{cl}(\mathrm{AG}(R)) = 3$. If $R \cong \mathbb{Z}_{27}$, $\mathbb{Z}_3[x]/(x^3)$, $\mathbb{Z}_9[x]/(3x,x^2-3)$ or $\mathbb{Z}_9[x]/(3x,x^2-6)$, then its annihilator graph is isomorphic to K^8. Therefore $\mathrm{cl}(\mathrm{AG}(R)) = 8$. Now by the above discussion the result holds. □

4. Extension rings

In this section, we compare some properties of the annihilator graph $\mathrm{AG}(R)$ with the graphs $\mathrm{AG}(R[x])$ and $\mathrm{AG}(S^{-1}R)$. Note that McCoy's theorem states that $f(x) \in R[x]$ is a zero-divisor if and only if there is a nonzero element $r \in R$ such that $rf(x) = 0$. Also it is proved that a polynomial $f(x)$ over a commutative ring R is nilpotent if and only if each coefficient of $f(x)$ is nilpotent (cf. [11]).

Proposition 4.1. *Let R be a finite commutative ring with $|Z(R)^*| > 1$ and $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$. Then the annihilator graph $\mathrm{AG}(R)$ is a complete graph if and only if R is a local ring.*

Proof. First assume that the annihilator graph $\mathrm{AG}(R)$ is a complete graph. We shall show that R is a local ring. If R is a finite commutative ring and $Z(R)$ is an ideal of R, then R is a local ring with $Z(R) = \mathrm{Nil}(R)$ its unique maximal ideal. So it is enough to show that $Z(R)$ is an ideal of R. Let $|Z(R)^*| = 2$. So $Z(R) = \{0,x,y\}$ where $x \neq y$. If $xy \neq 0$, then $x^2 = y^2 = 0$. Hence $Z(R) = \mathrm{Nil}(R)$. Therefore $Z(R)$ is an ideal of R. Now, suppose that $xy = 0$. Then the zero-divisor graph $\Gamma(R)$ is a complete graph. Moreover, in [8], Theorem 2.10, it was shown that for any finite commutative ring R, if $\Gamma(R)$ is complete, then either $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or R is a local ring with characteristic p or p^2 and $|\Gamma(R)| = p^s - 1$, where p is a prime and $s \geqslant 1$. Thus the result follows. Now assume that $|Z(R)^*| \geqslant 3$. Let x, y be distinct elements in $Z(R)^*$. It is enough to show that $x + y \in Z(R)$. Since $\Gamma(R)$ is connected, there is a nonzero element $r \in R$ such that $rx = 0$ (or $ry = 0$). Now, because $\mathrm{AG}(R)$ is a complete graph, $r - y$ is an edge of $AG(R)$. So $\mathrm{ann}_R(ry) \neq \mathrm{ann}_R(r) \cup \mathrm{ann}_R(y)$. Thus there exists $r' \in R$ such that $r'ry = 0$ and $r'r \neq 0$ and $r'y \neq 0$. Hence $r'r(x + y) = 0$. Therefore $x + y \in Z(R)$.

Conversely, since for a finite local ring we have $Z(R) = \mathrm{Nil}(R)$, the result follows from [12], Theorem 3.10. □

Theorem 4.2. *Let R be a finite commutative ring with $|Z(R)^*| > 1$ and $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$. If $\mathrm{AG}(R)$ is complete, then $\mathrm{AG}(R[x])$ is also complete.*

Proof. It is enough to show that every zero-divisor element in $R[x]$ is nilpotent. Then, by [12], Theorem 3.10, $\mathrm{AG}(R[x])$ is complete. Since $\mathrm{AG}(R)$ is complete, by Proposition 4.1, R is a local ring. So $Z(R) = \mathrm{Nil}(R)$. Now, let $f(x) \in Z(R[x])$, where $f(x) = a_0 + a_1x + \ldots + a_nx^n$. Thus $a_0, \ldots, a_n \in \mathrm{Nil}(R)$, which implies that $f(x)$ is nilpotent. □

Recall that the diameter of a graph G, denoted by $\mathrm{diam}(G)$, is equal to $\sup\{\mathrm{d}(a, b) : a, b \in V(G)\}$, where $\mathrm{d}(a, b)$ is the length of the shortest path connecting a and b.

Corollary 4.3. *Let R be a finite commutative ring with $|Z(R)^*| > 1$ and $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$. Then $\mathrm{diam}(\mathrm{AG}(R)) = \mathrm{diam}(\mathrm{AG}(R[x]))$.*

Proof. By [12], Theorem 2.2, we have $\mathrm{diam}(\mathrm{AG}(R)) = 1$ or 2. Assume that $\mathrm{diam}(\mathrm{AG}(R)) = 1$. So $\mathrm{AG}(R)$ is a complete graph. Therefore, by Theorem 4.2, the annihilator graph $\mathrm{AG}(R[x])$ is also a complete graph. Hence $\mathrm{diam}(\mathrm{AG}(R[x])) = 1$. Now if $\mathrm{diam}(\mathrm{AG}(R)) = 2$, then there are distinct elements $a, b \in Z(R)^*$, such that $a - b$ is not an edge of $\mathrm{AG}(R)$. We show that $a - b$ is not an edge of $\mathrm{AG}(R[x])$ either. Since $a - b$ is not an edge of $\mathrm{AG}(R)$, $\mathrm{ann}_R(ab) = \mathrm{ann}_R(a) \cup \mathrm{ann}_R(b)$. So, $\mathrm{ann}_R(a) \subseteq \mathrm{ann}_R(b)$ or $\mathrm{ann}_R(b) \subseteq \mathrm{ann}_R(a)$. Now (without loss of generality), we set $\mathrm{ann}_R(a) \subseteq \mathrm{ann}_R(b)$. So $\mathrm{ann}_R(ab) = \mathrm{ann}_R(b)$. Suppose that $f(x) \in \mathrm{ann}_{R[x]}(ab)$ where $f(x) = a_0 + a_1x + \ldots + a_nx^n$. Hence $aba_0 = aba_1 = \ldots = aba_n = 0$. Since $\mathrm{ann}_R(ab) = \mathrm{ann}_R(b)$, $ba_0 = ba_1 = \ldots = ba_n = 0$. Thus $f(x) \in \mathrm{ann}_{R[x]}(b)$. Hence $\mathrm{ann}_{R[x]}(ab) \subseteq \mathrm{ann}_{R[x]}(b)$. Therefore $\mathrm{ann}_{R[x]}(ab) = \mathrm{ann}_{R[x]}(a) \cup \mathrm{ann}_{R[x]}(b)$, so $a - b$ is not an edge of $\mathrm{AG}(R[x])$, and since $\mathrm{diam}(\mathrm{AG}(R[x])) \leqslant 2$, we conclude that $\mathrm{diam}(\mathrm{AG}(R[x])) = 2$. □

Theorem 4.4. *Let R be a commutative ring. If R is not an integral domain, then the annihilator graph $\mathrm{AG}(R[x])$ is not planar.*

P r o o f. First suppose that R is not a reduced ring. So there exists a nonzero nilpotent element $a \in R$. Let n be the least positive integer such that $a^n = 0$. Then one can find a copy of K^5 with vertex set $\{a, ax, ax^2, ax^3, ax^4\}$ in $\mathrm{AG}(R[x])$, and so $\mathrm{AG}(R[x])$ is not planar. Now assume that R is a reduced ring. Hence there exist $a, b \in R$ such that $a \neq b$ and $ab = 0$. Then, by Figure 7, the graph $\mathrm{AG}(R[x])$ has a copy of $K^{3,3}$, and so $\mathrm{AG}(R[x])$ is not planar. $\square$

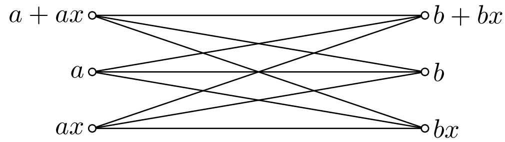

Figure 7.

The following corollary immediately follows from Theorem 4.4.

Corollary 4.5. *Let R be a commutative ring. If R is not an integral domain, then* $\mathrm{gr}(\mathrm{AG}(R[x])) \in \{3, 4\}$.

In the rest of the section, we study the annihilator graph of the ring of fractions $S^{-1}R$, where S is a multiplicatively closed subset of R. It is obvious that if $r \in Z(R)$, then $r/s \in Z(S^{-1}R)$ for every $s \in S$. Now, let $r/s \in Z(S^{-1}R)$. Thus there is a nonzero element $r'/s' \in S^{-1}R$ such that $(r/s) \cdot (r'/s') = 0/1$. So there exists $u \in S$ such that $urr' = 0$. Clearly $ur' \neq 0$, because otherwise $r'/s' = 0/1$. Thus $r \in Z(R)$.

Proposition 4.6. *Let R be a commutative ring. If r and r' are arbitrary elements of R such that* $\mathrm{ann}_R(r) \subseteq \mathrm{ann}_R(r')$, *then* $\mathrm{ann}_{S^{-1}R}(r/s) \subseteq \mathrm{ann}_{S^{-1}R}(r'/s')$ *for every $s, s' \in S$.*

P r o o f. Assume that $\mathrm{ann}_R(r) \subseteq \mathrm{ann}_R(r')$, and suppose on the contrary that $\mathrm{ann}_{S^{-1}R}(r/s) \nsubseteq \mathrm{ann}_{S^{-1}R}(r'/s')$. Then there is $r''/s'' \in S^{-1}R$ such that $(r/s) \times (r''/s'') = 0/1$ and $(r'/s')(r''/s'') \neq 0/1$. So there exists $u \in S$ such that $urr'' = 0$, and, for every $v \in S$, $vr'r'' \neq 0$. So $ur'' \in \mathrm{ann}_R(r)$. Thus $ur'' \in \mathrm{ann}_R(r')$. So we have $ur'r'' = 0$, which is the required contradiction. $\square$

Lemma 4.7. *Let R be a commutative ring. If a_1/s_1 is adjacent to a_2/s_2 in* $\mathrm{AG}(S^{-1}R)$, *then either a_1 is adjacent to a_2 or a_1s_2 is adjacent to a_2s_1 in* $\mathrm{AG}(R)$ *for every $s_1, s_2 \in S$.*

P r o o f. First assume that $a_1 \neq a_2$. Since $a_1/s_1 - a_2/s_2$ is an edge in $\mathrm{AG}(S^{-1}R)$, there is b/s in $\mathrm{AG}(S^{-1}R)$ such that $(b/s)(a_1a_2/s_1s_2) = 0$, $(b/s)(a_1/s_1) \neq 0$ and $(b/s)(a_2/s_2) \neq 0$. Hence there exists $v \in S$ such that $vba_1a_2 = 0$, $vba_1 \neq 0$ and

$vba_2 \neq 0$, and so a_1 is adjacent to a_2 in $\mathrm{AG}(R)$. Now assume that $a_1 = a_2$. Since $a_1/s_1 \neq a_2/s_2$, we have $a_1s_2 \neq a_2s_1$. Also a_1s_2/s_1s_2 is adjacent to a_2s_1/s_1s_2 in $\mathrm{AG}(S^{-1}R)$, and so a_1s_2 is adjacent to a_2s_1 in $\mathrm{AG}(R)$. □

By Lemma 4.7, one can see that if $\mathrm{AG}(S^{-1}R)$ is a complete graph, then $\mathrm{AG}(R)$ is complete.

Lemma 4.8. *Let R be a commutative ring. If $\mathrm{ann}_R(x_1) = \mathrm{ann}_R(x_2)$, then x_1 and x_2 have the same neighbours in $\mathrm{AG}(R)$.*

Proof. Suppose that x is adjacent to x_1 in $\mathrm{AG}(R)$. So we have $\mathrm{ann}_R(xx_1) \neq \mathrm{ann}_R(x) \cup \mathrm{ann}_R(x_1)$. Hence there is x' such that $x'xx_1 = 0$, $x'x_1 \neq 0$ and $x'x \neq 0$. Now, since $\mathrm{ann}_R(x_1) = \mathrm{ann}_R(x_2)$, we have $x'xx_2 = 0$ and $x'x_2 \neq 0$. Therefore $\mathrm{ann}_R(xx_2) \neq \mathrm{ann}_R(x) \cup \mathrm{ann}_R(x_2)$, and so x is adjacent to x_2 in $\mathrm{AG}(R)$. Also, if x is adjacent to x_2 in $\mathrm{AG}(R)$, then similarly x is adjacent to x_1 in $\mathrm{AG}(R)$. So x_1 and x_2 have the same neighbours in $\mathrm{AG}(R)$. □

Lemma 4.9. *Let R be a commutative ring. Suppose that s is an arbitrary element in S. If $r \in Z(R)$, then of r/s and $r/1$ have the same neighbours in $\mathrm{AG}(S^{-1}R)$.*

Proof. By Lemma 4.8, it is enough to show that $\mathrm{ann}_{S^{-1}R}(r/s) = \mathrm{ann}_{S^{-1}R}(r/1)$. So if $a/t \in \mathrm{ann}_{S^{-1}R}(r/s)$, then we have $(a/t)(r/s) = 0/1$. Hence there exists $u \in S$ such that $uar = 0$. Also $(a/t)(r/1) = ar/t = aru/(tu) = 0/1$, and so $a/t \in \mathrm{ann}_{S^{-1}R}(r/1)$. Now, if $a/t \in \mathrm{ann}_{S^{-1}R}(r/1)$, then there exists $u \in S$ such that $uar = 0$. Also $(a/t)(r/s) = ar/(ts) = aru/(tsu) = 0/1$. Therefore $a/t \in \mathrm{ann}_{S^{-1}R}(r/s)$. □

Let $T(R) = S^{-1}R$ be the total quotient ring of R, where $S = R - Z(R)$. In [7], Theorem 2.2, Anderson and Shapiro showed that the graphs $\Gamma(R)$ and $\Gamma(T(R))$ are isomorphic. For $x, y \in R$, they defined a relation $\sim$ as follows: $x \sim y$ if $\mathrm{ann}_R(x) = \mathrm{ann}_R(y)$. Clearly $\sim$ is an equivalence relation on R. Let $T = T(R)$. Denote the equivalence relations on $Z(R)^*$ and $Z(T)^*$ by $\sim_R$ and $\sim_T$, respectively, and denote their equivalence classes by $[a]_R$ and $[a]_T$, respectively. They proved that there is a bijection between equivalence classes of $\Gamma(T(R))$ and $\Gamma(R)$, and they defined a bijection $\varphi\colon Z(R)^* \to Z(T)^*$ by $\varphi(x) = \varphi_\alpha(x)$, where $\varphi_\alpha\colon [a_\alpha] \to [a_\alpha/1]$ is a bijection and $x \in [a_\alpha]$. In the next theorem, using the above notation, we show that $\mathrm{AG}(R)$ is isomorphic to $\mathrm{AG}(T(R))$.

Theorem 4.10. *Let R be a commutative ring. Then the graphs $\mathrm{AG}(R)$ and $\mathrm{AG}(T(R))$ are isomorphic.*

P r o o f. By the proof of [7], Theorem 2.2, we have the bijection $\varphi\colon Z(R)^* \to Z(T)^*$ defined by $\varphi(x) = \varphi_\alpha(x)$, where $\varphi_\alpha\colon [a_\alpha] \to [a_\alpha/1]$ is a bijection and $x \in [a_\alpha]$. Thus we only need to show that x and y are adjacent in $\mathrm{AG}(R)$ if and only if $\varphi(x)$ and $\varphi(y)$ are adjacent in $\mathrm{AG}(T(R))$; i.e., $\mathrm{ann}_R(xy) \neq \mathrm{ann}_R(x) \cup \mathrm{ann}_R(y)$ if and only if $\mathrm{ann}_T(\varphi(x)\varphi(y)) \neq \mathrm{ann}_T(\varphi(x)) \cup \mathrm{ann}_T(\varphi(y))$. Let $x \in [a]_R, y \in [b]_R$, $l \in [c]_R$, $w \in [a/1]_T$, $z \in [b/1]_T$ and $t \in [c/1]_T$. We need only to show that $xyl = 0$, $xl \neq 0$ and $yl \neq 0$ if and only if $wzt = 0$, $wt \neq 0$ and $zt \neq 0$. Note that $\mathrm{ann}_T(x) = \mathrm{ann}_T(a) = \mathrm{ann}_T(w)$, $\mathrm{ann}_T(y) = \mathrm{ann}_T(b) = \mathrm{ann}_T(z)$ and $\mathrm{ann}_T(l) = \mathrm{ann}_T(c) = \mathrm{ann}_T(t)$. Hence

$$\begin{aligned} xyl = 0 &\Leftrightarrow xy \in \mathrm{ann}_T(l) = \mathrm{ann}_T(t) \Leftrightarrow xyt = 0 \Leftrightarrow xt \in \mathrm{ann}_T(y) = \mathrm{ann}_T(z) \\ &\Leftrightarrow xtz = 0 \Leftrightarrow tz \in \mathrm{ann}_T(x) = \mathrm{ann}_T(w) \Leftrightarrow wzt = 0. \end{aligned}$$

Since φ is an isomorphism between the graphs $\Gamma(R)$ and $\Gamma(T(R))$, we have $xl \neq 0$ and $yl \neq 0$ if and only if $wt \neq 0$ and $zt \neq 0$. □

Theorem 4.11. *Let R be a finite commutative ring. If $\mathfrak{p}$ is a prime ideal of R, then the annihilator graph $\mathrm{AG}(S^{-1}R)$ is complete, where $S = R - \mathfrak{p}$.*

P r o o f. Since $R_\mathfrak{p}$ is a finite local ring, $Z(R_\mathfrak{p}) = \mathrm{Nil}(R_\mathfrak{p})$. So, by [12], Theorem 3.10, $\mathrm{AG}(R_\mathfrak{p})$ is complete. □

5. Annihilator graphs of $\mathbb{Z}_n$

In this section, we examine the existence of cut-vertices and cut-sets in $\mathrm{AG}(\mathbb{Z}_n)$. We determine the reduced rings whose annihilator graphs are bipartite graphs. Also we show that $\mathrm{AG}(\mathbb{Z}_n)$ is bipartite for a certain n.

Theorem 5.1. *Let $n \geqslant 6$. Then $r \in \mathbb{Z}_n$ is a cut-vertex of $\mathrm{AG}(\mathbb{Z}_n)$ if and only if $2r = n$ and r is a prime integer.*

P r o o f. First assume that r is a cut-vertex of $\mathrm{AG}(\mathbb{Z}_n)$. Since the zero-divisor graph is the (spanning) subgraph of the annihilator graph $\mathrm{AG}(R)$, every cut-vertex in $\mathrm{AG}(R)$ is a cut-vertex in $\Gamma(R)$. Therefore, by [17], Lemma 2.2, $n = 2r$. Now, we prove that r is a prime integer. Since r is a cut-vertex of the annihilator graph $\mathrm{AG}(\mathbb{Z}_n)$, there exist vertices α and β such that $\alpha - r - \beta$ is the only path that connects α to β. Now, since α is not adjacent to β in $\mathrm{AG}(\mathbb{Z}_n)$, hence by [12], Lemma 2.1, $\mathrm{ann}_R(\alpha) \subseteq \mathrm{ann}_R(\beta)$ or $\mathrm{ann}_R(\beta) \subseteq \mathrm{ann}_R(\alpha)$. Without loss of generality, we may assume $\mathrm{ann}_R(\alpha) \subseteq \mathrm{ann}_R(\beta)$. Let t be a nonzero element in $\mathbb{Z}_n \setminus \{r\}$. If $t\alpha = 0$, then $t\beta = 0$. So we have the path $\alpha - t - \beta$, which is impossible. Now, since $\alpha \in Z(\mathbb{Z}_n)$, we have $\alpha r = 0$. Therefore $\mathrm{ann}_R(\alpha) = \{0, r\}$. If r is not prime, then there are positive integers $q, q' \neq 1$ such that $qq' = r$. Hence we have $\operatorname{ann}_R(\alpha q) \neq \operatorname{ann}_R(\alpha) \cup \operatorname{ann}_R(q)$ and $\operatorname{ann}_R(\alpha q') \neq \operatorname{ann}_R(\alpha) \cup \operatorname{ann}_R(q')$, which implies that α is adjacent to q and q' in $\mathrm{AG}(\mathbb{Z}_n)$. If $q\beta = 0$ or $q'\beta = 0$, then we have the path $\alpha - q - \beta$ or $\alpha - q' - \beta$, which is impossible. So $q\beta \neq 0$ and $q'\beta \neq 0$. In this situation one can easily check that $\operatorname{ann}_R(\beta q) \neq \operatorname{ann}_R(\beta) \cup \operatorname{ann}_R(q)$ and $\operatorname{ann}_R(\beta q') \neq \operatorname{ann}_R(\beta) \cup \operatorname{ann}_R(q')$, and therefore we have the paths $\alpha - q - \beta$ and $\alpha - q' - \beta$, which is again impossible. Thus r is prime.

Conversely, by [12], Theorem 3.8, $\mathrm{AG}(\mathbb{Z}_n) \cong K^{1,m}$ for some $m \geqslant 1$. Thus it contains a cut-vertex. □

Theorem 5.2. *If $|\mathrm{Min}(\mathbb{Z}_n)| = 2$ and every minimal prime ideal has at least 3 elements, then the nonzero elements in a minimal prime ideal of $\mathbb{Z}_n$ with minimal cardinality form a cut-set of $\mathrm{AG}(\mathbb{Z}_n)$.*

Proof. Since $|\mathrm{Min}(\mathbb{Z}_n)| = 2$ and every minimal prime ideal has at least 3 elements, we have $n = pq$, where p and q are distinct prime integers with $p, q \neq 2$. So let $A = \{x \in \mathbb{Z}_n\colon\ p \mid x,\ q \nmid x\}$ and $B = \{x \in \mathbb{Z}_n\colon\ q \mid x,\ p \nmid x\}$. One can easily see that $A \cup \{0\}$ and $B \cup \{0\}$ are minimal prime ideals of $\mathbb{Z}_n$ and $\mathrm{AG}(\mathbb{Z}_n) \cong K^{m,n}$, where $|A| = m$ and $|B| = n$ and A, B are two parts in $\mathrm{AG}(\mathbb{Z}_n)$. □

In the next theorem, we determine some conditions under which $\mathrm{AG}(\mathbb{Z}_n)$ is weakly perfect.

Theorem 5.3. *Suppose that $\mathbb{Z}_n$ is a finite reduced ring. If $|\mathrm{Min}(\mathbb{Z}_n)| \leqslant 3$, then $\mathrm{AG}(\mathbb{Z}_n)$ is weakly perfect.*

Proof. Since $|\mathrm{Min}(\mathbb{Z}_n)| \leqslant 3$ and $\mathbb{Z}_n$ is a reduced ring, we have $n = p$, pq or pqr where p, q and r are distinct prime integers. If $n = p$, then $Z(\mathbb{Z}_n) = \{0\}$. If $n = pq$, then $\mathbb{Z}_n = \mathbb{Z}_{pq} \cong \mathbb{Z}_p \times \mathbb{Z}_q \cong K_1 \times K_2$, where K_1 and K_2 are finite fields with p and q elements, respectively. So $\mathrm{AG}(K_1 \times K_2) \cong K^{p-1,q-1}$. Hence $\mathrm{cl}(\mathrm{AG}(\mathbb{Z}_{pq})) = \chi(\mathrm{AG}(\mathbb{Z}_{pq})) = 2$. If $n = pqr$, then $\mathbb{Z}_n = \mathbb{Z}_{pqr} \cong \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r \cong K_1 \times K_2 \times K_3$, where K_1, K_2 and K_3 are finite fields with p, q and r elements, respectively. Therefore, by Lemma 3.1, $\mathrm{cl}(\mathrm{AG}(\mathbb{Z}_{pqr})) = \chi(\mathrm{AG}(\mathbb{Z}_{pqr})) = 3$. □

Theorem 5.4. *Let R be a commutative reduced ring. Then $\mathrm{AG}(R)$ is bipartite if and only if there exist two distinct prime ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$ of R such that $\mathfrak{p}_1 \cap \mathfrak{p}_2 = \{0\}$. In addition, if $\mathrm{AG}(R)$ is bipartite, then it is a complete bipartite graph.*

Proof. Suppose that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are distinct prime ideals of R such that $\mathfrak{p}_1 \cap \mathfrak{p}_2 = \{0\}$. Then, in view of the proof of [3], Theorem 2.4, we have $Z(R) = \mathfrak{p}_1 \cup \mathfrak{p}_2$. Set $V_1 = \mathfrak{p}_1 \setminus \{0\}$ and $V_2 = \mathfrak{p}_2 \setminus \{0\}$. Now, we show that $\mathrm{AG}(R)$ is bipartite with two parts V_1 and V_2. If $a, b \in V_1$ and a is adjacent to b, then $\operatorname{ann}_R(ab) \neq \operatorname{ann}_R(a) \cup \operatorname{ann}_R(b)$. Hence there exists a nonzero element $c \in R$ such that $abc = 0$, $ac \neq 0$ and $bc \neq 0$. Since $\mathfrak{p}_1 \cap \mathfrak{p}_2 = \{0\}$ and $a, b \in V_1$, we have $c \in \mathfrak{p}_2$. So $ac \in \mathfrak{p}_1 \cap \mathfrak{p}_2$, which is a contradiction. Also, for every $a \in V_1$ and $b \in V_2$, we have $ab = 0$ since $\mathfrak{p}_1 \cap \mathfrak{p}_2 = \{0\}$. Therefore $\mathrm{AG}(R)$ is a complete bipartite graph.

Conversely, suppose that $\mathrm{AG}(R)$ is bipartite. Since, by [12], Lemma 2.1, $\Gamma(R)$ is a (spanning) subgraph of $\mathrm{AG}(R)$, we have that $\Gamma(R)$ is bipartite. Hence, by [3], Theorem 2.4, there exist two distinct prime ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$ of R such that $\mathfrak{p}_1 \cap \mathfrak{p}_2 = \{0\}$. □

Theorem 5.5. *The graph $\mathrm{AG}(\mathbb{Z}_n)$ is bipartite if and only if one of the following holds.*

(i) *$n = 4$ or 9.*

(ii) *$n = p_1p_2$, where p_1 and p_2 are distinct prime integers.*

(iii) *$n = 4p$, where p is a prime integer and $p \neq 2$.*

Proof. First assume that the graph $\mathrm{AG}(\mathbb{Z}_n)$ is bipartite. If there exist at least three distinct prime integers in divisors of n, say p_1, p_2 and p_3, then p_1 is adjacent to p_2 and p_3 and p_2 is adjacent to p_3. So, we have the cycle $p_1 - p_2 - p_3 - p_1$ of length three. Therefore $\mathrm{AG}(\mathbb{Z}_n)$ is not bipartite. Hence there exist at most two distinct prime integers in divisors of n. Now let $n = p_1^r p_2^s$ for some distinct prime integers p_1, p_2 and nonzero positive integers r and s. Set $A_{p_1p_2} = \{r \in \mathbb{N}\colon\ r \mid n,\ p_1 \mid r,\ p_2 \mid r\}$. Since $A_{p_1p_2}$ is a complete subgraph of $\mathrm{AG}(\mathbb{Z}_n)$ and $|A_{p_1p_2}| = p_1^{r-1}p_2^{s-1}$, we have $r, s \leqslant 2$. If $r = s = 2$, then $n = p_1^2p_2^2$, and so p_1 is adjacent to p_2^2 and p_1p_2, and p_2^2 is adjacent to p_1p_2. Thus $\mathrm{AG}(\mathbb{Z}_n)$ is not bipartite. If $r = 2$ and $s = 1$, then $n = p_1^2p_2$. Since $A_{p_1p_2}$ is a complete subgraph of $\mathrm{AG}(\mathbb{Z}_n)$ and $|A_{p_1p_2}| = p_1$, we have $p_1 = 2$. So $n = 4p_2$, where $p_2 \neq 2$. Also if $r = 1$ and $s = 2$, then similarly $n = 4p_1$, where $p_1 \neq 2$. If $r = 1$ and $s = 1$, then $n = p_1p_2$. If $n = p_1^r$, then $|Z(\mathbb{Z}_n)| = p_1^{r-1}$. Also, $\mathrm{AG}(\mathbb{Z}_n)$ is a complete graph with $p_1^{r-1} - 1$ vertices. So $\mathrm{AG}(\mathbb{Z}_n)$ is bipartite if and only if $p_1^{r-1} - 1 \leqslant 2$. Hence $p_1^{r-1} = 3$ or $p_1^{r-1} = 2$. If $p_1^{r-1} = 3$, then $p_1 = 3$ and $r = 2$. Thus $n = 9$. If $p_1^{r-1} = 2$, then $p_1 = 2$ and $r = 2$. Thus $n = 4$.

Conversely, one can easily check that $\mathrm{AG}(\mathbb{Z}_4) \cong K^1$, $\mathrm{AG}(\mathbb{Z}_9) \cong K^2$, $\mathrm{AG}(\mathbb{Z}_{p_1p_2}) \cong K^{p_1-1,p_2-1}$, where p_1 and p_2 are distinct prime integers and $\mathrm{AG}(\mathbb{Z}_{4p}) \cong K^{3,2p-2}$, where p is prime a integer and $p \neq 2$. □

Lemma 5.6. *The graph $\mathrm{AG}(\mathbb{Z}_{p^n})$ is weakly perfect, where p is a prime integer. In addition, $\mathrm{AG}(\mathbb{Z}_{p^n})$ has chromatic number $p^{n-1} - 1$.*

P r o o f. Clearly $Z(\mathbb{Z}_{p^n})$ is an ideal of $\mathbb{Z}_{p^n}$. Then $Z(\mathbb{Z}_{p^n}) = \mathrm{Nil}(\mathbb{Z}_{p^n})$. So by [12], Theorem 3.10, $\mathrm{AG}(\mathbb{Z}_{p^n})$ is a complete graph with $p^{n-1} - 1$ vertices. Hence $\mathrm{AG}(\mathbb{Z}_{p^n})$ is weakly perfect and has chromatic number $p^{n-1} - 1$. □

Lemma 5.7. *The graphs* $\mathrm{AG}(\mathbb{Z}_{p_1p_2})$ *and* $\mathrm{AG}(\mathbb{Z}_{p_1p_2p_3})$ *for some distinct prime integers* p_1, p_2, p_3 *are weakly perfect. In addition,*

$$\chi(\mathrm{AG}(\mathbb{Z}_{p_1p_2})) = 2 \quad \text{and} \quad \chi(\mathrm{AG}(\mathbb{Z}_{p_1p_2p_3})) = 3.$$

P r o o f. Let $R \cong \mathbb{Z}_{p_1p_2}$. Then one can easily check that $\mathrm{AG}(R)$ is bipartite. So $\mathrm{AG}(R)$ is weakly perfect and $\chi(\mathrm{AG}(R)) = 2$. Let $R \cong \mathbb{Z}_{p_1p_2p_3}$. Then $R \cong K_1 \times K_2 \times K_3$, where K_1, K_2 and K_3 are fields with $|K_1| = p_1$, $|K_2| = p_2$ and $|K_3| = p_3$. Hence, by Lemma 3.1, $\mathrm{AG}(R)$ is weakly perfect and $\chi(\mathrm{AG}(R)) = 3$. □

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Authors' addresses: Mojgan Afkhami, Department of Mathematics, University of Neyshabur, P. O. Box 91136–899, Neyshabur, Iran, e-mail: mojgan.afkhami@yahoo.com; Kazem Khashyarmanesh, Zohreh Rajabi, Department of Pure Mathematics, Ferdowsi University of Mashhad, P. O. Box 1159–91775, Mashhad, Iran, e-mail: Khashyar@ipm.ir, rajabi261@yahoo.com.