Antonio M. Oller-Marcén
On a certain class of arithmetic functions

Mathematica Bohemica, Vol. 142 (2017), No. 1, 21–25

Persistent URL: http://dml.cz/dmlcz/146006

ON A CERTAIN CLASS OF ARITHMETIC FUNCTIONS

Antonio M. Oller-Marcén, Zaragoza

Received October 20, 2014. First published October 18, 2016.
Communicated by Dagmar Medková

Abstract. A homothetic arithmetic function of ratio K is a function $f\colon \mathbb{N} \to R$ such that $f(Kn) = f(n)$ for every $n \in \mathbb{N}$. Periodic arithmetic funtions are always homothetic, while the converse is not true in general. In this paper we study homothetic and periodic arithmetic functions. In particular we give an upper bound for the number of elements of $f(\mathbb{N})$ in terms of the period and the ratio of f.

Keywords: arithmetic function; periodic function; homothetic function

MSC 2010: 11A25, 11B99

1. Introduction

Throughout the paper R will denote any ring. Recall that an arithmetic function $f\colon \mathbb{N} \to R$ is said to be *periodic* of period T if $f(n) = f(n+T)$ for every $n \in \mathbb{N}$ and T is minimal with this property. The class of periodic arithmetic functions has received much attention in the literature [3], [6], [5], [7], [8].

Let $f\colon \mathbb{N} \to R$ be a periodic function of period T and let K be an integer such that $K \equiv 1 \pmod{T}$; i.e., $K = \alpha T + 1$ for some α. Then, for every $n \in \mathbb{N}$ we have that

$$f(Kn) = f((\alpha T + 1)n) = f(n + (\alpha T)n) = f(n).$$

This fact motivates the following definition.

Definition 1.1. A function $f\colon \mathbb{N} \to R$ is said to be *homothetic* of ratio K if $f(Kn) = f(n)$ for every $n \in \mathbb{N}$.

With our notation we have just seen that periodic functions are always homothetic. Nevertheless, the converse is trivially false as shown by the following easy example.

DOI: 10.21136/MB.2017.0071-14

E x a m p l e 1.1. Consider the function $f\colon \mathbb{N} \to \mathbb{R}$ given by

$$f(n) = \begin{cases} 1, & \text{if } n \text{ is a power of 2,} \\ 0, & \text{otherwise.} \end{cases}$$

Clearly f is homothetic of ratio any power of 2, while it is not periodic since there is no T with the property that $2^a + T$ is a power of 2 for every a.

A more interesting example of homothetic non-periodic sequence is given by the following construction. Let p be any prime and let us denote by $\mathfrak{S}_p(n)$ the last nonzero digit of n^n in base p. Then $\mathfrak{S}_p(pn) = \mathfrak{S}_p(n)$ and $\mathfrak{S}_p$ is not periodic [4]. Also Liouville's function [1], given by $\lambda(n) = (-1)^{r_1+\dots+r_s}$ if $n = p_1^{r_1} \dots p_s^{r_s}$ gives us an example of homothetic non-periodic function, where any perfect square is a ratio for λ.

Although we have seen that a periodic function with period T is always homothetic of ratio $T+1$, we can give examples of homothetic functions of ratio K such that they are not periodic of period $K-1$. Thus, if f is both periodic of period T and homothetic of ratio K, the period T and the ratio K need not be related.

2. The range of a homothetic and periodic function

Clearly the range of a periodic function of period T has at most T different elements. In this section we are interested in giving a bound for the number of elements of the range of a function which is both periodic of period T and homothetic of ratio K.

A first easy result in this direction that motivates the general question is following lemma.

Lemma 2.1. *Let $f\colon \mathbb{N} \to R$ be a periodic function of period T which is also homothetic of ratio T. Then f is constant.*

P r o o f. Given $n \in \mathbb{N}$ we have that

$$f(n) = f(Tn) = f(Tn + T) = f(T(n+1)) = f(n+1).$$

$\square$

Let f be a function which is both periodic of period T and homothetic of ratio K. We consider the homomorphism (an automorphism if K is coprime to T) $L_K\colon \mathbb{Z}/T\mathbb{Z} \to \mathbb{Z}/T\mathbb{Z}$ given by the multiplication by K; i.e., $L_K(\overline{n}) = \overline{Kn}$. It is clear that, in order to study the maximum number of elements of $f(\mathbb{N})$, we have to study the orbits of the elements of $\mathbb{Z}/T\mathbb{Z}$ under this homomorphism. This is because if $\overline{n} = L_K(\overline{n}')$, then clearly $f(n) = f(n')$.

Moreover, if $T = p_1^{r_1} \dots p_s^{r_s}$ is the prime power decomposition of T, since $\mathbb{Z}/T\mathbb{Z} \cong \mathbb{Z}/p_1^{r_1}\mathbb{Z} \times \dots \times \mathbb{Z}/p_s^{r_s}\mathbb{Z}$ we can restrict ourselves to the case when T is a prime power.

Before we proceed let us introduce some notation. Given $n \in \mathbb{Z}$ and p a prime such that p does not divide n we will denote by $\mathrm{ord}_{p^m}(n)$ the order of n in $\mathbb{Z}/p^m\mathbb{Z}$; i.e., $n^{\mathrm{ord}_{p^m}(n)} \equiv 1 \pmod{p^m}$ and $\mathrm{ord}_{p^m}(n)$ is minimal with this property.

Proposition 2.1. *Let $f\colon \mathbb{N} \to R$ be a periodic function of period p^m which is also homothetic of ratio K, with $m > 0$ and p a prime such that $p \mid K$. Under these conditions, f is constant.*

Proof. We can put $K = p^\kappa K'$ with $\kappa > 0$ and we consider two cases:
If $\kappa \geqslant m$: Let $n \in \mathbb{N}$. Then, in this case we have that

$$\begin{aligned} f(n) &= f(Kn) = f(Kn + p^m) = f(Kn + p^m p^{\kappa - m} K') \\ &= f(K(n+1)) = f(n+1). \end{aligned}$$

If $\kappa < m$: In this case we can find an integer r such that $\kappa r \geqslant m$. Then we get that

$$\begin{aligned} f(n) &= f(K^r n) = f(K^r n + p^m) = f(K^r n + p^m p^{\kappa r - m} (K')^r) \\ &= f(K^r(n+1)) = f(n+1). \end{aligned}$$

Thus, in both cases f is constant and the proof is complete. $\square$

Remark 2.1. Observe that Lemma 2.1 is just a particular case of the proposition above.

Now we turn to the opposite case.

Proposition 2.2. *Let $f\colon \mathbb{N} \to R$ be a periodic function of period p^m which is also homothetic of ratio K, with $m > 0$ and p a prime such that p does not divide K. Under these conditions,*

$$\# f(\mathbb{N}) \leqslant \frac{p^{m-1}(p-1)}{\mathrm{ord}_{p^m}(K)} + p^{m-1}.$$

Proof. Let $L_K\colon \mathbb{Z}/p^m\mathbb{Z} \to \mathbb{Z}/p^m\mathbb{Z}$ be the automorphism given by $L_K(\overline{n}) = \overline{Kn}$. This automorphism has at most $p^m - \varphi(p^m) = p^{m-1}$ fixed points. The rest of elements of $\mathbb{Z}/p^m\mathbb{Z}$ are grouped in orbits of length $\mathrm{ord}_{p^m}(K)$. Thus, there are $\varphi(p^m)/\mathrm{ord}_{p^m}(K)$ of such orbits.

Hence, we have seen that L_K has exactly $(p^{m-1}(p-1))/\mathrm{ord}_{p^m}(K) + p^{m-1}$ different orbits, and the proof is complete. $\square$

These two propositions lead to the main result of the paper.

Teorem 2.1. *Let $f\colon \mathbb{N} \to R$ be a periodic function of period T which is also homothetic of ratio K. Put $T = p_1^{r_1} \dots p_s^{r_s}$. Then:*

$$\#f(\mathbb{N}) \leqslant \prod_{p_i \nmid K} \Big(\frac{p_i^{r_i-1}(p_i-1)}{\operatorname{ord}_{p_i^{r_i}}(K)} + p_i^{r_i-1} \Big).$$

R e m a r k 2.2. Assume that $f\colon \mathbb{N} \to R$ is periodic of period T. In this case we already know that $\#f(\mathbb{N}) \leqslant T$. If, in addition, f is homothetic of ratio K and $T = p_1^{r_1} \dots p_s^{r_s}$ we have just seen that

$$\#f(\mathbb{N}) \leqslant \prod_{p_i \nmid K} \Big(\frac{p_i^{r_i-1}(p_i-1)}{\operatorname{ord}_{p_i^{r_i}}(K)} + p_i^{r_i-1} \Big) \leqslant K$$

Clearly, the equality holds if and only if $\operatorname{ord}_{p_i^{r_i}}(K) = 1$ for every $i \in \{1, \dots, s\}$; i.e., if and only if $K \equiv 1 \pmod{T}$. Note that this is the case (recall the Introduction) when the ratio K comes from the period T.

We will conclude this section and the paper with two corollaries giving particular cases. But before that we will introduce some notation. Recall that the *Dedekind ψ function* [2] is given by

$$\psi(n) = n \prod_{p \mid n} \Big(1 + \frac{1}{p}\Big).$$

It can be easily seen that $\psi(n)$ also admits the following expression:

$$\psi(n) = \frac{n}{\operatorname{rad}(n)} \prod_{p \mid n} (p+1).$$

Corollary 2.1. *Let $f\colon \mathbb{N} \to R$ be a periodic function of period T which is also homothetic of ratio $T-1$. If T is odd, then $\#f(\mathbb{N}) \leqslant \psi(T)/2^s$ where s is the number of different prime factors of T.*

P r o o f. If $T = p_1^{r_1} \dots p_s^{r_s}$ it is clear that $\operatorname{ord}_{p_i^{r_i}}(T-1) = 2$. Moreover, p_i does not divide $T-1$ for any $i \in \{1, \dots, s\}$. With this in mind, an application of Theorem 2.1 gives:

$$\begin{aligned}\#f(\mathbb{N}) &\leqslant \prod_{i=1}^{s} \Big(\frac{p_i^{r_i-1}(p_i-1)}{2} + p_i^{r_i-1} \Big) = \prod_{i=1}^{s} \frac{p_i^{r_i-1}(p_i+1)}{2} \\ &= \frac{T}{2^s \operatorname{rad}(T)} \prod_{i=1}^{s} (p_i+1) = \frac{\psi(T)}{2^s}.\end{aligned}$$

$\square$

Corollary 2.2. *Let $f\colon \mathbb{N} \to R$ be a periodic function of period T which is also homothetic of ratio $T-1$. If T is even, then $\#f(\mathbb{N}) \leqslant \frac{1}{3}\psi(T)2^{1-s}$ where s is the number of different prime factors of T.*

P r o o f. The proof goes as in the previous corollary, but observe that in this case $p_i = 2$ for some i and $\operatorname{ord}_{p_i^{r_i}}(T-1) = 1$. We give no further details. $\square$

References

[1] *T. M. Apostol*: Introduction to Analytic Number Theory. Undergraduate Texts in Mathematics, Springer, New York, 1976. zbl MR

[2] *E. Cohen*: A property of Dedekind's ψ-function. Proc. Am. Math. Soc. *12* (1961), 996. zbl MR doi

[3] *A. K. Ghiyasi*: Constants in inequalities for the mean values of some periodic arithmetic functions. Mosc. Univ. Math. Bull. *63* (2008), 265–269; translation from Vest. Mosk. Univ. Mat. Mekh. *63* (2008), 44–48. zbl MR doi

[4] *J. M. Grau, A. M. Oller-Marcén*: On the last digit and the last non-zero digit of n^n in base b. Bull. Korean Math. Soc. *51* (2014), 1325–1337. zbl MR doi

[5] *T. Hessami Pilehrood, K. Hessami Pilehrood*: On a conjecture of Erdős. Math. Notes *83* (2008), 281–284; translation from Mat. Zametki *83* (2008), 312–315. zbl MR doi

[6] *Q.-Z. Ji, C.-G. Ji*: On the periodicity of some Farhi arithmetical functions. Proc. Am. Math. Soc. *138* (2010), 3025–3035. zbl MR doi

[7] *U. Rausch*: Character sums in algebraic number fields. J. Number Theory *46* (1994), 179–195. zbl MR doi

[8] *J. Steuding*: Dirichlet series associated to periodic arithmetic functions and the zeros of Dirichlet L-functions. Analytic and Probabilistic Methods in Number Theory. Proc. Int. Conf., Palanga, Lithuania, 2001 (A. Dubickas et al., eds.). TEV, Vilnius, 2002, pp. 282–296. zbl MR

Author's address: *Antonio Miguel Oller-Marcén*, Centro Universitario de la Defensa de Zaragoza, Academia General Militar. Ctra. de Huesca s/n., C. P. 50090 Zaragoza, Spain, e-mail: `oller@unizar.es`.