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MINIMAL REEB VECTOR FIELDS ON ALMOST KENMOTSU MANIFOLDS

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Abstract. A necessary and sufficient condition for the Reeb vector field of a three dimensional non-Kenmotsu almost Kenmotsu manifold to be minimal is obtained. Using this result, we obtain some classifications of some types of (k,μ,ν)-almost Kenmotsu manifolds. Also, we give some characterizations of the minimality of the Reeb vector fields of (k,μ,ν)-almost Kenmotsu manifolds. In addition, we prove that the Reeb vector field of an almost Kenmotsu manifold with conformal Reeb foliation is minimal.

Keywords: almost Kenmotsu manifold; Reeb vector field; minimal vector field; harmonic vector field; Lie group

MSC 2010: 53D15, 53C25, 53C43

1. Introduction

Let (M,g) be a Riemannian manifold of dimension m and (T^1M,g_S) its unit tangent sphere bundle furnished with the standard Sasakian metric g_S (not to be confused with the well-known Sasakian structure in contact geometry). Then every unit tangent vector field V on M defines an immersion from (M,g) into (T^1M,g_S). When M is compact and orientable, the energy and the volume of V are just the energy of the corresponding immersion and the volume of the submanifold (M,V^*g_S) of the unit tangent sphere bundle, respectively. From this, one may obtain two functions on the space $\mathfrak{X}^1(M)$ of all unit vector fields on M. Generally, V is said to be harmonic and minimal if it is a critical point of the energy function and the

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volume function defined on $\mathfrak{X}^1(M)$, respectively. We remark that two similar notions were introduced and studied by Gil-Medrano [7] even if M is non-compact and non-orientable. We also observe that V is minimal if and only if the corresponding submanifold (M, V^*g_S) of (T^1M, g_S) is minimal. However, harmonic vector field does not necessarily imply harmonic maps. Many authors have studied the minimality and harmonicity of the unit vector fields on several kinds of Riemannian manifolds, see for example [2], [8] and [9].

In the framework of almost contact geometry, the minimality and harmonicity of the Reeb vector field were also studied by many authors from different points of view. In [10], González-Dávila and Vanhecke proved that the Reeb vector field of a Kenmotsu manifold is minimal. Also, the same authors in [11] characterized the minimality of the Reeb vector field of a three dimensional contact metric manifold. A complete classification of contact metric manifolds of dimension three such that the Reeb vector field is minimal and harmonic was obtained by Perrone in [20]. Also, Koufogiorgos et al. in [15] proved that the harmonicity of the Reeb vector field ξ of a contact metric three-manifold implies that ξ belongs to the (k, μ, ν)-nullity distribution on an open and dense subset. With regard to the three dimensional cosymplectic manifolds, Perrone in [22], [21] proved that the Reeb vector field is minimal if and only if it is harmonic, and this is also equivalent to that it is an eigenvector field of the Ricci operator. The studies of the harmonicity of the Reeb vector field of almost Kenmotsu manifolds are rare. Perrone in [21] obtained a necessary and sufficient condition for the Reeb vector field of an α-Kenmotsu manifold to be harmonic, that is, it is an eigenvector field of the Ricci operator. In addition, the author proved that the Reeb vector field of an α-Kenmotsu manifold ($\alpha \neq 0$) never defines a harmonic map. As a special case of trans-Sasakian manifolds, the harmonicity of the Reeb vector field of β-Kenmotsu manifolds was also presented by Vergara-Diaz and Wood in [24].

In the present paper, we start to study the minimality of the Reeb vector field of an almost Kenmotsu manifold. After some necessary preliminaries regarding almost contact metric manifolds and minimal and harmonic unit vector fields are given in Section 2, in Section 3 we present a necessary condition for the Reeb vector field of a three dimensional almost Kenmotsu manifold to be minimal. As an application of the above result, some classification results for almost Kenmotsu manifolds of dimension three under additional conditions are also obtained. Finally, following González-Dávila and Vanhecke [10], we characterize the minimality of the Reeb vector field of almost Kenmotsu manifolds of dimension greater than three with conformal Reeb foliations.

2.1. Minimal and harmonic vector fields. Let (M, g) be a Riemannian manifold of dimension m and (T^1M, g_S) its unit tangent sphere bundle furnished with the standard Sasakian metric g_S. Then the induced metric from g_S on M via a unit vector field V can be written as

$$(V^*g_S)(X,Y) = g(X,Y) + g(\nabla_X V, \nabla_Y V) \tag{2.1}$$

for any vector fields X and Y on M, where ∇ denotes the Levi-Civita connection of the metric g. Let $\mathfrak{X}^1(M)$ denote the set of all unit vector fields on M and $V \in \mathfrak{X}^1(M)$, then we may define a $(1,1)$-type tensor field L_V on M by

$$L_V = \operatorname{id} + (\nabla V)^t \circ \nabla V, \tag{2.2}$$

where id is the identity map, and hence we may write $V^*g_S = g(L_V\cdot, \cdot)$. When M is compact and orientable, the volume of V is the volume of the corresponding submanifold (M, V^*g_S) of (T^1M, g_S) and can be written as

$$\operatorname{Vol}(V) = \int_M f(V)\, \mathrm{d}v_g,$$

where $f(V) = \sqrt{\det(L_V)}$. We define another $(1,1)$-type tensor field K_V by

$$K_V = f(V)(L_V)^{-1} \circ (\nabla V)^t. \tag{2.3}$$

According to Gil-Medrano and Llinares-Fuster [8], V is a critical point for the volume function if and only if the 1-form

$$\omega_V(X) = \operatorname{trace}\{Y \to (\nabla_Y K_V)X\} \tag{2.4}$$

vanishes on the distribution $\mathcal{D}^V$ determined by all vector fields orthogonal to V. Following Gil-Medrano [7], such a critical point is said to be a minimal vector field even if M is non-compact and non-orientable. It is well-known that the minimality of V is equivalent to that of the corresponding submanifold.

Moreover, the energy of V is the energy of the map from (M, g) into (T^1M, g_S) and can be written as

$$E(V) = \frac{m}{2} \operatorname{Vol}(M, g) + \frac{1}{2} \int_M \|\nabla V\|^2\, \mathrm{d}v_g.$$

Following Gil-Medrano [7], a unit vector field V is a critical point for the energy function if and only if the 1-form

$$\varrho_V(X) = \operatorname{trace}\{Y \to (\nabla_Y(\nabla V)^t)X\} \tag{2.5}$$

vanishes on the distribution $\mathcal{D}^V$. A unit vector field V satisfying this condition is said to be harmonic.

Furthermore, the map $V\colon (M,g) \to (T^1M, g_S)$ defines a harmonic map if and only if V is a harmonic vector field and, in addition, the 1-form

$$\bar{\varrho}_V(X) = \operatorname{trace}\{Y \to R(\nabla_Y V, V)X\} \tag{2.6}$$

vanishes for any vector field X on M, where R denotes the Riemannian curvature tensor defined by $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$.

2.2. Almost Kenmotsu manifolds. According to Blair [1], an almost contact metric structure on a smooth differentiable manifold M^{2n+1} of dimension $2n+1$ is a (φ,ξ,η,g)-structure satisfying

$$\begin{gathered}\varphi^2 = -\operatorname{id} + \eta\otimes\xi, \quad \eta(\xi) = 1,\\ g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y)\end{gathered} \tag{2.7}$$

for any vector fields X and Y, where φ is a $(1,1)$-type tensor field, and ξ is a tangent vector field called the characteristic or the Reeb vector field and η is a 1-form called the contact form. A Riemannian manifold M^{2n+1} furnished with an almost contact metric structure is called an almost contact metric manifold and is denoted by $(M^{2n+1},\varphi,\xi,\eta,g)$.

According to Janssens and Vanhecke [13], an almost Kenmotsu manifold is an almost contact metric manifold $(M^{2n+1},\varphi,\xi,\eta,g)$ such that η is closed and $\mathrm{d}\Phi = 2\eta\wedge\Phi$, where the fundamental 2-form Φ of the almost contact metric manifold M^{2n+1} is defined by $\Phi(X,Y) = g(X,\varphi Y)$ for any vector fields X and Y on M^{2n+1}. An almost contact metric manifold such that $\mathrm{d}\eta = \Phi$ or $\mathrm{d}\eta = 0$, $\mathrm{d}\Phi = 0$ is said to be a contact metric manifold or an almost cosymplectic manifold, respectively (see Blair [1]). Given an almost contact metric manifold $(M^{2n+1},\varphi,\xi,\eta,g)$, we define on the product $M^{2n+1}\times\mathbb{R}$ an almost complex structure J by

$$J\Big(X, f\frac{\mathrm{d}}{\mathrm{d}t}\Big) = \Big(\varphi X - f\xi, \eta(X)\frac{\mathrm{d}}{\mathrm{d}t}\Big),$$

where X denotes a vector field tangent to M^{2n+1}, t is the coordinate of $\mathbb{R}$ and f is a $\mathcal{C}^\infty$-function on $M^{2n+1}\times\mathbb{R}$. We denote by $[\varphi,\varphi]$ the Nijenhuis tensor of φ (see

Blair [1]). If $[\varphi, \varphi] = -2\mathrm{d}\eta \otimes \xi$ (or equivalently, the almost complex structure J is integrable), then the almost contact metric structure is said to be normal.

A normal almost Kenmotsu manifold is called a Kenmotsu manifold (see [13], [14]). It is well-known that on an almost Kenmotsu manifold the normality condition holds if and only if the relation

$$(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X \tag{2.8}$$

holds for any vector fields X, Y. Moreover, a normal contact metric or almost cosymplectic manifold is said to be a Sasakian or cosymplectic manifold, respectively.

The following three symmetric operators $l = R(\cdot, \xi)\xi$, $h = \mathcal{L}_\xi \varphi/2$ and $h' = h \circ \varphi$ play key roles in the studies of the geometry of the almost Kenmotsu manifolds, where $\mathcal{L}$ is the Lie differentiation. From Dileo and Pastore [4], [5], we now collect some properties of almost Kenmotsu manifolds as follows:

$$h\xi = l\xi = 0, \quad \operatorname{tr} h = \operatorname{tr}(h') = 0, \quad h\varphi + \varphi h = 0, \tag{2.9}$$

$$\nabla \xi = h' + \operatorname{id} - \eta \otimes \xi, \tag{2.10}$$

$$\varphi l \varphi - l = 2(h^2 - \varphi^2), \tag{2.11}$$

$$\nabla_\xi h = -\varphi - 2h - \varphi h^2 - \varphi l, \tag{2.12}$$

$$\operatorname{tr}(l) = S(\xi, \xi) = g(Q\xi, \xi) = -2n - \operatorname{tr} h^2, \tag{2.13}$$

where S denotes the Ricci curvature tensor and Q the associated Ricci operator. Throughout this paper, we denote by $\mathcal{D}$ the distribution $\mathcal{D} = \ker \eta$, which is of dimension $2n$. Then it is easy to check that each integral manifold of $\mathcal{D}$, with the restriction of φ, admits an almost Kähler structure. If the associated almost Kähler structure is integrable, or equivalently (see [5]),

$$(\nabla_X \varphi)Y = g(\varphi X + hX, Y)\xi - \eta(Y)(\varphi X + hX) \tag{2.14}$$

for any vector fields X, Y, then we say that M^{2n+1} is CR-integrable. Obviously, equations (2.8) and (2.14) yield the following.

Proposition 2.1. *An almost Kenmotsu manifold is Kenmotsu if and only if it is CR-integrable and h is vanishing. In particular, a three dimensional almost Kenmotsu manifold is Kenmotsu if and only if h is vanishing.*

3. Minimal Reeb vector fields on almost Kenmotsu manifolds

In this section, let $(M^3, \varphi, \xi, \eta, g)$ be a three dimensional almost Kenmotsu manifold. Let $\mathcal{U}_1$ be the open subset of M^3 such that $h \neq 0$ and $\mathcal{U}_2$ the open subset of M^3 which is defined as $\mathcal{U}_2 = \{p \in M^3\colon\ h = 0 \text{ in a neighborhood of } p\}$. Therefore, $\mathcal{U}_1 \cup \mathcal{U}_2$ is an open and dense subset of M^3 and there exists a local orthonormal basis $\{\xi, e, \varphi e\}$ of three smooth unit eigenvectors of h for any point $p \in \mathcal{U}_1 \cup \mathcal{U}_2$. On $\mathcal{U}_1$, we may set $he = \lambda e$ and hence $h\varphi e = -\lambda\varphi e$, where λ is a positive function on $\mathcal{U}_1$. Note that the eigenvalue function λ is continuous on M^3 and smooth on $\mathcal{U}_1 \cup \mathcal{U}_2$.

Lemma 3.1 ([3], Lemma 6). *On $\mathcal{U}_1$ we have*

$$\begin{aligned} &\nabla_\xi \xi = 0, \quad \nabla_\xi e = a\varphi e, \quad \nabla_\xi \varphi e = -ae, \\ &\nabla_e \xi = e - \lambda\varphi e, \quad \nabla_e e = -\xi - b\varphi e, \quad \nabla_e \varphi e = \lambda\xi + be, \\ &\nabla_{\varphi e} \xi = -\lambda e + \varphi e, \quad \nabla_{\varphi e} e = \lambda\xi + c\varphi e, \quad \nabla_{\varphi e}\varphi e = -\xi - ce, \end{aligned} \tag{3.1}$$

where a, b, c are smooth functions.

Applying Lemma 3.1 in the Jacobi identity

$$[[\xi, e], \varphi e] + [[e, \varphi e], \xi] + [[\varphi e, \xi], e] = 0$$

yields that

$$\begin{aligned} e(\lambda) - \xi(b) - e(a) + c(\lambda - a) - b = 0, \\ \varphi e(\lambda) - \xi(c) + \varphi e(a) + b(\lambda + a) - c = 0. \end{aligned} \tag{3.2}$$

Moreover, applying Lemma 3.1, we have (see also [3]) the following.

Lemma 3.2. *On $\mathcal{U}_1$, the Ricci operator can be written as*

$$\begin{aligned} Q\xi &= -2(\lambda^2 + 1)\xi - (\varphi e(\lambda) + 2\lambda b)e - (e(\lambda) + 2\lambda c)\varphi e, \\ Qe &= -(\varphi e(\lambda) + 2\lambda b)\xi - (e(c) + \varphi e(b) + b^2 + c^2 + 2\lambda a + 2)e + (\xi(\lambda) + 2\lambda)\varphi e, \\ Q\varphi e &= -(e(\lambda) + 2\lambda c)\xi + (\xi(\lambda) + 2\lambda)e - (e(c) + \varphi e(b) + b^2 + c^2 - 2\lambda a + 2)\varphi e, \end{aligned}$$

with respect to the local basis $\{\xi, e, \varphi e\}$.

We remark that González-Dávila and Vanhecke [10], Proposition 3.12, proved that the Reeb vector field of a Kenmotsu manifold is minimal. In what follows, applying Proposition 2.1 and Lemmas 3.1 and 3.2, we present a characterization of the minimality of the Reeb vector field of a three dimensional non-Kenmotsu almost Kenmotsu manifold.

Theorem 3.1. *The Reeb vector field of a three dimensional non-Kenmotsu almost Kenmotsu manifold is minimal if and only if on $\mathcal{U}_1$ the relation*

$$\begin{cases} e(\lambda) = \frac{1}{4}\lambda(\lambda^2+2)\eta(Qe) - \frac{1}{2}\lambda^2\eta(Q\varphi e), \\ \varphi e(\lambda) = \frac{1}{4}\lambda(\lambda^2+2)\eta(Q\varphi e) - \frac{1}{2}\lambda^2\eta(Qe) \end{cases} \tag{3.3}$$

holds, where the eigenvalue function λ of h is positive and smooth on nonempty $\mathcal{U}_1$.

Proof. Let (M^3,φ,ξ,η,g) be a three dimensional strictly almost Kenmotsu manifold. From (2.10) we see easily that $\nabla\xi$ is a symmetric operator with respect to the metric g. Then it follows from equations (2.1) and (2.2) that

$$(\nabla\xi)^t = (\nabla\xi) = h' + \mathrm{id} - \eta\otimes\xi \quad \text{and} \quad L_\xi = h^2 + 2h' + 2\,\mathrm{id} - \eta\otimes\xi. \tag{3.4}$$

Also, L_ξ can be presented using the local basis $\{\xi, e, \varphi e\}$ as follows:

$$L_\xi(\xi) = \xi, \quad L_\xi(e) = (\lambda^2+2)e - 2\lambda\varphi e, \quad L_\xi(\varphi e) = (\lambda^2+2)\varphi e - 2\lambda e. \tag{3.5}$$

Therefore, a simple computation using (3.5) gives that

$$f(\xi) = \sqrt{\det(L_\xi)} = \sqrt{\lambda^4+4}. \tag{3.6}$$

Making use of relation (3.5), we obtain $(L_\xi)^{-1}$ which is expressed in terms of the local basis $\{\xi, e, \varphi e\}$ as follows:

$$\begin{aligned} &(L_\xi)^{-1}(\xi) = \xi, \quad (L_\xi)^{-1}(e) = \frac{\lambda^2+2}{\lambda^4+4}e + \frac{2\lambda}{\lambda^4+4}\varphi e, \\ &(L_\xi)^{-1}(\varphi e) = \frac{2\lambda}{\lambda^4+4}e + \frac{\lambda^2+2}{\lambda^4+4}\varphi e. \end{aligned} \tag{3.7}$$

Putting relations (3.4)–(3.7) into equation (2.3) yields that

$$\begin{aligned} &K_\xi(\xi) = 0, \quad K_\xi(e) = \frac{2-\lambda^2}{\sqrt{\lambda^4+4}}e - \frac{\lambda^3}{\sqrt{\lambda^4+4}}\varphi e, \\ &K_\xi(\varphi e) = \frac{2-\lambda^2}{\sqrt{\lambda^4+4}}\varphi e - \frac{\lambda^3}{\sqrt{\lambda^4+4}}e. \end{aligned} \tag{3.8}$$

In order to simplify the notation, in what follows we set

$$\alpha = \frac{2-\lambda^2}{\sqrt{\lambda^4+4}} \quad \text{and} \quad \beta = \frac{\lambda^3}{\sqrt{\lambda^4+4}}. \tag{3.9}$$

Thus, applying Lemma 3.1 and relations (3.8) and (3.9), we obtain the formulas

$$\begin{aligned}
(\nabla_\xi K_\xi)(e) &= (\xi(\alpha) + 2a\beta)e - \xi(\beta)\varphi e,\\
(\nabla_e K_\xi)(e) &= -(\alpha + \lambda\beta)\xi + (e(\alpha) - 2b\beta)e - e(\beta)\varphi e,\\
(\nabla_{\varphi e} K_\xi)(e) &= (\lambda\alpha + \beta)\xi + (\varphi e(\alpha) + 2c\beta)e - \varphi e(\beta)\varphi e,
\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}
(\nabla_\xi K_\xi)(\varphi e) &= -\xi(\beta)e + (\xi(\alpha) - 2a\beta)\varphi e,\\
(\nabla_e K_\xi)(\varphi e) &= (\lambda\alpha + \beta)\xi - e(\beta)e + (e(\alpha) + 2b\beta)\varphi e,\\
(\nabla_{\varphi e} K_\xi)(\varphi e) &= -(\alpha + \lambda\beta)\xi - \varphi e(\beta)e + (\varphi e(\alpha) - 2c\beta)\varphi e.
\end{aligned} \tag{3.11}$$

Using relation (2.4) and taking into account (3.10), we have

$$\omega_\xi(e) = e(\alpha) - 2b\beta - \varphi e(\beta). \tag{3.12}$$

Similarly, it follows from relations (2.4) and (3.11) that

$$\omega_\xi(\varphi e) = -e(\beta) + \varphi e(\alpha) - 2c\beta. \tag{3.13}$$

Note that $h \neq 0$ and hence $\mathcal{U}_1$ is a nonempty subset of M^3; from Lemma 3.2 we have that

$$\eta(Qe) = -(\varphi e(\lambda) + 2\lambda b) \quad \text{and} \quad \eta(Q\varphi e) = -(e(\lambda) + 2\lambda c). \tag{3.14}$$

Putting $\alpha = (2-\lambda^2)/\sqrt{\lambda^4+4}$ and $\beta = \lambda^3/\sqrt{\lambda^4+4}$ into equations (3.12) and (3.13), and making use of (3.14), we obtain (3.3). This completes the proof. □

As an application of Theorem 3.1, we obtain the following result.

Theorem 3.2. *The Reeb vector field of a three dimensional non-Kenmotsu (k,μ,ν)-almost Kenmotsu manifold is minimal if and only if* $\mathrm{d}k \wedge \eta = 0$.

P r o o f. By a three dimensional (k,μ,ν)-almost Kenmotsu manifold M^3 we mean an almost Kenmotsu manifold such that the Reeb vector field ξ belongs to the (k,μ,ν)-nullity distribution, that is

$$R(X,Y)\xi = k(\eta(Y)X-\eta(X)Y)+\mu(\eta(Y)hX-\eta(X)hY) + \nu(\eta(Y)h'X-\eta(X)h'Y) \tag{3.15}$$

for any vector fields X, Y, where k, μ, ν are smooth functions on M^3. Using (2.9), from (3.15) we have that ξ is an eigenvector field of the Ricci operator with eigenvalue $2k$. Moreover, putting $Y=\xi$ into (3.15) gives that

$$l = -k\varphi^2 + \mu h + \nu h', \tag{3.16}$$

and using this in equation (2.11) yields that

$$h^2 = (k+1)\varphi^2. \tag{3.17}$$

Applying Proposition 2.1, from (3.17) we have that M^3 is non-Kenmotsu if and only if $h \neq 0$, or equivalently, $k < -1$. Also, from (3.17) we see that the positive eigenvalue function λ of h is given by $\lambda = \sqrt{-k-1}$. Using this and $Q\xi = 2k\xi$ in relation (3.3), we conclude that ξ is minimal if and only if k is invariant along the distribution $\mathcal{D}$, or equivalently, $\mathrm{d}k \wedge \eta = 0$. □

Remark 3.1. On an almost Kenmotsu manifold, any (k,μ,ν)-condition reduces to a generalized (k,μ)-condition when $\nu = 0$ and a generalized $(k,\nu)'$-condition when $\mu = 0$ (see [19] and [23]). If both k and μ are constant, then a generalized $(k,\mu)'$-condition or generalized (k,μ)-condition is just the $(k,\mu)'$-condition or (k,μ)-condition, respectively. For more details see [5].

Remark 3.2. Pastore and Saltarelli in [19] constructed some generalized (k,μ) and $(k,\mu)'$-almost Kenmotsu manifolds in any odd dimensions. Moreover, some local classification results of three dimensional generalized (k,μ) and $(k,\mu)'$-almost Kenmotsu manifolds such that $\mathrm{d}k \wedge \eta = 0$ and $k < -1$ were also obtained by Saltarelli [23]. Thus, applying Theorem 3.2, we find many examples of three dimensional non-Kenmotsu almost Kenmotsu manifolds for which ξ is minimal. See, for example, [19], Section 6, and also [23], Remarks 4.1, 5.1.

Applying again Theorem 3.2, we give now some classification results for three dimensional (k,μ,ν)-almost Kenmotsu manifolds.

Proposition 3.1. *Let M^3 be a three dimensional non-Kenmotsu $(k,0,\nu)$-almost Kenmotsu manifold. If the Reeb vector field ξ is minimal and k is invariant along ξ, then M^3 is locally isometric to a non-unimodular Lie group equipped with a left invariant almost Kenmotsu structure.*

P r o o f. If on a three dimensional non-Kenmotsu $(k, 0, \nu)$-almost Kenmotsu manifold the Reeb vector field ξ is minimal and k is invariant along ξ, then it follows from Theorem 3.2 that k is a constant less than -1. Since on any (k, μ, ν)-almost Kenmotsu manifold we have equations (3.16) and (3.17), using this in equation (2.12) we have that

$$\nabla_\xi h = -(\nu + 2)h. \tag{3.18}$$

We consider a unit eigenvector field e of h with a positive constant eigenvalue λ. Then the action of (3.18) on e gives that $h\nabla_\xi e = \lambda \nabla_\xi e + \lambda(\nu + 2)e$, and the inner product of this with e gives that $\nu = -2$. Therefore, M^3 is a $(k, 0, -2)$-almost Kenmotsu manifold with k a constant. Finally, from [5], Theorem 5.1, we observe that a non-Kenmotsu $(k, -2)'$-almost Kenmotsu manifold of dimension three is locally isometric to a non-unimodular Lie group. This completes the proof. □

Corollary 3.1. *A three dimensional non-Kenmotsu $(-2, 0, \nu)$-almost Kenmotsu manifold with minimal Reeb vector field is locally isometric to the Riemannian product $\mathbb{H}^2(-4) \times \mathbb{R}$.*

P r o o f. From [5], Theorem 5.1, we know that a $(k, \mu)'$-almost Kenmotsu manifold is locally isometric to either the Riemannian warped product $\mathbb{H}^{n+1}(k-2\lambda) \times_f \mathbb{R}^n$ or the warped product $\mathbb{B}^{n+1}(k + 2\lambda) \times_{f'} \mathbb{R}^n$, where $n \geqslant 1$, and $f = ce^{(1-\lambda)t}$ and $f' = c'e^{(1+\lambda)t}$ for certain constants c and c'. Then the proof follows directly from proofs of Proposition 3.1 and Theorem 3.2. □

Proposition 3.2. *Let M^3 be a three dimensional $(k, \mu, 0)$-almost Kenmotsu manifold. Then the Reeb vector field ξ is minimal and k is invariant along ξ if and only if M^3 is a Kenmotsu manifold.*

P r o o f. Suppose that M^3 is a three dimensional non-Kenmotsu $(k, \mu, 0)$-almost Kenmotsu manifold. By Proposition 2.1 we have $h \neq 0$ and hence by (3.17) we have $k < -1$. Therefore, applying Theorem 3.2, if ξ is minimal and $\xi(k) = 0$, we obtain that k is a constant. In this context, from [19], Proposition 3.2, we have that $\xi(k) = -4(k + 1)$. Since k is a constant, it follows that $k = -1$, a contradiction. The converse follows directly from Theorem 3.1. Moreover, from (2.10) we have that $R(X, Y)\xi = \eta(X)(Y + h'Y) - \eta(Y)(X + h'X) + (\nabla_X h')Y - (\nabla_Y h')X$ and using $h = 0$ in this equation gives that $R(X, Y)\xi = -\eta(Y)X + \eta(X)Y$. This completes the proof. □

Applying Theorem 3.1, we now characterize the minimality of the Reeb vector field ξ of a class of almost Kenmotsu manifolds.

Proposition 3.3. *The Reeb vector field of a three dimensional non-Kenmotsu almost Kenmotsu manifold with η-parallel h' is minimal if and only if it is an eigenvector field of the Ricci operator.*

Proof. The tensor field h' is said to be η-parallel if $g((\nabla_X h')Y, Z) = 0$ for any vector fields $X, Y, Z \in \mathcal{D}$. From [6], Proposition 3, we know that the parallelism of h' on an almost Kenmotsu manifold implies that the eigenvalues of h' are constant along the distribution $\mathcal{D}$. In view of $h^2 = h'^2$ and the fact that the eigenvalue function λ of h on $\mathcal{U}_1$ is positive, it follows from Theorem 3.1 that ξ is minimal if and only if $(\lambda^2 + 2)\eta(Qe) = 2\lambda\eta(Q\varphi e)$ and $(\lambda^2 + 2)\eta(Q\varphi e) = 2\lambda\eta(Qe)$. It follows that $\eta(Qe) = \eta(Q\varphi e) = 0$ and this is equivalent to ξ being an eigenvector field of the Ricci operator. □

Since a local conformal almost cosymplectic structure $(\varphi, \xi, \eta, \omega, g)$ reduces to an almost Kenmotsu structure if $\omega = \eta$ (see Olszak [17], Theorem 3.1), then the characterization of the harmonicity of the Reeb vector field of almost Kenmotsu manifolds is given as follows.

Lemma 3.3 ([21], Theorem 4.1). *The Reeb vector field of an almost Kenmotsu manifold is harmonic if and only if it is an eigenvector field of the Ricci operator.*

Remark 3.3. Following Proposition 2.1 and the proof of Proposition 3.2, on a three dimensional Kenmotsu manifold M^3 we have $R(X,Y)\xi = -\eta(Y)X + \eta(X)Y$ for any vector fields X, Y. Thus, by Lemma 3.3, we state that the Reeb vector field of M^3 is harmonic.

Following González-Dávila and Vanhecke [10], [11], we have the next result.

Lemma 3.4 ([10], [12]). *Any three dimensional non-unimodular Lie group admits a left invariant minimal and harmonic unit vector field.*

Applying the above Lemmas 3.3 and 3.4, we present a classification of three dimensional almost Kenmotsu manifolds.

Theorem 3.3. *Let M^3 be a three dimensional non-Kenmotsu almost Kenmotsu manifold. Then the Reeb vector field ξ of M^3 is minimal and harmonic, and both the scalar curvature r and $\|Q\|$ are invariant along ξ if and only if M^3 is locally isometric to a non-unimodular Lie group equipped with a left invariant almost Kenmotsu structure.*

Proof. Applying Lemma 3.3 and Theorem 3.1, if the Reeb vector field of a three dimensional non-Kenmotsu almost Kenmotsu manifold M^3 is minimal and harmonic, we obtain that

$$\eta(Qe) = \eta(Q\varphi e) = 0 \quad \text{and} \quad e(\lambda) = \varphi e(\lambda) = 0, \tag{3.19}$$

where the eigenvalue function λ of h on the nonempty subset $\mathcal{U}_1$ is positive. Then, using relation (3.19) in Lemma 3.2 gives that

$$b = c = 0. \tag{3.20}$$

Using this in relation (3.2) yields that

$$e(a) = \varphi e(a) = 0. \tag{3.21}$$

In this case, we obtain from Lemma 3.2 that

$$\begin{aligned} Q\xi &= -2(\lambda^2+1)\xi, \\ Qe &= -2(\lambda a+1)e + (\xi(\lambda)+2\lambda)\varphi e, \\ Q\varphi e &= (\xi(\lambda)+2\lambda)e + 2(\lambda a-1)\varphi e. \end{aligned} \tag{3.22}$$

Applying Lemma 3.1, from relations (3.19)–(3.22) we have

$$\begin{aligned} (\nabla_\xi Q)\xi &= -4\lambda\xi(\lambda)\xi, \\ (\nabla_e Q)e &= \lambda(\xi(\lambda)+2a)\xi + e(\xi(\lambda))\varphi e, \\ (\nabla_{\varphi e} Q)\varphi e &= \lambda(\xi(\lambda)-2a)\xi + \varphi e(\xi(\lambda))e. \end{aligned} \tag{3.23}$$

Substituting relation (3.23) into the well-known formula

$$\operatorname{div} Q = \frac{1}{2}\operatorname{grad}(r),$$

where grad denotes the usual gradient operator with respect to g, and using relation (3.19), we obtain that the scalar curvature r is a constant if and only if it is invariant along ξ, or equivalently, $\operatorname{tr}(h^2)$ is invariant along ξ.

Therefore, under the hypotheses of the theorem, using (3.22) we obtain that the scalar curvature $r = -2(\lambda^2+3)$ is a global constant, where we have used that λ is continuous. In this context, it follows from relation (3.22) that $\|Q\|^2 = 4(\lambda^4 + 2\lambda^2(a^2+2)+3)$. When $\|Q\|$ is invariant along ξ, taking into account relation (3.21) we obtain that a is also a global constant. Consequently, applying Lemma 3.1, we have that

$$[\xi, e] = (\lambda+a)\varphi e - e, \quad [e, \varphi e] = 0, \quad [\varphi e, \xi] = (a-\lambda)e + \varphi e.$$

Following Milnor [16], we state that M^3 is locally isometric to a non-unimodular Lie group equipped with a left invariant almost Kenmotsu structure. Finally, the converse follows directly from Lemma 3.4. We also refer the reader to Dileo and Pastore [5], Theorem 5.2 for the construction of the almost Kenmotsu structure on 3-dimensional non-unimodular Lie groups. □

We close this paper by discussing the minimality of the Reeb vector field on almost Kenmotsu manifolds of dimension greater than three. First, we need the following lemma.

Lemma 3.5 ([18]). *The Reeb foliation of an almost Kenmotsu manifold is conformal if and only if* $h = 0$.

By Proposition 2.1 and using Lemma 3.5, we obtain the following result, which is a generalization of [10], Proposition 3.12.

Proposition 3.4. *The Reeb vector field of an almost Kenmotsu manifold with conformal Reeb foliation is minimal.*

Proof. Let M^{2n+1} be an almost Kenmotsu manifold of dimension greater than 3 whose Reeb foliation $\ker\varphi$ is conformal. From Lemma 3.5 we have $h = 0$. Using this, we obtain from Dileo and Pastore [4], Theorem 2 that M^{2n+1} is locally isometric to the warped product $C \times_{e^t} N$, where N is an almost Kähler manifold and the Reeb vector field ξ is tangent to the open interval C with coordinate t. On the other hand, following [10], Proposition 3.8, we observe that the unit vector field $\partial/\partial t$ tangent to $\mathbb{R}$ is a minimal vector field on a warped product $\mathbb{R} \times_f M'$. This completes the proof. □

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References

[1] *D. E. Blair*: Riemannian Geometry of Contact and Symplectic Manifolds. Progress in Mathematics 203, Birkhäuser, Boston, 2010. zbl MR doi

[2] *E. Boeckx, L. Vanhecke*: Harmonic and minimal vector fields on tangent and unit tangent bundles. Differ. Geom. Appl. *13* (2000), 77–93. zbl MR doi

[3] *J. T. Cho, M. Kimura*: Reeb flow symmetry on almost contact three-manifolds. Differ. Geom. Appl. *35* (2014), 266–273. zbl MR doi

[4] *G. Dileo, A. M. Pastore*: Almost Kenmotsu manifolds and local symmetry. Bull. Belg. Math. Soc. Simon Stevin *14* (2007), 343–354. zbl MR

[5] *G. Dileo, A. M. Pastore*: Almost Kenmotsu manifolds and nullity distributions. J. Geom. *93* (2009), 46–61. zbl MR doi

[6] *G. Dileo, A. M. Pastore*: Almost Kenmotsu manifolds with a condition of η-parallelism. Differ. Geom. Appl. *27* (2009), 671–679. zbl MR doi
[7] *O. Gil-Medrano*: Relationship between volume and energy of vector fields. Differ. Geom. Appl. *15* (2001), 137–152. zbl MR doi
[8] *O. Gil-Medrano, E. Llinares-Fuster*: Minimal unit vector fields. Tohoku Math. J., II. *54* (2002), 71–84. zbl MR doi
[9] *H. Gluck, W. Ziller*: On the volume of a unit vector field on the three-sphere. Comment. Math. Helv. *61* (1986), 177–192. zbl MR doi
[10] *J. C. González-Dávila, L. Vanhecke*: Examples of minimal unit vector fields. Ann. Global Anal. Geom. *18* (2000), 385–404. zbl MR doi
[11] *J. C. González-Dávila, L. Vanhecke*: Minimal and harmonic characteristic vector fields on three-dimensional contact metric manifolds. J. Geom. *72* (2001), 65–76. zbl MR doi
[12] *J. C. González-Dávila, L. Vanhecke*: Invariant harmonic unit vector fields on Lie groups. Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) *5* (2002), 377–403. zbl MR
[13] *D. Janssens, L. Vanhecke*: Almost contact structures and curvature tensors. Kodai Math. J. *4* (1981), 1–27. zbl MR doi
[14] *K. Kenmotsu*: A class of almost contact Riemannian manifolds. Tohoku Math. J., II. Ser. *24* (1972), 93–103. zbl MR doi
[15] *T. Koufogiorgos, M. Markellos, V. J. Papantoniou*: The harmonicity of the Reeb vector field on contact metric 3-manifolds. Pac. J. Math. *234* (2008), 325–344. zbl MR doi
[16] *J. W. Milnor*: Curvature of left invariant metrics on Lie groups. Adv. Math. *21* (1976), 293–329. zbl MR doi
[17] *Z. Olszak*: Local conformal almost cosymplectic manifolds. Colloq. Math. *57* (1989), 73–87. zbl MR
[18] *A. M. Pastore, V. Saltarelli*: Almost Kenmotsu manifolds with conformal Reeb foliation. Bull. Belg. Math. Soc. Simon Stevin *18* (2011), 655–666. zbl MR
[19] *A. M. Pastore, V. Saltarelli*: Generalized nullity distributions on almost Kenmotsu manifolds. Int. Electron. J. Geom. *4* (2011), 168–183. zbl MR
[20] *D. Perrone*: Harmonic characteristic vector fields on contact metric three-manifolds. Bull. Aust. Math. Soc. *67* (2003), 305–315. zbl MR doi
[21] *D. Perrone*: Almost contact metric manifolds whose Reeb vector field is a harmonic section. Acta Math. Hung. *138* (2013), 102–126. zbl MR doi
[22] *D. Perrone*: Minimal Reeb vector fields on almost cosymplectic manifolds. Kodai Math. J. *36* (2013), 258–274. zbl MR doi
[23] *V. Saltarelli*: Three-dimensional almost Kenmotsu manifolds satisfying certain nullity conditions. Bull. Malays. Math. Sci. Soc. *38* (2015), 437–459. zbl MR doi
[24] *E. Vergara-Diaz, C. M. Wood*: Harmonic almost contact structures. Geom. Dedicata *123* (2006), 131–151. zbl MR doi

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